



# LAPLACE

## TRANSFORM

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# LAPLACE TRANSFORM



Edisi Pertama

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Hakcipta terpelihara. Tiada bahagian terbitan ini boleh diterbitkan semula atau ditukar dalam apa jua bentuk dengan cara apa jua sama ada secara elektronik, mekanikal, fotokopi, rakaman dan sebagainya sebelum mendapat kebenaran bertulis daripada Ketua Pengarah Perpustakaan Negara Malaysia.

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# Preface

Assalamualaikum w.b.t and Greetings,

Bismillahirrahmanirrahim. Alhamdulillah, thank to Allah for his bounty, the Laplace Transform E-Book has successfully produced. This e-book is designed mainly for students who need a clear understanding of the basic of Laplace Transform. It is also suitable for those who are studying independently. Students are given examples with step by step on how to solve related Laplace Transform problem. At the end of each sub-topics, there are exercise and tutorial questions with answers to enhance understanding of the sub-topic discussed. It is hope that the material in this e-book will benefit educators and students.

We would like to take this opportunity to express our heartfelt gratitude to everyone who assisted, guided, and supported us in completing this e-book. Their perspectives and advice are beneficial. We hope the students will use this e-book wisely in developing their mathematics skills problem solving and thinking.



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# Introduction

## HISTORICAL BACKGROUND

Laplace transform, in mathematics, is a particular integral transform invented by the French mathematician Pierre-Simon Laplace (1749–1827), and systematically developed by the British physicist Oliver Heaviside (1850–1925), to simplify the solution of many differential equations that describe physical processes.

## WHAT IS LAPLACE TRANSFORM?

The Laplace transform is a powerful tool used in many branches of science and engineering to study and solve differential equations, systems of linear equations, and other complex functions.

It is a mathematical operation that is used to “transform” a variable (such as  $x, y, z$  in space, or  $t$  at time) to a parameter of  $s$ . It finds wide applications in various areas of physics, electrical engineering, control engineering, optics, mathematics and signal processing.

The commonly used notation for the Laplace Transform are as follows :

- $L\{f(t)\}$  or  $\mathcal{L}\{f(t)\}$
- $L(f)$  or  $\mathcal{L}(f)$
- $\bar{f}(s)$  or  $f(s)$



# Topic

## APPLY DEFINITION OF LAPLACE TRANSFORM

At the end of this topic, you should be able to:

- Derive Laplace Transform from First Principles.



# Apply Definition of Laplace Transform

The Laplace transform of a function  $f(t)$ , defined for all real numbers  $t \geq 0$ , is the function  $F(s)$ , which is a unilateral transform defined by:

“

$$F(s) = L\{f(t)\} \\ = \int_0^{\infty} e^{-st} f(t) dt$$

”

How to apply the definition of Laplace Transform ?

**Step 1** : Substitute  $f(t)$  into the formula

$$F(s) = \int_0^{\infty} e^{-st} f(t) dt$$

**Step 2** : Solve the exponent (using **law of indices** )

$$e^m \times e^n = e^{m+n}$$

**Step 3** : Factorize  $t$  (with  $-ve$  outside the bracket) (if applicable)

**Step 4** : Integrate



**01****EXAMPLE**

Find the Laplace Transform for the following function by using the **Definition of Laplace Transform**.

a)  $f(t) = 5$



**Step 1 :** Substitute  $f(t)$  into the formula

$$\begin{aligned} F(s) &= \int_0^{\infty} e^{-st} f(t) dt \\ &= \int_0^{\infty} e^{-st} (5) dt \end{aligned}$$

**Step 2 :** Solve the exponents, (using law of indices:  $e^m \times e^n = e^{m+n}$ )

$$= 5 \int_0^{\infty} e^{-st} dt$$

**Step 3 :** Factorize  $t$  (with  $-ve$  outside the bracket)

$$= 5 \int_0^{\infty} e^{-(s)t} dt$$

**Step 4 :** Integrate

$$\begin{aligned} &= 5 \left[ \frac{e^{-st}}{-s} \right]_0^{\infty} \\ &= 5 \left[ \frac{e^{-s(\infty)}}{-s} - \frac{e^{-s(0)}}{-s} \right] \\ &= 5 \left[ 0 - \frac{1}{-s} \right] \\ &= 5 \left[ \frac{1}{s} \right] \\ &= \frac{5}{s} \end{aligned}$$



**02****EXAMPLE**

Find the Laplace Transform for the following function by using the **Definition of Laplace Transform**.

b)  $f(t) = 2e^{3t}$



**Step 1 :** Substitute  $f(t)$  into the formula

$$\begin{aligned} F(s) &= \int_0^{\infty} e^{-st} f(t) dt \\ &= 2 \int_0^{\infty} e^{-st} e^{3t} dt \end{aligned}$$

**Step 2 :** Solve the exponents, (using law of indices:  $e^m \times e^n = e^{m+n}$ )

$$= 2 \int_0^{\infty} e^{-st+3t} dt$$

**Step 3 :** Factorize  $t$  (with  $-ve$  outside the bracket)

$$= 2 \int_0^{\infty} e^{-(s-3)t} dt$$

**Step 4 :** Integrate

$$\begin{aligned} &= 2 \left[ \frac{e^{-(s-3)t}}{-(s-3)} \right]_0^{\infty} \\ &= 2 \left[ \frac{e^{-(s-3)(\infty)}}{-(s-3)} - \frac{e^{-(s-3)(0)}}{-(s-3)} \right] \\ &= 2 \left[ 0 - \frac{1}{-(s-3)} \right] \\ &= \frac{2}{(s-3)} \end{aligned}$$



**03****EXAMPLE**

Find the Laplace Transform for the following function by using the **Definition of Laplace Transform**.

c)  $f(t) = 5e^{4t} + k$  (where  $k$  is constant)



**Step 1 :** Substitute  $f(t)$  into the formula

$$\begin{aligned} F(s) &= \int_0^{\infty} e^{-st} f(t) dt \\ &= \int_0^{\infty} e^{-st} (5e^{4t} + k) dt \end{aligned}$$

**Step 2 :** Solve the exponents, (using law of indices:  $e^m \times e^n = e^{m+n}$ )

$$\begin{aligned} F(s) &= \int_0^{\infty} 5e^{-st} e^{4t} + ke^{-st} dt \\ &= \int_0^{\infty} 5e^{-st+4t} + ke^{-st} dt \end{aligned}$$

**Step 3 :** Factorize  $t$  (with  $-ve$  outside the bracket)

$$F(s) = 5 \int_0^{\infty} e^{-(s-4)t} dt + k \int_0^{\infty} e^{-st} dt$$

**Step 4 :** Integrate

$$\begin{aligned} F(s) &= 5 \left[ \frac{e^{-(s-4)t}}{-(s-4)} \right]_0^{\infty} + k \left[ \frac{e^{-st}}{-s} \right]_0^{\infty} \\ &= 5 \left[ \frac{e^{-(s-4)(\infty)}}{-(s-4)} - \frac{e^{-(s-4)(0)}}{-(s-4)} \right] + k \left[ \frac{e^{-s(\infty)}}{-s} - \frac{e^{-s(0)}}{-s} \right] \\ &= 5 \left[ 0 - \frac{1}{-(s-4)} \right] + k \left[ 0 - \frac{1}{-s} \right] \\ &= \frac{5}{s-4} + \frac{k}{s} \end{aligned}$$





Find the Laplace Transform for the following function by using the Definition of Laplace Transform

d)  $f(t) = 2t$



**Step 1 :** Substitute  $f(t)$  as stated in question into the formula

$$\begin{aligned} F(s) &= \int_0^{\infty} e^{-st} f(t) dt \\ &= \int_0^{\infty} e^{-st} 2t dt \end{aligned}$$

**Step 2 :** Integrate by using integration by part & definite integration

$$\begin{aligned} u &= 2t & dv &= \int e^{-st} dt \\ \frac{du}{dt} &= 2 & v &= \frac{e^{-st}}{-s} \\ du &= 2dt \end{aligned}$$

$$\begin{aligned} \int u dv &= uv - \int v du \\ \int_0^{\infty} e^{-st} 2t dt &= \left[ 2t \cdot \frac{e^{-st}}{-s} \right]_0^{\infty} - \int_0^{\infty} \frac{e^{-st}}{-s} \cdot 2 dt \\ &= \left[ 2t \cdot \frac{e^{-st}}{-s} \right]_0^{\infty} + 2 \int_0^{\infty} \frac{e^{-st}}{s} dt \\ &= \left[ 2t \cdot \frac{e^{-st}}{-s} \right]_0^{\infty} - 2 \left[ \frac{e^{-st}}{s^2} \right]_0^{\infty} \\ &= 2 \left[ \infty \cdot \frac{e^{-s(\infty)}}{-s} - \frac{e^{-s(\infty)}}{s^2} \right] - 2 \left[ 0 \cdot \frac{e^{-s(0)}}{-s} - \frac{e^{-s(0)}}{s^2} \right] \\ &= 0 - 2 \left[ 0 - \frac{1}{s^2} \right] \\ &= \frac{2}{s^2} \end{aligned}$$



## 01

## EXERCISE



Find the Laplace Transform for the following function by using the **Definition of Laplace Transform**.

1.  $f(t) = -3$

2.  $f(t) = 4t$

3.  $f(t) = e^{-5t}$

4.  $f(t) = 2e^{2t} + \sqrt{3}$

5.  $f(t) = \sqrt{e^{2t}} + n$  where  $n$  is constant

6.  $f(t) = t^3$



[Click here for answer!](#)

## 01

## TUTORIAL



By using the **Definition of Laplace Transform**, find the Laplace Transform for the following functions:

1.  $f(t) = 2e^{\frac{t}{2}}$

2.  $f(t) = \sqrt{16e^{8t^2}}$

3.  $f(t) = (e^t + 5)^2$

4.  $f(t) = 2 \cos 3t$



[Click here for solution!](#)



# Topic

## APPLY LAPLACE TRANSFORM

**At the end of this topic, you should be able to:**

- Solve Laplace Transform by using Table of Laplace Transforms
- Apply Laplace Transform properties



# Using Table of Laplace Transforms

The following table of Laplace Transforms shows that each expression in the right-hand column is obtained from the infinite integral solved in the Definition of the Laplace Transform.

LAPLACE TRANSFORM					
No.	f(t)	F(s)	No.	f(t)	F(s)
1	$a$	$\frac{a}{s}$	13	$e^{-at}\sin \omega t$	$\frac{\omega}{(s+a)^2 + \omega^2}$
2	$at$	$\frac{a}{s^2}$	14	$e^{-at}\cos \omega t$	$\frac{s+a}{(s+a)^2 + \omega^2}$
3	$t^n$	$\frac{n!}{s^{n+1}}$	15	$\sinh \omega t$	$\frac{\omega}{s^2 - \omega^2}$
4	$e^{at}$	$\frac{1}{s-a}$	16	$\cosh \omega t$	$\frac{s}{s^2 - \omega^2}$
5	$e^{-at}$	$\frac{1}{s+a}$	17	$e^{at}\sinh \omega t$	$\frac{\omega}{(s-a)^2 - \omega^2}$
6	$te^{-at}$	$\frac{1}{(s+a)^2}$	18	$e^{-at}\sinh \omega t$	$\frac{\omega}{(s+a)^2 - \omega^2}$
7	$t^n e^{at}$ $n = 1, 2, 3$	$\frac{n!}{(s-a)^{n+1}}$	19	$e^{-at}\cosh \omega t$	$\frac{s+a}{(s+a)^2 - \omega^2}$
8	$t^n \cdot f(t)$	$(-1)^n \frac{d^n}{ds^n} [F(s)]$	20	$f_1(t) + f_2(t)$	$F_1(s) + F_2(s)$
9	$\sin \omega t$	$\frac{\omega}{s^2 + \omega^2}$	21	$\int_0^t f(u) du$	$\frac{F(s)}{s}$
10	$\cos \omega t$	$\frac{s}{s^2 + \omega^2}$	22	$f(t-a)u(t-a)$	$e^{-as}F(s)$
11	$t \sin \omega t$	$\frac{2\omega s}{(s^2 + \omega^2)^2}$	23	First derivative $\frac{dy}{dt}, y'(t)$	$sY(s) - y(0)$
12	$t \cos \omega t$	$\frac{s^2 - \omega^2}{(s^2 + \omega^2)^2}$	24	Second derivative $\frac{d^2y}{dt^2}, y''(t)$	$s^2Y(s) - sy(0) - y'(0)$



# Properties of Laplace Transform

Three (3) of the basic properties of Laplace transform are :

“  
Linearity  
”

“  
First Shifting  
”

“  
Multiplication of  $t^n$   
”

## Property 1: Linearity

The Laplace transform is a linear operation; that is, for any functions  $f$  and  $g$  whose transforms exist, and any constant  $k$  exists.

- If  $\mathcal{L}f(t)$  and  $\mathcal{L}g(t)$  exist, then
$$\mathcal{L}(f(t) \pm g(t)) = \mathcal{L}f(t) \pm \mathcal{L}g(t)$$
- If  $k$  is constant, then
$$\mathcal{L}k\{f(t)\} = k\mathcal{L}\{f(t)\}$$

“ “  
**How to determine the Laplace Transform for a function ?**

**Step 1 :** Expand or separate the function (if applicable)

**Step 2 :** Put the Laplace Transform symbol on each term

**Step 3 :** Refer to the Laplace Transform table for each term  
” ”



01

## EXAMPLE



Determine the Laplace Transform of the following function

a)  $f(t) = e^{3t} + 2t^3 + 2$



**Step 1 :** Put the Laplace Transform symbol on each term

$$F(s) = \mathcal{L}e^{3t} + 2\mathcal{L}t^3 + \mathcal{L}2$$

**Step 2 :** Refer to the Laplace Transform table for each term

Laplace Transform Table	
$f(t)$	$F(s)$
$a$	$\frac{a}{s}$
$e^{at}$	$\frac{1}{s-a}$
$t^n$	$\frac{n!}{s^{n+1}}$

$$\begin{aligned}
 F(s) &= \mathcal{L}e^{3t} + 2\mathcal{L}t^3 + \mathcal{L}2 \\
 &= \frac{1}{s-3} + 2 \cdot \frac{n!}{s^{n+1}} + \frac{a}{s} \\
 &= \frac{1}{s-3} + 2 \cdot \frac{3!}{s^{3+1}} + \frac{2}{s} \\
 &= \frac{1}{s-3} + \frac{12}{s^4} + \frac{2}{s}
 \end{aligned}$$



**02****EXAMPLE**

Determine the Laplace Transform of the following function

b)  $f(t) = 3e^{2t} + \cos 5t$



**Step 1 :** Put the Laplace Transform symbol

$$F(s) = \mathcal{L}3e^{2t} + \mathcal{L}\cos 5t$$

**Step 2 :** Refer to the Laplace Transform table for each term

Laplace Transform Table	
$f(t)$	$F(s)$
$e^{at}$	$\frac{1}{s-a}$
$\cos \omega t$	$\frac{s}{s^2 + \omega^2}$

$$\begin{aligned} F(s) &= \mathcal{L} 3 e^{2t} + \mathcal{L} \cos 5t \\ &= 3 \cdot \frac{1}{s-a} + \frac{s}{s^2 + \omega^2} \\ &= \frac{3}{s-2} + \frac{s}{s^2 + 5^2} \\ &= \frac{3}{s-2} + \frac{s}{s^2 + 25} \end{aligned}$$



## 03

## EXAMPLE



Determine the Laplace Transform of the following function

$$c) f(t) = (t^2 + 1)(t^4 - 2)$$



**Step 1 :** Expand the function

$$f(t) = t^6 - 2t^2 + t^4 - 2$$

**Step 2 :** Put the Laplace Transform symbol

$$F(s) = \mathcal{L}t^6 - 2\mathcal{L}t^2 + \mathcal{L}t^4 - \mathcal{L}2$$

**Step 3 :** Refer to the Laplace Transform table for each term

Laplace Transform Table	
$f(t)$	$F(s)$
$a$	$\frac{a}{s}$
$t^n$	$\frac{n!}{s^{n+1}}$

$$\begin{aligned}
 F(s) &= \mathcal{L}t^6 - 2\mathcal{L}t^2 + \mathcal{L}t^4 - \mathcal{L}2 \\
 &= \frac{n!}{s^{n+1}} - 2 \cdot \frac{n!}{s^{n+1}} + \frac{n!}{s^{n+1}} - \frac{a}{s} \\
 &= \frac{6!}{s^{6+1}} - 2 \cdot \frac{2!}{s^{2+1}} + \frac{4!}{s^{4+1}} - \frac{2}{s} \\
 &= \frac{720}{s^7} - \frac{4}{s^3} + \frac{24}{s^5} - \frac{2}{s}
 \end{aligned}$$



## 04

## EXAMPLE



Determine the Laplace Transform of the following function

$$d) f(t) = \frac{4e^{7t}+3}{e^{4t}}$$



**Step 1 :** Separate the function

$$\begin{aligned} f(t) &= \frac{4e^{7t}}{e^{4t}} + \frac{3}{e^{4t}} \\ &= 4e^{3t} + 3e^{-4t} \end{aligned}$$

**Step 2 :** Put the Laplace Transform symbol

$$F(s) = 4\mathcal{L}e^{3t} + 3\mathcal{L}e^{-4t}$$

**Step 3 :** Refer to the Laplace Transform table for each term

Laplace Transform Table	
$f(t)$	$F(s)$
$e^{at}$	$\frac{1}{s-a}$
$e^{-at}$	$\frac{1}{s+a}$

$$\begin{aligned} F(s) &= 4\mathcal{L}e^{3t} + 3\mathcal{L}e^{-4t} \\ &= 4 \cdot \frac{1}{s-3} + 3 \cdot \frac{1}{s+4} \\ &= \frac{4}{s-3} + \frac{3}{s+4} \end{aligned}$$



## 02

## EXERCISE



Determine the Laplace Transform of the following functions.

1.  $f(t) = e^{-2t} + 3t + 1$

2.  $f(t) = 2e^{4t} + \sin 2t$

3.  $f(t) = (2t^2 + 1)^2$

4.  $f(t) = (t - 3)(2t + 3)$

5.  $f(t) = \frac{e^{10t} + 3}{2e^{5t}}$

6.  $f(t) = \frac{\cos 5t}{4} + \sin 10t - 2$



[Click here for answer !](#)

## 02

## TUTORIAL



By using the **Table of Laplace Transform**, find the Laplace Transform for the following functions:

1.  $f(t) = 3t^2(e^{2t} - t)^2$

2.  $F(s) = 3 \mathcal{L} t \cos 2t - 5 \mathcal{L} t \sin 4t + \mathcal{L} t^3 e^{4t}$

3.  $f(t) = \frac{2 \tan 2t}{\cos 2t} + e^{-3t} \sinh 5t + \frac{t^4}{2}$

4.  $f(t) = \frac{9t^4 + 6t^3 + t^2}{e^{5t}}$



[Click here for solution !](#)



# Properties of Laplace Transform

## Property 2: First Shifting

The **first shifting theorem** provides a convenient way of calculating the Laplace transform of functions that are of the form  $e^{at} \cdot f(t)$  where  $a$  is a constant and  $f(t)$  is a given function. In formula,

$$\mathcal{L}f(t) = F(s)$$

$$\mathcal{L}e^{at}f(t) = F(s - a)$$



How to solve the Laplace Transform ?

**Step 1 :** From  $\mathcal{L}e^{at}f(t)$ , identify  $a$  and  $f(t)$

**Step 2 :** Find the Laplace Transform of  $f(t)$

→ Refer Laplace Transform Table

**Step 3 :** Substitute  $s$  with  $s - a$



**01****EXAMPLE**

Determine the Laplace Transform of the following function

a)  $f(t) = e^{2t}t^3$



**Step 1** : Identify  $a$  and  $f(t)$

$$\mathcal{L}e^{at}f(t) = \mathcal{L}e^{2t}t^3$$

$$a = 2 \quad \text{and} \quad f(t) = t^3$$

$$\therefore \mathcal{L}e^{2t}t^3 = F(s - 2)$$

**Step 2** : Find the Laplace Transform of  $f(t)$  → Refer Laplace Transform Table

Laplace Transform Table	
$f(t)$	$F(s)$
$t^n$	$\frac{n!}{s^{n+1}}$

$$\begin{aligned} F(s) &= \mathcal{L}t^3 \\ &= \frac{3!}{s^{3+1}} \\ &= \frac{6}{s^4} \end{aligned}$$

**Step 3** : Substitute  $s$  with  $s - a$  into  $\frac{6}{s^4}$

$$\begin{aligned} \text{Thus, } F(s - a) &= F(s - 2) \\ &= \frac{6}{(s - 2)^4} \end{aligned}$$

$$\text{Hence, } \mathcal{L}e^{2t}t^3 = \frac{6}{(s-2)^4}$$



## 02

## EXAMPLE



Determine the Laplace Transform of the following function

$$\text{b) } f(t) = \frac{\sin 3t}{e^{5t}}$$



Change the form of the function

$$f(t) = e^{-5t} \sin 3t$$

**Step 1 :** Identify  $a$  and  $f(t)$

$$\mathcal{L}e^{at}f(t) = \mathcal{L}e^{-5t} \sin 3t$$

$$a = -5 \quad \text{and} \quad f(t) = \sin 3t,$$

$$\therefore \mathcal{L}e^{-5t} \sin 3t = F(s - (-5)) = F(s + 5)$$

**Step 2 :** Find the Laplace Transform of  $f(t) \rightarrow$  Refer Laplace Transform Table

Laplace Transform Table	
$f(t)$	$F(s)$
$\sin \omega t$	$\frac{\omega}{s^2 + \omega^2}$

$$F(s) = \mathcal{L}\sin 3t$$

$$= \frac{3}{s^2 + 3^2}$$

$$= \frac{3}{s^2 + 9}$$

**Step 3 :** Substitute  $s$  with  $s + a$  into  $\frac{3}{s^2+9}$

Thus,  $F(s + a) = F(s + 5)$

$$\begin{aligned} \therefore \mathcal{L}e^{-5t} \sin 3t &= \frac{3}{(s + 5)^2 + 9} \\ &= \frac{3}{s^2 + 10s + 25 + 9} \\ &= \frac{3}{s^2 + 10s + 34} \end{aligned}$$





## 03

## EXAMPLE



Determine the Laplace Transform of the following function

$$c) \quad f(t) = 3t + e^{4t} \cosh 2t$$

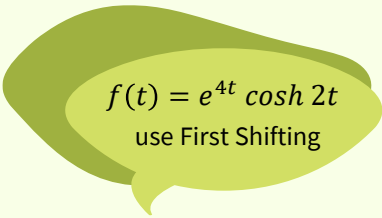

**Solution**


**Step 1 :** Identify  $a$  and  $f(t)$  for First Shifting

$$\mathcal{L}e^{at}f(t) = \mathcal{L}e^{4t} \cosh 2t$$

$$a = 4 \quad \text{and} \quad f(t) = \cosh 2t,$$

$$\therefore \mathcal{L}e^{4t} \cosh 2t = F(s - 4)$$



$$f(t) = e^{4t} \cosh 2t$$

use First Shifting

**Step 2 :** Find the Laplace Transform of  $f(t)$  → Refer Laplace Transform Table

Laplace Transform Table	
$f(t)$	$F(s)$
$at$	$\frac{a}{s^2}$
$\cosh \omega t$	$\frac{s}{s^2 - \omega^2}$

$$F(s) = \mathcal{L} 3t + \mathcal{L} \cosh 2t$$

$$= \frac{3}{s^2} + \frac{s}{s^2 - 2^2}$$

$$= \frac{3}{s^2} + \frac{s}{s^2 - 4}$$

**Step 3 :** Substitute **all**  $s$  with  $(s - a)$  into  $\frac{s}{s^2 - 4}$

$$\text{Thus, } F(s - a) = F(s - 4)$$

$$\begin{aligned} \therefore \mathcal{L}3t + \mathcal{L}e^{4t} \cosh 2t &= \frac{3}{s^2} + \frac{s - 4}{(s - 4)^2 - 4} \\ &= \frac{3}{s^2} + \frac{s - 4}{s^2 - 8s + 16 - 4} \\ &= \frac{3}{s^2} + \frac{s - 4}{s^2 - 8s + 12} \end{aligned}$$



## 04

## EXAMPLE



Determine the Laplace Transform of the following function

$$d) \quad f(t) = e^{3t}(3 - 2t^2)$$



Expand the function

$$\therefore f(t) = 3e^{3t} - 2e^{3t}t^2$$

$$f(t) = 2e^{3t}t^2$$

use First Shifting

**Step 1 :** Identify  $a$  and  $f(t)$  for First Shifting

$$\mathcal{L}e^{at}f(t) = \mathcal{L}2e^{3t}t^2$$

$$a = 3 \quad \text{and} \quad f(t) = 2t^2$$

$$\therefore \mathcal{L}2e^{3t}t^2 = F(s - 3)$$

**Step 2 :** Find the Laplace Transform of  $f(t) \rightarrow$  Refer Laplace Transform Table

Laplace Transform Table	
$f(t)$	$F(s)$
$t^n$	$\frac{n!}{s^{n+1}}$
$e^{at}$	$\frac{1}{s - a}$

$$\begin{aligned} F(s) &= \mathcal{L}3e^{3t} - \mathcal{L}2t^2 \\ &= \frac{3}{s - 3} - 2 \cdot \frac{2!}{s^{2+1}} \\ &= \frac{3}{s - 3} - \frac{4}{s^3} \end{aligned}$$

**Step 3 :** Substitute  $s$  with  $(s - a)$  into  $\frac{4}{s^3}$

Thus,  $F(s - a) = F(s - 3)$

$$\therefore \mathcal{L}3e^{3t} - \mathcal{L}2e^{3t}t^2 = \frac{3}{s - 3} - \frac{4}{(s - 3)^3}$$



## 03

## EXERCISE



By using **first shifting theorem**, determine the Laplace Transform of the following functions:

1.  $f(t) = 3t^4 e^{-2t}$

2.  $f(t) = \frac{\sinh t}{2e^{3t}}$

3.  $f(t) = (e^t + t^3)^2$

4.  $f(t) = \cos 5t + \frac{t^2}{e^t}$

5.  $f(t) = \frac{e^{5t} + 4t^3}{e^{2t}}$

6.  $f(t) = (t + 1)(2 - e^{4t})$



[Click here for answer !](#)

## 03

## TUTORIAL



Find the Laplace Transform for the following functions by using **first shifting theorem** method:

1.  $f(t) = (e^{3t} + t^2)^2$

2.  $f(t) = \frac{t^3 + 2t^2 + t}{e^{3t}}$

3.  $f(t) = 3e^t + e^{4t} \cosh 2t$

4.  $f(t) = \frac{e^{2t} \sinh \sqrt{5}t}{3}$



[Click here for solution !](#)



# Properties of Laplace Transform

## Property 3 : Multiplication of $t^n$

If  $\mathcal{L}\{f(t)\} = F(s)$ , then

$$\begin{aligned}\mathcal{L}t^n f(t) &= (-1)^n \frac{d^n F(s)}{ds^n} \\ &= (-1)^n F^n(s)\end{aligned}$$

where  $n = 1, 2, 3, \dots$

**How to solve the Laplace Transform : Multiplication of  $t^n$ ?**

**Step 1 :** From  $\mathcal{L}t^n f(t)$ , identify  $n$  and  $f(t)$

**Step 2 :** Find the Laplace Transform of  $f(t) \rightarrow$  Refer Laplace Transform Table

**Step 3 :** Differentiate  $F(s)$  (repeat the differentiation based on the power of  $t$ )

**Step 4 :** Substitute into the formula  $\mathcal{L}t^n f(t) = (-1)^n F^n(s)$



01

## EXAMPLE



Find the Laplace Transform of the following function by using **Multiplication of  $t^n$**

a)  $\mathcal{L}(5t)$


**Solution**


**Step 1 :** From  $\mathcal{L}t^n f(t)$ , identify  $n$  and  $f(t)$

$$\mathcal{L}t^n f(t) = \mathcal{L}(5t)$$

$$n = 1 \quad \text{and} \quad f(t) = 5$$

Since  $n = 1$ , then  
do the **first order**  
**differentiation** only

**Step 2 :** Find the Laplace Transform of  $f(t) \rightarrow$  Refer Laplace Transform Table

Laplace Transform Table	
$f(t)$	$F(s)$
$a$	$\frac{a}{s}$

$$\begin{aligned} F(s) &= \mathcal{L}5 \\ &= \frac{5}{s} \end{aligned}$$

**Step 3 :** Differentiate  $F(s)$  (repeat the differentiation based on the power of  $t$ )

$$\begin{aligned} F(s) &= \frac{5}{s} \\ &= 5s^{-1} \end{aligned}$$

$$\begin{aligned} F'(s) &= (-1) 5s^{-2} \\ &= -\frac{5}{s^2} \end{aligned}$$

**Step 4 :** Substitute into the formula  $\mathcal{L}t^n f(t) = (-1)^n F^n(s)$

$$\begin{aligned} \therefore \mathcal{L}(5t) &= (-1) \left( -\frac{5}{s^2} \right) \\ &= \frac{5}{s^2} \end{aligned}$$



## 02

## EXAMPLE



Find the Laplace Transform of the following function by using **Multiplication of  $t^n$**

b)  $\mathcal{L}(t \cosh 2t)$



**Step 1 :** From  $\mathcal{L}t^n f(t)$ , identify  $n$  and  $f(t)$

$$\mathcal{L}t^n f(t) = \mathcal{L}(t \cosh 2t)$$

$$n = 1 \quad \text{and} \quad f(t) = \cosh 2t$$

Since  $n = 1$ , then do the **first order differentiation** only

**Step 2 :** Find the Laplace Transform of  $f(t) \rightarrow$  Refer Laplace Transform Table

Laplace Transform Table	
$f(t)$	$F(s)$
$\cosh \omega t$	$\frac{s}{s^2 - \omega^2}$

$$\begin{aligned} F(s) &= \mathcal{L} \cosh 2t \\ &= \frac{s}{s^2 - 2^2} \\ &= \frac{s}{s^2 - 4} \end{aligned}$$

**Step 3 :** Differentiate  $F(s)$  (repeat the differentiation based on the power of  $t$ )

$$\begin{aligned} F(s) &= \frac{s}{s^2 - 4} \\ F'(s) &= \frac{(s^2 - 4)(1) - s(2s)}{(s^2 - 4)^2} \\ &= \frac{s^2 - 4 - 2s^2}{(s^2 - 4)^2} \\ &= \frac{-4 - s^2}{(s^2 - 4)^2} = \frac{-(4 + s^2)}{(s^2 - 4)^2} \end{aligned}$$

Use **Quotient Rule** to integrate

$$F'(s) = \frac{v \frac{du}{ds} - u \frac{dv}{ds}}{v^2}$$

**Step 4 :** Substitute into the formula  $\mathcal{L}t^n f(t) = (-1)^n F^n(s)$

$$\begin{aligned} \therefore \mathcal{L}(t \cosh 2t) &= (-1) \left( \frac{-(4 + s^2)}{(s^2 - 4)^2} \right) \\ &= \frac{4 + s^2}{(s^2 - 4)^2} \end{aligned}$$





## 03

## EXAMPLE



Find the Laplace Transform of the following function by using **Multiplication of  $t^n$**

c)  $f(t) = t^2 e^{3t}$

 Solution 

**Step 1 :** From  $\mathcal{L}t^n f(t)$ , identify  $n$  and  $f(t)$

$$\mathcal{L}t^n f(t) = \mathcal{L}(t^2 e^{3t})$$

$$n = 2 \quad \text{and} \quad f(t) = e^{3t}$$

Since  $n = 2$ , then do the **second order differentiation**

**Step 2 :** Find the Laplace Transform of  $f(t) \rightarrow$  Refer Laplace Transform Table

Laplace Transform Table	
$f(t)$	$F(s)$
$e^{at}$	$\frac{1}{s-a}$

$$\begin{aligned} F(s) &= \mathcal{L}e^{3t} \\ &= \frac{1}{s-3} \end{aligned}$$

**Step 3 :** Differentiate  $F(s)$  (repeat the differentiation based on the power of  $t$ )

$$F(s) = \frac{1}{s-3}$$

$$= (s-3)^{-1}$$

$$F'(s) = (-1)(s-3)^{-2}(1)$$

$$= -\frac{1}{(s-3)^2}$$

$$F''(s) = (-2)(-1)(s-3)^{-3}(1)$$

$$= \frac{2}{(s-3)^3}$$

**Step 4 :** Substitute into the formula  $\mathcal{L}t^n f(t) = (-1)^n F^n(s)$

$$\therefore \mathcal{L}(t^2 e^{3t}) = (-1)^2 \left( \frac{2}{(s-3)^3} \right)$$

$$= \frac{2}{(s-3)^3}$$





Find the Laplace Transform of the following function by using **Multiplication of  $t^n$**

d)  $\mathcal{L}(t^2 \sin 4t)$

**Solution**

**Step 1 :** From  $\mathcal{L}t^n f(t)$ , identify  $n$  and  $f(t)$

$$\mathcal{L}t^n f(t) = \mathcal{L}(t^2 \sin 4t)$$

$$n = 2 \quad \text{and} \quad f(t) = \sin 4t$$

Since  $n = 2$ , then do the **second order differentiation**

**Step 2 :** Find the Laplace Transform of  $f(t) \rightarrow$  Refer Laplace Transform Table

Laplace Transform Table	
$f(t)$	$F(s)$
$\sin \omega t$	$\frac{\omega}{s^2 + \omega^2}$

$$\begin{aligned} F(s) &= \mathcal{L}\sin 4t \\ &= \frac{4}{s^2 + 4^2} \\ &= \frac{4}{s^2 + 16} \end{aligned}$$

**Step 3 :** Differentiate  $F(s)$  (repeat the differentiation based on the power of  $t$ )

$$\begin{aligned} F(s) &= \frac{4}{s^2 + 16} \\ &= 4(s^2 + 16)^{-1} \end{aligned}$$

Use **Quotient Rule** to integrate

$$F''(s) = \frac{v \frac{du}{ds} - u \frac{dv}{ds}}{v^2}$$

$$\begin{aligned} F'(s) &= (-1)4(s^2 + 16)^{-2}(2s) \\ &= -8s(s^2 + 16)^{-2} \\ &= \frac{-8s}{(s^2 + 16)^2} \end{aligned}$$

$$\begin{aligned} F''(s) &= \frac{(s^2 + 16)^2 (-8) - (-8s)(2)(s^2 + 16)(2s)}{((s^2 + 16)^2)^2} \\ &= \frac{-8(s^2 + 16)^2 + 32s^2(s^2 + 16)}{(s^2 + 16)^4} \\ &= \frac{8(s^2 + 16)[(s^2 + 16) - 4s^2]}{(s^2 + 16)^4} = \frac{-8(-3s^2 + 16)}{(s^2 + 16)^3} \end{aligned}$$

**Step 4 :** Substitute into the formula  $\mathcal{L}t^n f(t) = (-1)^n F^n(s)$

$$\begin{aligned} \therefore \mathcal{L}(t^2 \sin 4t) &= (-1)^2 \left( \frac{-8(-3s^2 + 16)}{(s^2 + 16)^3} \right) \\ &= \frac{24s^2 - 128}{(s^2 + 16)^3} \end{aligned}$$



## 04

## EXERCISE



Find the Laplace Transform of the following functions by using **Multiplication of  $t^n$** :

1.  $\mathcal{L}\left(\frac{2}{3}t\right)$

2.  $f(t) = 5e^{4t}t^2$

3.  $\mathcal{L}(t^2 \cos 2t)$

4.  $\mathcal{L}(t \sinh t)$

5.  $f(t) = 5 + 2t + t^2 \sin 2t$

6.  $f(t) = t(e^{-3t} + 2)$



[Click here for answer !](#)

## 04

## TUTORIAL



By using the **Multiplication of  $t^n$**  method find the Laplace Transform for the following functions:

1.  $f(t) = e^{4t}(3t + \cos 2t)$

2.  $\mathcal{L}(t \sin \pi t)$

3.  $f(t) = \frac{3e^{\sqrt{5}t}t^2}{2}$

4.  $f(t) = t^2(e^t + 1)^2$



[Click here for solution !](#)



# Topic

## APPLY INVERSE LAPLACE TRANSFORM

**At the end of this topic, you should be able to:**

- Solve Inverse Laplace Transforms by using Table of Laplace Transform
- Apply Inverse Laplace Transforms properties
- Solve differential equation by using Laplace Transform



# Inverse Laplace Transform

To perform an **Inverse Laplace Transform**, we use the table of Laplace transforms to look up the corresponding inverse transform of the Laplace function. We're simply reversing the process of Laplace transforms using the table of Laplace transform. Instead of  $F(s)$ , we're now finding the expression for  $f(t)$ , which is mathematically written as:

$$f(t) = \mathcal{L}^{-1}\{F(s)\}$$

In summary, the **Inverse Laplace Transform** is a powerful mathematical tool that allows us to recover a function's original expression or behavior in the time domain, given its Laplace transform in the  $s$ -domain. The inverse Laplace transform is an essential tool in many areas of physics, engineering, and mathematics, and is used to solve a wide range of problems, including circuit analysis, control systems, signal processing, and more.

Three (3) basic properties of Inverse Laplace Transform are :



# Properties of Inverse Laplace Transform

## Property 1: Linearity

The **linearity** property of the inverse Laplace transform states that the Inverse Laplace Transform of this combination is the same linear combination of the inverse Laplace transforms of the individual functions. In other words, the inverse Laplace transform is a linear operator.

Mathematically, the linearity property of the inverse Laplace transform can be expressed as:

If given  $\mathcal{L}f(t) = F(s)$ , then :

$$\begin{aligned} \text{i) } \mathcal{L}^{-1}(F(s) \pm G(s)) \\ &= \mathcal{L}^{-1}F(s) \pm \mathcal{L}^{-1}G(s) \\ &= f(t) \pm g(t) \end{aligned}$$

$$\begin{aligned} \text{ii) } \mathcal{L}^{-1}kF(s) \\ &= k\mathcal{L}^{-1}F(s) \\ &= kf(t) \end{aligned}$$

### How to determine the Inverse Laplace Transform for a function ?

**Step 1** : Put the Inverse Laplace Transform symbol on each term

**Step 2** : Compare the denominator of the question with the Laplace Table & state the  $F(s)$  according to the table.

**Step 3** : Compare the numerator with the question & State the answer by referring to the Laplace Transform table



01

## EXAMPLE



Determine the Inverse Laplace Transform of the following function

$$a) F(s) = \frac{3}{s} + \frac{4}{s^3} + \frac{s}{s^2+4}$$



**Step 1 :** Put the Inverse Laplace Transform symbol on each term

$$f(t) = \mathcal{L}^{-1} \frac{3}{s} + \mathcal{L}^{-1} \frac{4}{s^3} + \mathcal{L}^{-1} \frac{s}{s^2+4}$$

**Step 2 :** Compare the denominator of the question with the Laplace Table & State the  $F(s)$  according to the table.

Laplace Transform Table	
$f(t)$	$F(s)$
$a$	$\frac{a}{s}$
$t^n$	$\frac{n!}{s^{n+1}}$
$\cos \omega t$	$\frac{s}{s^2 + \omega^2}$

$$\begin{aligned} f(t) &= \mathcal{L}^{-1} \frac{a}{s} + \mathcal{L}^{-1} \frac{n!}{s^{n+1}} + \mathcal{L}^{-1} \frac{s}{s^2 + \omega^2} \\ &= \mathcal{L}^{-1} \frac{3}{s} + 2 \mathcal{L}^{-1} \frac{2!}{s^{2+1}} + \mathcal{L}^{-1} \frac{s}{s^2 + 2^2} \end{aligned}$$

**Step 3 :** Compare the numerator with the question & State the answer by referring to the Laplace Transform table

$$\therefore f(t) = 3 + 2t^2 + \cos 2t$$



## 02

## EXAMPLE



Determine the Inverse Laplace Transform of the following function

$$b) F(s) = \frac{6}{s^2-9} + \frac{1}{s^4} + \frac{5}{s-4}$$



**Step 1 :** Put the Inverse Laplace Transform symbol on each term

$$f(t) = \mathcal{L}^{-1} \frac{6}{s^2-9} + \mathcal{L}^{-1} \frac{1}{s^4} + \mathcal{L}^{-1} \frac{5}{s-4}$$

**Step 2 :** Compare the denominator of the question with the Laplace Table & State the  $F(s)$  according to the table.

Laplace Transform Table	
$f(t)$	$F(s)$
$t^n$	$\frac{n!}{s^{n+1}}$
$e^{at}$	$\frac{1}{s-a}$
$\sinh \omega t$	$\frac{\omega}{s^2 - \omega^2}$

$$\begin{aligned} f(t) &= \mathcal{L}^{-1} \frac{\omega}{s^2 + \omega^2} + \mathcal{L}^{-1} \frac{n!}{s^{n+1}} + \mathcal{L}^{-1} \frac{1}{s-a} \\ &= \mathcal{L}^{-1} \frac{3}{s^2 + 3^2} + \mathcal{L}^{-1} \frac{3!}{s^{3+1}} + \mathcal{L}^{-1} \frac{1}{s-4} \end{aligned}$$

**Step 3 :** Compare the numerator with the question & state the answer by referring to the Laplace Transform table

$$\begin{aligned} \therefore f(t) &= 2 \cdot \mathcal{L}^{-1} \frac{3}{s^2 + 3^2} + \frac{1}{6} \cdot \mathcal{L}^{-1} \frac{3!}{s^{3+1}} + 5 \cdot \mathcal{L}^{-1} \frac{1}{s-4} \\ &= 2 \sinh 3t + \frac{1}{6} t^3 + 5e^{4t} \end{aligned}$$



## 03

## EXAMPLE



Determine the Inverse Laplace Transform of the following function

$$c) F(s) = \frac{1-5s}{s^2+49}$$



If the numerator has an operation and the denominator is in the table, the function **NEED to be separated**.

$$F(s) = \frac{1}{s^2 + 49} - \frac{5s}{s^2 + 49}$$

**Step 1 :** Put the Inverse Laplace Transform symbol on each term

$$f(t) = \mathcal{L}^{-1} \frac{1}{s^2 + 49} - \mathcal{L}^{-1} \frac{5s}{s^2 + 49}$$

**Step 2 :** Compare the denominator of the question with the Laplace Table & State the **F(s)** according to the table.

Laplace Transform Table	
$f(t)$	$F(s)$
$\cos \omega t$	$\frac{s}{s^2 + \omega^2}$
$\sin \omega t$	$\frac{\omega}{s^2 + \omega^2}$

$$\begin{aligned} f(t) &= \mathcal{L}^{-1} \frac{\omega}{s^2 + \omega^2} - \mathcal{L}^{-1} \frac{s}{s^2 + a^2} \\ &= \mathcal{L}^{-1} \frac{7}{s^2 + 7^2} - \mathcal{L}^{-1} \frac{s}{s^2 + 7^2} \end{aligned}$$

**Step 3 :** Compare the numerator with the question & State the answer by referring to the Laplace Transform table

$$\begin{aligned} \therefore f(t) &= \frac{1}{7} \mathcal{L}^{-1} \frac{7}{s^2 + 7^2} - 5 \mathcal{L}^{-1} \frac{s}{s^2 + 7^2} \\ &= \frac{1}{7} \sin 7t - 5 \cos 7t \end{aligned}$$



## 04

## EXAMPLE



Determine the Inverse Laplace Transform of the following function

$$d) F(s) = \frac{2}{s^2+9} - \frac{3}{s^5} + \frac{16s}{2s^2-32}$$



The function  $\frac{16s}{2s^2-32}$  need to be **factorized**.

$$= \frac{16s}{2(s^2 - 16)} = \frac{8s}{s^2 - 16}$$

**Step 1 :** Put the Inverse Laplace Transform symbol on each term

$$f(t) = \mathcal{L}^{-1} \frac{2}{s^2+9} - \mathcal{L}^{-1} \frac{3}{s^5} + \mathcal{L}^{-1} \frac{8s}{s^2-16}$$

**Step 2 :** Compare the denominator of the question with the Laplace Table & State the **F(s)** according to the table.

Laplace Transform Table	
$f(t)$	$F(s)$
$t^n$	$\frac{n!}{s^{n+1}}$
$\sin \omega t$	$\frac{\omega}{s^2 + \omega^2}$
$\cosh \omega t$	$\frac{s}{s^2 - \omega^2}$

$$\begin{aligned} f(t) &= \mathcal{L}^{-1} \frac{\omega}{s^2 + \omega^2} - \mathcal{L}^{-1} \frac{n!}{s^{n+1}} + \mathcal{L}^{-1} \frac{s}{s^2 - \omega^2} \\ &= \mathcal{L}^{-1} \frac{3}{s^2 + 3^2} - \mathcal{L}^{-1} \frac{4!}{s^{4+1}} + \mathcal{L}^{-1} \frac{s}{s^2 - 4^2} \end{aligned}$$

**Step 3 :** Compare the numerator with the question & State the answer by referring to the Laplace Transform table

$$\begin{aligned} \therefore f(t) &= \frac{2}{3} \cdot \mathcal{L}^{-1} \frac{3}{s^2 + 3^2} - \frac{3}{24} \cdot \mathcal{L}^{-1} \frac{4!}{s^{4+1}} + 8 \cdot \mathcal{L}^{-1} \frac{s}{s^2 - 4^2} \\ &= \frac{2}{3} \sin 3t - \frac{1}{8} t^4 + 8 \cosh 4t \end{aligned}$$



## 05

## EXERCISE



Determine the **Inverse Laplace Transform** for the followings by using Laplace Transform table:

$$1. \quad F(S) = \frac{2s - 4}{s^2 + 16}$$

$$2. \quad F(S) = \frac{2}{s^2} + \frac{s - 5}{s^2 - 25} - \frac{24}{s^5}$$

$$3. \quad F(S) = \frac{2s - 5}{s^2 + 25}$$

$$4. \quad F(S) = \frac{5}{s + 3} + \frac{3s}{s^2 + 16} - \frac{4}{s}$$

$$5. \quad F(S) = \frac{7}{s^5} + \frac{10}{2s^2 - 8}$$

$$6. \quad F(S) = \frac{24}{s^6} + \frac{16}{2s^2 - 32}$$



[Click here for answer!](#)

## 05

## TUTORIAL



By using the **Laplace Transform Table**, find the Inverse Laplace for the following functions:

$$1. \quad F(s) = \frac{11}{s^4} + \frac{3}{s - 4} - \frac{2s - 6}{s^2 + 49}$$

$$2. \quad f(t) = \mathcal{L}^{-1} \frac{4}{s^2} + \mathcal{L}^{-1} \frac{2s - 5}{s^2 - 25}$$

$$3. \quad f(t) = \mathcal{L}^{-1} \frac{10}{2s^2 - 8} - \mathcal{L}^{-1} \frac{2}{s - 11}$$

$$4. \quad F(s) = \frac{10}{(s + 5)^2 - 16} - \frac{4}{(s - 3)^5}$$



[Click here for solution!](#)



# Properties of Inverse Laplace Transform

## Property 2: Completing the Square

Solving Inverse Laplace Transform using **Completing the Square Method** if the below circumstances occur:

- The **denominator** is **NOT** in the Laplace Transform Table.
- The **denominator** **CANNOT BE** factorized using try and error method.

To solve, the **denominator** **CAN BE** factorize using the formula:

$$\left(s + \frac{b}{2}\right)^2 - \left(\frac{b}{2}\right)^2 + c$$

**How to determine the Inverse Laplace Transform for a function ?**

**Step 1 :** Factorize the denominator (*when necessary*) using formula Completing the Square

**Step 2 :** Write/replace the new denominator into the question.

**Step 3 :** Compare the denominator with the Laplace Transform Table.

**Step 4 :** Write the answer by referring to the Laplace Transform table.

Remember that Completing the square is a method for converting a quadratic expression of the form  $ax^2 + bx + c$  to the form  $a(x \pm h)^2 \pm k$




01

## EXAMPLE



Determine the Inverse Laplace Transform of the following function

$$a) F(s) = \frac{4}{s^2 + 4s + 8}$$



Solution



**DON'T FORGET!**

The denominator **CANNOT** be factorize using try & error method



**Step 1 :** Factorize the denominator (*when necessary*) using formula

Completing the Square

SOLUTION	STEPS
$s^2 + 4s + 8$	
$\left(s + \frac{4}{2}\right)^2 - \left(\frac{4}{2}\right)^2 + 8$	Factorize
$(s + 2)^2 - (2)^2 + 8$	Simplify
$(s + 2)^2 + 4$	

$$\text{Formula: } \left(s + \frac{b}{2}\right)^2 - \left(\frac{b}{2}\right)^2 + c$$

**Step 2 :** Write/replace the new denominator into the question.

$$f(t) = \mathcal{L}^{-1} \frac{4}{(s + 2)^2 + 4}$$

**Step 3 :** Compare the denominator with the Laplace Transform Table.

Laplace Transform Table	
$f(t)$	$F(s)$
$e^{-at} \sin \omega t$	$\frac{\omega}{(s + a)^2 + \omega^2}$

- State the **numerator** according to the table
- Compare the **numerator** of the question & change to make it the same

$$\begin{aligned} f(t) &= \mathcal{L}^{-1} \frac{\omega}{(s + a)^2 + \omega^2} \\ &= 2 \cdot \mathcal{L}^{-1} \frac{2}{(s + 2)^2 + 2^2} \end{aligned}$$

**Step 4 :** Write the answer by referring to the Laplace Transform table.

$$f(t) = 2 e^{-2t} \sin 2t$$



## 02

## EXAMPLE



Determine the Inverse Laplace Transform of the following function

$$b) F(s) = \frac{1}{s^2 + 2s + 17}$$



**DON'T FORGET!**

The denominator **CANNOT** be factorize using try & error method



**Step 1 :** Factorize the denominator (*when necessary*) using formula

Completing the Square

SOLUTION	STEPS
$s^2 + 2s + 17$	
$\left(s + \left(\frac{2}{2}\right)\right)^2 - \left(\frac{2}{2}\right)^2 + 17$	Factorize
$(s + 1)^2 - (1)^2 + 17$	Simplify
$(s + 1)^2 + 16$	

$$\text{Formula: } \left(s + \frac{b}{2}\right)^2 - \left(\frac{b}{2}\right)^2 + c$$

**Step 2 :** Write/replace the new denominator into the question.

$$f(t) = \mathcal{L}^{-1} \frac{1}{(s + 1)^2 + 16}$$

**Step 3 :** Compare the denominator with the Laplace Transform Table.

Laplace Transform Table	
$f(t)$	$F(s)$
$e^{-at} \sin \omega t$	$\frac{\omega}{(s + a)^2 + \omega^2}$

- State the **numerator** according to the table
- Compare the **numerator** of the question & change to make it the same

$$\begin{aligned} f(t) &= \mathcal{L}^{-1} \frac{\omega}{(s + a)^2 + \omega^2} \\ &= \frac{1}{4} \cdot \mathcal{L}^{-1} \frac{4}{(s + 1)^2 + 4^2} \end{aligned}$$

**Step 4 :** Write the answer by referring to the Laplace Transform table.

$$f(t) = \frac{1}{4} e^{-t} \sin 4t$$



03

## EXAMPLE



Determine the Inverse Laplace Transform of the following function

$$c) F(s) = \frac{2s+3}{s^2-4s+11}$$


 Solution


 DON'T FORGET!

The denominator **CANNOT** be factorize using try & error method



**Step 1 :** Factorize the denominator (*when necessary*) using formula

Completing the Square

SOLUTION	STEPS
$s^2 - 4s + 11$	
$\left(s + \left(\frac{-4}{2}\right)\right)^2 - \left(\frac{-4}{2}\right)^2 + 11$	Factorize
$(s - 2)^2 - (-2)^2 + 11$	Simplify
$(s - 2)^2 + 7$	

$$\text{Formula: } \left(s + \frac{b}{2}\right)^2 - \left(\frac{b}{2}\right)^2 + c$$

Substitute  $s$  with  $s - a$  into the numerator & solve

SOLUTION
$2s + 3$
$2(s - 2) + 7$
$2s - 4 + 7 = 2s + 3$

**Step 2 :** Write/replace the new numerator & denominator into the question.

$$\begin{aligned} f(t) &= \mathcal{L}^{-1} \frac{2(s-2) + 7}{(s-2)^2 + 7} \\ &= \mathcal{L}^{-1} \frac{2(s-2)}{(s-2)^2 + 7} + \mathcal{L}^{-1} \frac{7}{(s-2)^2 + 7} \end{aligned}$$

If the numerator has an operation and the denominator is in the table, the function **NEED** to be separated



## 03

## EXAMPLE



**Step 3 :** Compare the denominator with the Laplace Transform Table.

Laplace Transform Table	
$f(t)$	$F(s)$
$e^{at} \sin \omega t$	$\frac{\omega}{(s-a)^2 + \omega^2}$
$e^{at} \cos \omega t$	$\frac{s-a}{(s-a)^2 + \omega^2}$

- State the **numerator** according to the table
- Compare the **numerator** of the question & change to make it the same

$$\begin{aligned}
 f(t) &= \mathcal{L}^{-1} \frac{s-a}{(s-a)^2 + \omega^2} + \mathcal{L}^{-1} \frac{\omega}{(s-a)^2 + \omega^2} \\
 &= 2 \cdot \mathcal{L}^{-1} \frac{(s-2)}{(s-2)^2 + \sqrt{7}^2} + \frac{7}{\sqrt{7}} \cdot \mathcal{L}^{-1} \frac{\sqrt{7}}{(s-4)^2 + \sqrt{7}^2}
 \end{aligned}$$

**Step 4 :** Write the answer by referring to the Laplace Transform table.

$$f(t) = 2 e^{2t} \cos \sqrt{7}t + \frac{7}{\sqrt{7}} e^{2t} \sin \sqrt{7}t$$



## 04

## EXAMPLE



Determine the Inverse Laplace Transform of the following function

$$d) F(s) = \frac{2s+7}{s^2+2s+10}$$



**DON'T FORGET!**

The denominator **CANNOT** be factorize using try & error method



**Step 1 :** Factorize the denominator and numerator (when necessary) using formula Completing the Square

SOLUTION	STEPS
$s^2 + 2s + 10$	
$\left(s + \frac{2}{2}\right)^2 - \left(\frac{2}{2}\right)^2 + 10$	Factorize
$(s + 1)^2 - (1)^2 + 10$	Simplify
$(s + 1)^2 + 9$	

$$\text{Formula: } \left(s + \frac{b}{2}\right)^2 - \left(\frac{b}{2}\right)^2 + c$$

Substitute  $s$  with  $s - a$  into the numerator & solve

SOLUTION
$2s + 7$
$2(s + 1) + 5$
$2s + 2 + 5 = 2s + 7$

**Step 2 :** Write/replace the new denominator & numerator into the question.

$$\begin{aligned} f(t) &= \mathcal{L}^{-1} \frac{2(s+1)+5}{(s+1)^2+9} \\ &= \mathcal{L}^{-1} \frac{2(s+1)}{(s+1)^2+9} + \mathcal{L}^{-1} \frac{5}{(s+1)^2+9} \end{aligned}$$

If the numerator has an operation and the denominator is in the table, the function **NEED** to be separated





**Step 3 :** Compare the denominator with the Laplace Transform Table.

Laplace Transform Table	
$f(t)$	$F(s)$
$e^{-at} \sin \omega t$	$\frac{\omega}{(s+a)^2 + \omega^2}$
$e^{-at} \cos \omega t$	$\frac{s+a}{(s+a)^2 + \omega^2}$

- State the **numerator** according to the table
- Compare the **numerator** of the question & change to make it the same

$$\begin{aligned}
 f(t) &= \mathcal{L}^{-1} \frac{s+a}{(s+a)^2 + \omega^2} + \mathcal{L}^{-1} \frac{\omega}{(s+a)^2 + \omega^2} \\
 &= 2 \cdot \mathcal{L}^{-1} \frac{(s+1)}{(s+1)^2 + 3^2} + \frac{5}{3} \mathcal{L}^{-1} \frac{3}{(s+1)^2 + 3^2}
 \end{aligned}$$

**Step 4 :** Write the answer by referring to the Laplace Transform table.

$$f(t) = 2e^{-t} \cos 3t + \frac{5}{3} e^{-t} \sin 3t$$



## 06

## EXERCISE



Determine the **Inverse Laplace Transform** for the followings by using Completing the Square method.

$$1. \quad F(S) = \frac{5}{s^2 + 4s + 13}$$

$$2. \quad F(S) = \frac{3}{s^2 + 6s + 10}$$

$$3. \quad F(S) = \frac{2s - 3}{s^2 - 6s + 18}$$

$$4. \quad F(S) = \frac{s + 6}{s^2 + 4s - 12}$$

$$5. \quad F(S) = \frac{3s - 1}{s^2 + 2s + 17}$$

$$6. \quad F(s) = \frac{3s - 1}{s^2 + 2s + 16}$$



[Click here for answer !](#)

## 06

## TUTORIAL



By using **Completing the Square method**, find the Inverse Laplace for the following functions:

$$1. \quad f(t) = \mathcal{L}^{-1} \frac{6}{s^2 + 4s + 13}$$

$$2. \quad F(s) = \frac{5}{s^2 - 6s + 10}$$

$$3. \quad f(t) = \mathcal{L}^{-1} \frac{2s}{s^2 - 6s + 18}$$

$$4. \quad F(s) = \frac{s + 6}{s^2 + 4s - 12}$$



[Click here for solution !](#)



# Properties of Inverse Laplace Transform

## Property 3 : Partial Fraction

**Partial Fraction of Inverse Laplace Transform** is a technique to split up a complicated fraction into forms that are matching the function in the Laplace Transform table.

Solving Inverse Laplace Transform using **Partial Fraction Method** if the below circumstances occur:

- o The **denominator** is **NOT** in the Laplace Transform Table.
- o The **denominator** **CAN BE** factorized into two or more function either with distinct real root, repeated real root or complex root.

This technique consist of **Partial Fraction for Linear Factor, Repeated Linear Factor** and **Quadratic. Factor.**

In what condition do we use **Partial Fraction of Inverse Laplace Transform**?:

**Condition 1: Linear Factor**, when denominator has **no power**.

$$\Rightarrow \frac{2x+1}{x(x-2)} = \frac{A}{x} + \frac{B}{x-2}$$

**Condition 2: Repeated Linear Factor**, when denominator **has power outside** the bracket.

$$\Rightarrow \frac{5-3x}{(x+1)^2} = \frac{A}{x+1} + \frac{B}{(x+1)^2}$$

**Condition 3: Quadratic Factor**, when denominator **has power inside** the bracket.

$$\Rightarrow \frac{2x+1}{(x^2-3)} = \frac{Ax+B}{(x^2-3)}$$



# Inverse Laplace Transform

## Property 3 : Partial Fraction

Terms in Partial Fraction $F(s)$	Inverse Laplace $f(t)$
$\frac{A}{s - a}$	$Ae^{at}$
$\frac{A}{(s - a)^2}$	$Ate^{at}$
$\frac{A}{(s - a)^3}$	$\frac{A}{2}t^2e^{at}$
$\frac{As + B}{s^2 + a^2} = \frac{As}{s^2 + a^2} + \frac{B}{s^2 + a^2}$	$A\cos at - B\sin at$

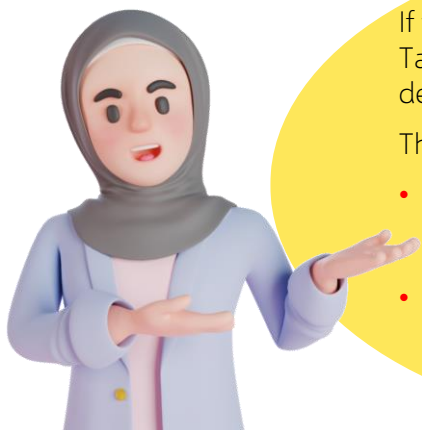
How to determine the Laplace Transform for partial fraction ??

**Step 1** : Write one partial fraction for each of the factors.

**Step 2** : Multiply through by the bottom so we no longer have fractions.

**Step 3** : Find the value of constants **A** and **B**.

**Step 4** : Substitute value of **A** and **B** & Write your answer by referring to the Laplace Transform table.



If the denominator **NOT FOUND** in the Laplace Table, then you need to test whether the denominator **can be FACTORIZE** or **NOT**.

Therefore, if it

- **CANNOT BE FACTORIZE**  
→ use Completing the Square Method
- **CAN BE FACTORIZE**  
→ use Partial Fraction Method



01

## EXAMPLE



## Partial Fraction for Linear Factor

Determine the Inverse Laplace Transform of the following function

$$\text{a) } f(t) = \frac{s+1}{s^2-s}$$



Solution

The denominator need to be **factorized**.

$$\frac{s+1}{s^2-s} = \frac{s+1}{s(s-1)}$$

**Step 1 :** Write one partial fraction for each of the factors

$$\frac{s+1}{s(s-1)} = \frac{A}{s} + \frac{B}{s-1}$$

**Step 2 :** Multiply through by the bottom so we no longer have fractions.

$$\left[ \frac{s+1}{s(s-1)} = \frac{A}{s} + \frac{B}{s-1} \right] \times s(s-1)$$

$$s+1 = A(s-1) + Bs$$

**Step 3 :** Find the value of constants **A** and **B**.

For root  $s$ , let  $s = 0$

$$(0) + 1 = A(0 - 1) + B(0)$$

$$1 = A(-1)$$

$$A = -1$$

For root  $(s - 1)$ , let  $s = 1$

$$(1) + 1 = (-1)(1 - 1) + B(1)$$

$$2 = B(1)$$

$$B = 2$$



**01****EXAMPLE**

**Step 4 :** Substitute value of **A** and **B** & Write the answer by referring to the Laplace Transform table.

$$\frac{s+1}{s(s-1)} = \frac{-1}{s} + \frac{2}{s-1}$$

$$\begin{aligned}\therefore f(t) &= \mathcal{L}^{-1} \frac{-1}{s} + \mathcal{L}^{-1} \frac{2}{s-1} \\ &= -\mathcal{L}^{-1} \frac{1}{s} + 2\mathcal{L}^{-1} \frac{1}{s-1} \\ &= -1 + 2e^t\end{aligned}$$





### Partial Fraction for Linear Factor

Determine the Inverse Laplace Transform of the following function

$$\text{b) } f(t) = \frac{s^3 + 4s - 2}{(s-3)(s-2)}$$



**Step 1 :** Write one partial fraction for each of the factors

$$\frac{s^3 + 4s - 2}{(s-3)(s-2)} = \frac{A}{s-3} + \frac{B}{s-2}$$

**Step 2 :** Multiply through by the bottom so we no longer have fractions.

$$\left[ \frac{s^3 + 4s - 2}{(s-3)(s-2)} = \frac{A}{s-3} + \frac{B}{s-2} \right] \times (s-3)(s-2)$$

$$s^3 + 4s - 2 = A(s-2) + B(s-3)$$

**Step 3 :** Find the value of constants **A** and **B**.

For root  $(s-2)$ , let  $s = 2$

$$(2)^3 + 4(2) - 2 = A(2-2) + B(2-3)$$

$$14 = B(-1)$$

$$B = -14$$

For root  $(s-3)$ , let  $s = 3$

$$(3)^3 + 4(3) - 2 = A(3-2) + B(3-3)$$

$$37 = A(1)$$

$$A = 37$$



**02****EXAMPLE**

**Step 4 :** Substitute value of **A** and **B** & Write the answer by referring to the Laplace Transform table.

$$\frac{s^3 + 4s - 2}{(s - 3)(s - 2)} = \frac{37}{s - 3} + \frac{-14}{s - 2}$$

$$\begin{aligned}\therefore f(t) &= \mathcal{L}^{-1} \frac{37}{s - 3} + \mathcal{L}^{-1} \frac{-14}{s - 2} \\ &= 37\mathcal{L}^{-1} \frac{1}{s - 3} - 14\mathcal{L}^{-1} \frac{1}{s - 2} \\ &= 37e^{3t} - 14e^{2t}\end{aligned}$$



03

## EXAMPLE



### Partial Fraction for Repeated Linear Factor

Determine the Inverse Laplace Transform of the following function

$$c) f(t) = \frac{s^2+4}{s^2(s+2)}$$



**Step 1 :** Write one partial fraction for each of the factors

$$\frac{s^2 + 4}{s^2(s + 2)} = \frac{A}{s} + \frac{B}{s^2} + \frac{C}{s + 2}$$

**Step 2 :** Multiply through by the bottom so we no longer have fractions.

$$\left[ \frac{s^2 + 4}{s^2(s + 2)} = \frac{A}{s} + \frac{B}{s^2} + \frac{C}{s + 2} \right] \times s^2(s + 2)$$

$$s^2 + 4 = As(s + 2) + B(s + 2) + Cs^2$$



**03****EXAMPLE**

**Step 3 :** Find the value of constants **A, B** and **C**.

For root  $s$ , let  $s = 0$

$$(0)^2 + 4 = A(0)(0 + 2) + B(0 + 2) + C(0)^2$$

$$4 = B(2)$$

$$B = 2$$

For root  $(s + 2)$ , let  $s = -2$

$$(-2)^2 + 4 = A(-2)(-2 + 2) + B(-2 + 2) + C(-2)^2$$

$$8 = C(4)$$

$$C = 2$$

Let  $s = 1$

$$(1)^2 + 4 = A(1)(1 + 2) + (2)(1 + 2) + (2)(1)^2$$

$$5 = A(3) + 8$$

$$A = -1$$

**Step 4 :** Substitute value of **A, B** and **C** & Write the answer by referring to the Laplace Transform table.

$$\frac{s^2 + 4}{s^2(s + 2)} = \frac{-1}{s} + \frac{2}{s^2} + \frac{2}{s + 2}$$

$$\therefore f(t) = \mathcal{L}^{-1} \frac{-1}{s} + \mathcal{L}^{-1} \frac{2}{s^2} + \mathcal{L}^{-1} \frac{2}{s + 2}$$

$$= -\mathcal{L}^{-1} \frac{1}{s} + \mathcal{L}^{-1} \frac{2}{s^2} + 2\mathcal{L}^{-1} \frac{1}{s + 2}$$

$$= -1 + 2t + 2e^{-2t}$$





### Partial Fraction for Repeated Linear Factor

Determine the Inverse Laplace Transform of the following function

$$d) f(t) = \frac{2s^2 - 3s + 5}{(s-2)(s-1)^2}$$



**Step 1 :** Write one partial fraction for each of the factors

$$\frac{2s^2 - 3s + 5}{(s-2)(s-1)^2} = \frac{A}{s-2} + \frac{B}{s-1} + \frac{C}{(s-1)^2}$$

**Step 2 :** Multiply through by the bottom so we no longer have fractions.

$$\left[ \frac{2s^2 - 3s + 5}{(s-2)(s-1)^2} = \frac{A}{s-2} + \frac{B}{s-1} + \frac{C}{(s-1)^2} \right] \times (s-2)(s-1)^2$$

$$2s^2 - 3s + 5 = A(s-1)^2 + B(s-2)(s-1) + C(s-2)$$

**Step 3 :** Find the value of constants **A**, **B** and **C**.

For root  $(s-1)$ , let  $s = 1$

$$2(1)^2 - 3(1) + 5 = A(1-1)^2 + B(1-2)(1-1) + C(1-2)$$

$$4 = C(-1)$$

$$C = -4$$

For root  $(s-2)$ , let  $s = 2$

$$2(2)^2 - 3(2) + 5 = A(2-1)^2 + B(2-2)(2-1) + C(2-2)$$

$$7 = A(1)$$

$$A = 7$$

Let  $s = 0$

$$2(0)^2 - 3(0) + 5 = A(0-1)^2 + B(0-2)(0-1) + C(0-2)$$

$$5 = 7 + B(2) + 8$$

$$B = -5$$



04

## EXAMPLE



**Step 4 :** Substitute value of **A, B** and **C** & Write the answer by referring to the Laplace Transform table.

$$\frac{2s^2 - 3s + 5}{(s - 2)(s - 1)^2} = \frac{7}{s - 2} + \frac{-5}{s - 1} + \frac{-4}{(s - 1)^2}$$

$$\begin{aligned}\therefore f(t) &= \mathcal{L}^{-1} \frac{7}{s - 2} + \mathcal{L}^{-1} \frac{-5}{s - 1} + \mathcal{L}^{-1} \frac{-4}{(s - 1)^2} \\ &= 7\mathcal{L}^{-1} \frac{1}{s - 2} - 5\mathcal{L}^{-1} \frac{1}{s - 1} - 4\mathcal{L}^{-1} \frac{1}{(s - 1)^2} \\ &= 7e^{2t} - 5e^t - 4te^t\end{aligned}$$



05

## EXAMPLE



### Partial Fraction for Quadratic Factor

Determine the Inverse Laplace Transform of the following function

$$e) f(t) = \frac{5s^2 - 6s + 7}{(s-1)(s^2+1)}$$



**Step 1 :** Write one partial fraction for each of the factors

$$\frac{5s^2 - 6s + 7}{(s-1)(s^2+1)} = \frac{A}{s-1} + \frac{Bs+C}{s^2+1}$$

**Step 2 :** Multiply through by the bottom so we no longer have fractions.

$$\left[ \frac{5s^2 - 6s + 7}{(s-1)(s^2+1)} = \frac{A}{s-1} + \frac{Bs+C}{s^2+1} \right] \times (s-1)(s^2+1)$$

$$5s^2 - 6s + 7 = A(s^2 + 1) + (Bs + C)(s - 1)$$

$$5s^2 - 6s + 7 = As^2 + A + Bs^2 - Bs + Cs - C$$

$$= (A + B)s^2 + (C - B)s + (A - C)$$





**Step 3 :** Find the value of constants **A, B** and **C**.

For root  $(s - 1)$ , let  $s = 1$

$$5(1)^2 - 6(1) + 7 = A((1)^2 + 1) + (B(1) + C)(1 - 1)$$

$$6 = A(2)$$

$$A = 3$$

associating corresponding coefficients,  $s^0$

$$A - C = 7$$

$$3 - C = 7$$

$$C = -4$$

associating corresponding coefficients,  $s^2$

$$A + B = 5$$

$$3 + B = 5$$

$$B = 2$$

**Step 4 :** Substitute value of **A, B** and **C** & Write the answer by referring to the Laplace Transform table.

$$\frac{5s^2 - 6s + 7}{(s - 1)(s^2 + 1)} = \frac{3}{s - 1} + \frac{2s - 4}{s^2 + 1}$$

$$\begin{aligned} \therefore f(t) &= \mathcal{L}^{-1} \frac{3}{s - 1} + \mathcal{L}^{-1} \frac{2s - 4}{s^2 + 1} \\ &= \mathcal{L}^{-1} \frac{3}{s - 1} + \mathcal{L}^{-1} \frac{2s}{s^2 + 1} + \mathcal{L}^{-1} \frac{-4}{s^2 + 1} \\ &= 3\mathcal{L}^{-1} \frac{1}{s - 1} + 2\mathcal{L}^{-1} \frac{s}{s^2 + 1} - 4\mathcal{L}^{-1} \frac{1}{s^2 + 1} \\ &= 3e^t + 2 \cos t - 4 \sin t \end{aligned}$$





## Partial Fraction for Quadratic Factor

Determine the Inverse Laplace Transform of the following function

$$f) f(t) = \frac{6s^2 + 12s - 14}{(s+3)(s^2 + 2s + 5)}$$



**Step 1 :** Write one partial fraction for each of the factors

$$\frac{6s^2 + 12s - 14}{(s + 3)(s^2 + 2s + 5)} = \frac{A}{s + 3} + \frac{Bs + C}{s^2 + 2s + 5}$$

**Step 2 :** Multiply through by the bottom so we no longer have fractions.

$$\left[ \frac{6s^2 + 12s - 14}{(s + 3)(s^2 + 2s + 5)} = \frac{A}{s + 3} + \frac{Bs + C}{s^2 + 2s + 5} \right] \times (s + 3)(s^2 + 2s + 5)$$

$$6s^2 + 12s - 14 = A(s^2 + 2s + 5) + (Bs + C)(s + 3)$$

$$6s^2 + 12s - 14 = As^2 + 2As + 5A + Bs^2 + 3Bs + Cs + 3C$$

$$= (A + B)s^2 + (2A + 3B + C)s + (5A + 3C)$$





**Step 3 :** Find the value of constants **A, B** and **C** .

For root  $(s + 3)$ , let  $s = -3$

$$6(-3)^2 + 12(-3) - 14 = A((-3)^2 + 2(-3) + 5) + (B(-3) + C)((-3) + 3)$$

$$4 = A(8)$$

$$A = \frac{1}{2}$$

associating corresponding coefficients,  $s^2$

$$A + B = 6$$

$$\frac{1}{2} + B = 6$$

$$B = \frac{11}{2}$$

associating corresponding coefficients,  $s^0$

$$5A + 3C = -14$$

$$5\left(\frac{1}{2}\right) + 3C = -14$$

$$C = -\frac{11}{2}$$

**Step 4 :** Substitute value of **A, B** and **C** & Write the answer by referring to the Laplace Transform table.

$$\frac{6s^2 + 12s - 14}{(s + 3)(s^2 + 2s + 5)} = \frac{1}{2} \frac{1}{s + 3} + \frac{11}{2} \frac{s - 1}{s^2 + 2s + 5}$$

$$\therefore f(t) = \frac{1}{2} \mathcal{L}^{-1} \frac{1}{s + 3} + \frac{11}{2} \mathcal{L}^{-1} \frac{s - 1}{s^2 + 2s + 5} \quad \text{Completing the Square method}$$

$$= \frac{1}{2} \mathcal{L}^{-1} \frac{2}{s + 3} + \frac{11}{2} \mathcal{L}^{-1} \frac{(s + 1) - 2}{(s + 1)^2 + 4}$$

$$= \frac{1}{2} \mathcal{L}^{-1} \frac{2}{s + 3} + \frac{11}{2} \mathcal{L}^{-1} \frac{(s + 1)}{(s + 1)^2 + 4} + \frac{11}{2} \mathcal{L}^{-1} \frac{-2}{(s + 2)^2 + 4}$$

$$= \frac{1}{2} \mathcal{L}^{-1} \frac{1}{s + 3} + \frac{11}{2} \mathcal{L}^{-1} \frac{(s + 2)}{(s + 2)^2 + 2^2} - \frac{11}{2} \mathcal{L}^{-1} \frac{2}{(s + 2)^2 + 2^2}$$

$$= \frac{1}{2} e^{-3t} + \frac{11}{2} e^{-t} \cos 2t - \frac{11}{2} e^{-t} \sin 2t$$



## 07

## EXERCISE



Determine the inverse Laplace Transform for the followings:

$$1. \quad F(S) = \frac{4s - 4}{s^2 + 2s - 15}$$

$$2. \quad F(S) = \frac{2s - 3}{(s + 2)(s^2 + 3s + 2)}$$

$$3. \quad F(S) = \frac{2s + 9}{(s + 4)^2}$$

$$4. \quad F(S) = \frac{2s^2 - 9}{s(s^2 - 9)}$$

$$5. \quad F(S) = \frac{5s + 11}{s^2 + 4s + 3}$$

$$6. \quad F(S) = \frac{5s^2 + 8s - 1}{(s + 3)(s^2 + 1)}$$



[Click here for answer!](#)

## 07

## TUTORIAL



Find the Inverse Laplace for the following functions by using **Partial Fraction method** :

$$1. \quad F(s) = \frac{3s}{(s + 2)(s - 1)}$$

$$2. \quad F(s) = \frac{s - 3}{s^2 - 6s + 5}$$

$$3. \quad F(s) = \frac{s + 3}{s^2 - 1}$$

$$4. \quad F(s) = \frac{4 + 2s - s^2}{s(s - 2)^2}$$

$$5. \quad F(s) = \frac{9s^2 + 5s - 1}{(s - 2)(s + 1)^2}$$

$$6. \quad F(s) = \frac{12s^2 + 24s + 9}{s^3 + 3s^2}$$

$$7. \quad F(s) = \frac{2s - 1}{s(s^2 + 4)}$$

$$8. \quad F(s) = \frac{5s^2 - 6s + 7}{(s^2 + 1)(s - 1)}$$

$$9. \quad F(s) = \frac{2s - 3}{(s + 2)(s^2 + 6s + 10)}$$



[Click here for solution!](#)



## Differential Equation using Laplace Transform

A differential equation is an equation that relates a function to its derivatives. Laplace transform is a mathematical tool that can be used to solve certain types of differential equations.

Property of the Laplace transform:

**First Order Derivative:**  $\frac{dy}{dt}$  or  $y'(t) = sY(s) - y(0)$

**Second Order Derivative:**  $\frac{d^2y}{dt^2}$  or  $y''(t) = s^2Y(s) - sy(0) - y'(0)$

To solve a differential equation using Laplace transform, we take the Laplace transform of both sides of the equation, and then solve for the transformed function. After that, we use inverse Laplace transform to obtain the solution in the time domain .

The Laplace Transform can be used to solve differential equations using a **four-step** process.

**Step 1:** Take the Laplace Transform of the differential equation using the derivative property (and, perhaps, others) as necessary.

**Step 2 :** Put initial conditions into the resulting equation.

**Step 3 :** Solve for the output variable.

**Step 4 :** Get result from Laplace Transform tables. If the result is in a form that is not in the tables, you'll need to use the Inverse Laplace Transform



01

## EXAMPLE



Solve the initial value problem for the following by using Laplace Transform:

$$y' + 2y = 12e^{3t}; \text{ given that } y(0) = 3$$



**Step 1 :** Take the Laplace Transform of the differential equation using the derivative property (and, perhaps, others) as necessary.

$$y' + 2y = 12e^{3t}$$

$$sY(s) - y(0) + 2Y(s) = 12 \cdot \frac{1}{s-3}$$

- $y'$  with  $sY(s) - y(0)$
- $y$  with  $Y(s)$
- $e^{3t}$  with  $\frac{1}{s-3}$

**Step 2 :** Put initial conditions into the resulting equation

$$sY(s) - 3 + 2Y(s) = \frac{12}{s-3}$$

$y(0)$  with 3

**Step 3 :** Solve the output variable

$$\begin{aligned} Y(s)(s+2) &= \frac{12}{s-3} + 3 \\ &= \frac{12 + 3(s-3)}{s-3} \\ &= \frac{12 + 3s - 9}{s-3} \\ &= \frac{3 + 3s}{s-3} \\ \therefore Y(s) &= \frac{3 + 3s}{(s-3)(s+2)} \end{aligned}$$



**01****EXAMPLE**

Solve by using Partial Fraction

$$\frac{3 + 3s}{(s - 3)(s + 2)} = \frac{A}{s - 3} + \frac{B}{s + 2}$$

$$3s + 3 = A(s + 2) + B(s - 3)$$

$$\text{Let } s = 3$$

$$12 = A(5)$$

$$A = \frac{12}{5}$$

$$\text{Let } s = -2$$

$$-3 = B(-5)$$

$$B = \frac{3}{5}$$

**Step 4 :** Write the answer by referring to the Laplace Transform table.

$$Y(S) = \frac{12}{5} \frac{1}{s - 3} + \frac{3}{5} \frac{1}{s + 2}$$

$$\begin{aligned} Y(t) &= \frac{12}{5} \mathcal{L}^{-1} \frac{1}{s - 3} + \frac{3}{5} \mathcal{L}^{-1} \frac{1}{s + 2} \\ &= \frac{12}{5} e^{3t} + \frac{3}{5} e^{-2t} \end{aligned}$$



**02****EXAMPLE**

Solve the initial value problem for the following by using Laplace Transform

$$2y' + y = 0; \text{ given that } y(0) = 1$$



**Step 1 :** Take the Laplace Transform of the differential equation using the derivative property (and, perhaps, others) as necessary.

$$2y' + y = 0$$

$$2[sY(s) - y(0)] + Y(s) = 0$$

$$2sY(s) - 2y(0) + Y(s) = 0$$

- $y'$  with  $sY(s) - y(0)$
- $y$  with  $Y(s)$

**Step 2 :** Put initial conditions into the resulting equation

$$sY(s) - 2(1) + Y(s) = 0$$

$y(0)$  with 1

**Step 3 :** Solve the output variable

$$Y(s)(2s + 1) - 2 = 0$$

$$Y(s)(2s + 1) = 2$$

$$Y(s) = \frac{2}{(2s + 1)}$$

$$= \frac{2}{2\left(s + \frac{1}{2}\right)}$$

$$= \frac{1}{\left(s + \frac{1}{2}\right)}$$

**Step 4 :** Write the answer by referring to the Laplace Transform table.

$$Y(t) = \mathcal{L}^{-1} \frac{1}{\left(s + \frac{1}{2}\right)}$$

$$= e^{-\frac{1}{2}t}$$



03

## EXAMPLE



Solve the initial value problem for the following by using Laplace Transform:

$$y''(t) - y(t) = \cos 2t; \text{ given that } y(0) = 0, y'(0) = 1.$$



**Step 1 :** Take the Laplace Transform of the differential equation using the derivative property (and, perhaps, others) as necessary.

$$y''(t) - y(t) = \cos 2t$$

$$s^2Y(s) - sy(0) - y'(0) - Y(s) = \frac{s}{s^2 + 4}$$

- $y''$  with  $s^2Y(s) - sy(0) - y'(0)$
- $y'$  with  $sY(s) - y(0)$
- $y$  with  $Y(s)$
- $\cos 2t$  with  $\frac{s}{s^2+4}$

**Step 2 :** Put initial conditions into the resulting equation

$$s^2Y(s) - s(0) - 1 - Y(s) = \frac{s}{s^2 + 4}$$

$y(0)$  with 0  
 $y'(0)$  with 1

**Step 3 :** Solve the output variable

$$Y(s)(s^2 - 1) - 1 = \frac{s}{s^2 + 4}$$

$$Y(s)(s^2 - 1) = \frac{s}{s^2 + 4} + 1$$

$$Y(s)(s^2 - 1) = \frac{s + s^2 + 4}{s^2 + 4}$$

$$\therefore Y(s) = \frac{s + s^2 + 4}{(s^2 + 4)(s^2 - 1)}$$



## 03

## EXAMPLE



Solve by using Partial Fraction

$$\frac{s + s^2 + 4}{(s^2 + 4)(s^2 - 1)} = \frac{s + s^2 + 4}{(s^2 + 4)(s - 1)(s + 1)} = \frac{As + B}{s^2 + 4} + \frac{C}{s - 1} + \frac{D}{s + 1}$$

$$s + s^2 + 4 = (As + B)(s^2 - 1) + C(s^2 + 4)(s + 1) + D(s^2 + 4)(s - 1)$$

$$= As^3 - As + Bs^2 - B + Cs^3 + 4Cs + Cs^2 + 4C + Ds^3 + 4Ds - 4Ds^2 - 4D$$

$$= (A + C + D)s^3 + (B + C - D)s^2 + (4C - A + 4D)s + (4C - B - 4D)$$

Let  $s = 1$

$$6 = C(5)(2)$$

$$6 = 10C$$

$$C = \frac{3}{5}$$

Let  $s = -1$

$$4 = D(5)(-2)$$

$$4 = -10D$$

$$D = -\frac{2}{5}$$

coefficients,  $s^3$

$$A + C + D = 0$$

$$A + \frac{3}{5} - \frac{2}{5} = 0$$

$$A = -\frac{1}{5}$$

coefficients,  $s^2$

$$B + C - D = 1$$

$$B + \frac{3}{5} - \left(-\frac{2}{5}\right) = 1$$

$$B = 0$$

**Step 4 :** Write the answer by referring to the Laplace Transform table.

$$Y(S) = \frac{s + s^2 + 4}{(s^2 + 4)(s^2 - 1)} = \frac{-\frac{1}{5}s + 0}{s^2 + 4} + \frac{\frac{3}{5}}{s - 1} + \frac{-\frac{2}{5}}{s + 1}$$

$$Y(t) = -\frac{1}{5}\mathcal{L}^{-1}\frac{s}{s^2 + 4} + \frac{3}{5}\mathcal{L}^{-1}\frac{1}{s - 1} - \frac{2}{5}\mathcal{L}^{-1}\frac{1}{s + 1}$$

$$= -\frac{1}{5}\mathcal{L}^{-1}\frac{s}{s^2 + 2^2} + \frac{3}{5}\mathcal{L}^{-1}\frac{1}{s - 1} - \frac{2}{5}\mathcal{L}^{-1}\frac{1}{s + 1}$$

$$= -\frac{1}{5}\cos 2t + \frac{3}{5}e^t - \frac{2}{5}e^{-t}$$





Solve the initial value problem for the following by using Laplace Transform:

$$y''(t) - 10y'(t) + 9y(t) = 5t; \text{ given that } y(0) = -1, y'(0) = 2.$$



**Step 1 :** Take the Laplace Transform of the differential equation using the derivative property (and, perhaps, others) as necessary.

$$y''(t) - 10y'(t) + 9y(t) = 5t$$

$$s^2Y(s) - sy(0) - y'(0) - 10[sY(s) - y(0)] + 9Y(s) = \frac{5}{s^2}$$

$$s^2Y(s) - sy(0) - y'(0) - 10sY(s) + 10y(0) + 9Y(s) = \frac{5}{s^2}$$

**Step 2 :** Put initial conditions into the resulting equation

$$s^2Y(s) - s(-1) - 2 - 10sY(s) + 10(-1) + 9Y(s) = \frac{5}{s^2}$$

$$s^2Y(s) + s - 2 - 10sY(s) - 10 + 9Y(s) = \frac{5}{s^2}$$

$$s^2Y(s) + s - 10sY(s) - 12 + 9Y(s) = \frac{5}{s^2}$$

**Step 3 :** Solve the output variable

$$Y(s)(s^2 - 10s + 9) - 12 + s = \frac{5}{s^2}$$

$$Y(s)(s^2 - 10s + 9) = \frac{5}{s^2} + 12 - s$$

$$Y(s)(s^2 - 10s + 9) = \frac{5 + 12s^2 - s^3}{s^2}$$

$$\therefore Y(s) = \frac{5 + 12s^2 - s^3}{s^2(s^2 - 10s + 9)}$$



## 04

## EXAMPLE



Solve by using Partial Fraction

$$\frac{5 + 12s^2 - s^3}{s^2(s^2 - 10s + 9)} = \frac{5 + 12s^2 - s^3}{s^2(s - 9)(s - 1)}$$

$$= \frac{A}{s} + \frac{B}{s^2} + \frac{C}{s - 9} + \frac{D}{s - 1}$$

$$5 + 12s^2 - s^3 = As(s - 9)(s - 1) + B(s - 9)(s - 1) + Cs^2(s - 1) + Ds^2(s - 9)$$

$$= As^3 - 10As^2 + 9As + Bs^2 - 10Bs + 9B + Cs^3 - Cs^2 + Ds^3 - 9Ds^2$$

$$= (A + C + D)s^3 + (B - 10A - C - 9D)s^2 + (9A - 10B)s + 9B$$

Let  $s = 1$

$$16 = D(1)(-8)$$

$$16 = -8D$$

$$D = -2$$

Let  $s = 9$

$$248 = C(81)(8)$$

$$248 = 648C$$

$$C = \frac{31}{81}$$

coefficients,  $s^0$

$$9B = 5$$

$$B = \frac{5}{9}$$

coefficients,  $s^1$

$$9A - 10B = 0$$

$$9A - 10\left(\frac{5}{9}\right) = 0$$

$$A = \frac{50}{81}$$

**Step 4 :** Write the answer by referring to the Laplace Transform table.

$$Y(S) = \frac{5 + 12s^2 - s^3}{s^2(s^2 - 10s + 9)}$$

$$= \frac{50}{81} \frac{1}{s} + \frac{5}{9} \frac{1}{s^2} + \frac{31}{81} \frac{1}{s - 9} + \frac{-2}{s - 1}$$

$$Y(t) = \frac{50}{81} \mathcal{L}^{-1} \frac{1}{s} + \frac{5}{9} \mathcal{L}^{-1} \frac{1}{s^2} + \frac{31}{81} \mathcal{L}^{-1} \frac{1}{s - 9} - 2 \mathcal{L}^{-1} \frac{1}{s - 1}$$

$$= \frac{50}{81} + \frac{5}{9}t + \frac{31}{81}e^{9t} - 2e^t$$



## 08

## EXERCISE



Determine the differential equation for the followings by By applying the inverse Laplace Transform :

1.  $y' - y = e^{3t}$  ;  $y(0) = 2$
2.  $2 - 2y' = 0$  ;  $y(0) = 1$
3.  $y'' - 3y' + 2y = e^{3t}$  ;  $y(0) = 1, y'(0) = 0$
4.  $y''(t) - 2y'(t) - 3y(t) = 0$  ;  $y(0) = 2, y'(0) = -1$
5.  $y''(t) + y'(t) - 2y(t) = 0$  ;  $y(0) = 0, y'(0) = 1$
6.  $y''(t) - y'(t) - 2y(t) = \sin 3t$  ;  $y(0) = 1, y'(0) = -1$



[Click here for answer !](#)

## 08

## TUTORIAL



By applying the inverse Laplace Transform, solve the differential equation for the followings:

1.  $2y' + y = 4e^{2t}$ , given  $y(0) = 3$
2.  $y'(t) - y(t) = \sin 2t$ , given  $y(0) = 0$
3.  $y'' - 3y' + 2y = 6$ , given  $y(0) = 2, y'(0) = 6$
4.  $y''(t) + 2y'(t) + 10y(t) = 5$ , given  $y(0) = 0, y'(0) = 1$

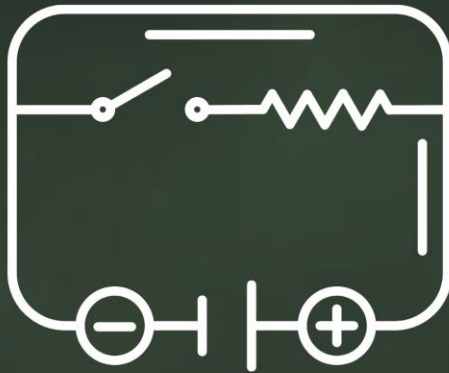


[Click here for solution !](#)



Topic

# LAPLACE TRANSFORM IN ELECTRICAL ENGINEERING



**Q1**

Given the input current is  $i(t)$  and the voltage across an inductor is  $v(t)$ .

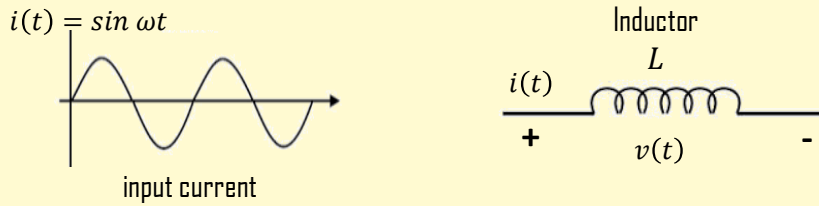


Figure 1

- a) Since the input current is  $i(t) = \sin \omega t$  and the exponential function for  $\sin \omega t = \frac{e^{i\omega t} - e^{-i\omega t}}{2i}$ , determine the Laplace Transform,  $I(s)$  by using  $F(s) = \int_0^{\infty} e^{-st} f(t) dt$
- b) If the voltage across the conductor is given by  $v(t) = L \frac{di(t)}{dt}$ , where  $L$  is the inductance value and the input current,  $i(t) = \sin \omega t$ , find the Laplace Transform,  $V(s)$  using the Laplace Transform Table.



**Q2**

A switching transformer introduced harmonics in the electrical power distribution system with the fundamental frequency  $50 \text{ Hz}$  supply, created distortion of the voltage waveform as depicted in *Figure 2*. Given the equation for the first harmonic is  $E_1 = 3t^4 - 2t^3 + 4e^{-3t} - \sin 5t + 3 \cos 2t$ . Find the Laplace Transform for the equation.

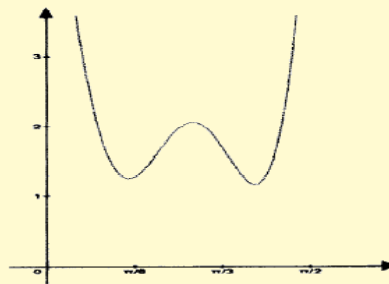


Figure 2



**Q3**

A closed-loop control system is shown in Figure 3.

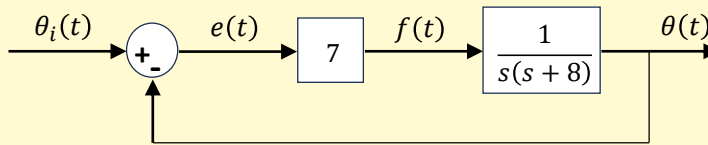


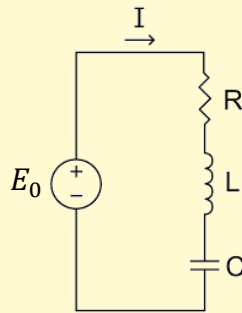
Figure 3

If  $K = 7$  and the input signal is a step response of  $\theta_i(s) = \frac{1}{s}$ , the closed-loop transfer function that incorporates the feedback relates the output  $\theta(s)$  to the input  $\theta_i(s)$ , is given by  $\theta(s) = \frac{7}{s(s^2+8s+7)}$ . Find the time-domain step response,  $\theta(t)$ .



**Q4**

Based on the RLC circuit shown in *Figure 4*, the equation derived from the Kirchoff's Law is given by  $L \frac{di}{dt} + Ri + \frac{1}{C} \int i dt = E_0$ .



**Figure 4**

- If  $i = \frac{d}{dt}(q(t))$ ,  $L = 1H$ ,  $R = 3\Omega$ ,  $C = 0.5$  and  $E_0 = 10$ , find the differential equation for the RLC circuit.
- If  $q(0) = 0$  and  $q'(0) = 0$ , solve the initial value problem for the differential equation in 3(a) using Laplace Transform.



# Reference

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ANSWER

1.  $-\frac{3}{s}$       2.  $\frac{4}{s^2}$       3.  $\frac{1}{s+5}$   
4.  $\frac{2}{s-2} + \frac{\sqrt{3}}{s}$       5.  $\frac{1}{s-1} + \frac{n}{s}$       6.  $\frac{6}{s^4}$

01

ANSWER

1.  $\frac{1}{s+2} + \frac{3}{s^2} + \frac{1}{s}$       2.  $\frac{2}{s-4} + \frac{2}{s^2+4}$       3.  $\frac{96}{s^5} + \frac{8}{s^3} + \frac{1}{s}$   
4.  $\frac{4}{s^3} - \frac{3}{s^2} - \frac{9}{s}$       5.  $\frac{1}{2(s-5)} + \frac{3}{2(s+5)}$       6.  $\frac{s}{4(s^2+25)} + \frac{10}{s^2+100} - \frac{2}{s}$

02

ANSWER

1.  $\frac{72}{(s+2)^5}$       2.  $\frac{1}{2(s^2+6s+8)}$       3.  $\frac{1}{s-2} + \frac{12}{(s-1)^4} + \frac{720}{s^7}$   
4.  $\frac{s}{s^2+25} + \frac{2}{(s+1)^3}$       5.  $\frac{1}{s-3} + \frac{24}{(s+2)^4}$       6.  $\frac{2}{s^2} - \frac{1}{(s-4)^2} + \frac{2}{s} - \frac{1}{s-4}$

03

ANSWER

1.  $\frac{2}{3s^2}$       2.  $\frac{10}{(s-4)^3}$       3.  $\frac{-2s(-s^2+12)}{(s^2+4)^3}$   
4.  $\frac{2s}{(s^2-1)^2}$       5.  $\frac{5}{s} + \frac{2}{s^2} - \frac{4(-3s^2+4)}{(s^2+4)^3}$       6.  $\frac{1}{(s+3)^2} + \frac{2}{s^2}$

04

ANSWER

1.  $f(t) = 2 \cos 4t - \sin 4t$       2.  $f(t) = 2t + \cosh 5t - \sinh 5t - t^4$       3.  $f(t) = 2 \cos 5t - \sin 5t$   
4.  $f(t) = 5e^{-3t} + 3 \cos 4t - 4$       5.  $f(t) = \frac{7}{24}t^4 + \frac{5}{2} \sinh 2t$       6.  $f(t) = \frac{1}{5}t^5 + 2 \sinh 4t$

05

ANSWER

1.  $f(t) = \frac{5}{3}e^{-2t} \sin 3t$       2.  $f(t) = 3e^{-3t} \sin t$       3.  $f(t) = 2e^{3t} \cos 3t + e^{3t} \sin 3t$   
4.  $f(t) = e^{-2t} \cosh 4t + e^{-2t} \sinh 4t$       5.  $f(t) = 3e^{-t} \cos 4t - e^{-t} \sin 4t$       6.  $f(t) = 3e^{-t} \cos \sqrt{15}t - \frac{4}{\sqrt{15}}e^{-t} \sin \sqrt{15}t$

06

ANSWER

1.  $f(t) = e^{3t} + 3e^{-5t}$       2.  $f(t) = -5e^{-t} + 5e^{-2t} + 7te^{-2t}$       3.  $f(t) = 2e^{-4t} + te^{-4t}$   
4.  $f(t) = 1 + \frac{1}{2}e^{3t} + \frac{1}{2}e^{-3t}$       5.  $f(t) = 3e^{-t} + 2e^{-3t}$       6.  $f(t) = 2e^{-3t} + 3 \cos t - \sin t$

07

ANSWER

1.  $y(t) = \frac{1}{2}e^{3t} + \frac{3}{2}e^t$       2.  $y(t) = e^{\frac{1}{2}t}$   
3.  $y(t) = e^{3t} + 3e^{-5t}$       4.  $y(t) = \frac{7}{4}e^{-t} + \frac{1}{4}e^{3t}$   
5.  $y(t) = \frac{1}{3}e^t - \frac{1}{3}e^{-2t}$       6.  $y(t) = \frac{3}{130} \cos 3t - \frac{3}{130} \sin 3t + \frac{9}{10}e^{-t} + \frac{1}{13}e^{2t}$

08



Find the Laplace Transform for the following function by using the Definition of Laplace Transform.

$$\begin{aligned}
 1. \quad f(t) &= 2e^{\frac{t}{2}} \\
 F(s) &= \int_0^{\infty} e^{-st} f(t) dt \\
 &= 2 \int_0^{\infty} e^{-st} e^{\frac{1}{2}t} dt \\
 &= 2 \int_0^{\infty} e^{-st + \frac{1}{2}t} dt \\
 &= 2 \int_0^{\infty} e^{-(s-\frac{1}{2})t} dt \\
 &= 2 \left[ \frac{e^{-(s-\frac{1}{2})t}}{-(s-\frac{1}{2})} \right]_0^{\infty} \\
 &= 2 \left[ \frac{e^{-(s-\frac{1}{2})(\infty)}}{-(s-\frac{1}{2})} - \frac{e^{-(s-\frac{1}{2})(0)}}{-(s-\frac{1}{2})} \right] \\
 &= 2 \left[ 0 - \frac{1}{-(s-\frac{1}{2})} \right] \\
 &= \frac{2}{s-\frac{1}{2}} \\
 &= \frac{4}{2s-1}
 \end{aligned}$$

$$\begin{aligned}
 2. \quad f(t) &= \sqrt{16e^{8t^2}} \\
 &= 4e^{4t} \\
 F(s) &= \int_0^{\infty} e^{-st} f(t) dt \\
 &= 4 \int_0^{\infty} e^{-st} e^{4t} dt \\
 &= 4 \int_0^{\infty} e^{-st+4t} dt \\
 &= 4 \int_0^{\infty} e^{-(s-4)t} dt \\
 &= 4 \left[ \frac{e^{-(s-4)t}}{-(s-4)} \right]_0^{\infty} \\
 &= 4 \left[ \frac{e^{-(s-4)(\infty)}}{-(s-4)} - \frac{e^{-(s-4)(0)}}{-(s-4)} \right] \\
 &= 4 \left[ 0 - \frac{1}{-(s-4)} \right] \\
 &= \frac{4}{s-4}
 \end{aligned}$$

$$\begin{aligned}
 3. \quad f(t) &= (e^t + 5)^2 \\
 f(t) &= (e^t + 5)(e^t + 5) \\
 &= e^{2t} + 10e^t + 25 \\
 F(s) &= \int_0^{\infty} e^{-st} f(t) dt \\
 &= \int_0^{\infty} e^{-st} (e^{2t} + 10e^t + 25) dt \\
 &= \int_0^{\infty} e^{-st} e^{2t} dt + 10 \int_0^{\infty} e^{-st} e^t dt + 25 \int_0^{\infty} e^{-st} dt \\
 &= \int_0^{\infty} e^{-st+2t} dt + 10 \int_0^{\infty} e^{-st+t} dt + 25 \int_0^{\infty} e^{-st} dt \\
 &= \int_0^{\infty} e^{-(s-2)t} dt + 10 \int_0^{\infty} e^{-(s-1)t} dt + 25 \int_0^{\infty} e^{-st} dt \\
 &= \left[ \frac{e^{-(s-2)t}}{-(s-2)} \right]_0^{\infty} + 10 \left[ \frac{e^{-(s-1)t}}{-(s-1)} \right]_0^{\infty} + 25 \left[ \frac{e^{-st}}{-s} \right]_0^{\infty} \\
 &= \left[ \frac{e^{-(s-2)(\infty)}}{-(s-2)} - \frac{e^{-(s-2)(0)}}{-(s-2)} \right] + 10 \left[ \frac{e^{-(s-1)(\infty)}}{-(s-1)} - \frac{e^{-(s-1)(0)}}{-(s-1)} \right] + 25 \left[ \frac{e^{-s(\infty)}}{-s} - \frac{e^{-s(0)}}{-s} \right] \\
 &= \left[ 0 - \frac{1}{-(s-2)} \right] + 10 \left[ 0 - \frac{1}{-(s-1)} \right] + 25 \left[ 0 - \frac{1}{-s} \right] \\
 &= \frac{1}{s-2} + \frac{10}{s-1} + \frac{25}{s}
 \end{aligned}$$

$$\begin{aligned}
 4. \quad f(t) &= 2 \cos 3t \\
 f(t) &= 2 \cos 3t \text{ (change to exponential form)} \\
 &= 2 \mathcal{L}e^{3it}
 \end{aligned}$$

$$\begin{aligned}
 F(s) &= \int_0^{\infty} e^{-st} f(t) dt \\
 &= \int_0^{\infty} e^{-st} \cdot 2 e^{3it} dt \\
 &= 2 \int_0^{\infty} e^{-st+3it} dt \\
 &= 2 \int_0^{\infty} e^{-(s-3i)t} dt \\
 &= 2 \left[ \frac{e^{-(s-3i)t}}{-(s-3i)} \right]_0^{\infty} \\
 &= 2 \left[ \frac{e^{-(s-3i)(\infty)}}{-(s-3i)} - \frac{e^{-(s-3i)(0)}}{-(s-3i)} \right] \\
 &= 2 \left[ 0 - \frac{1}{-(s-3i)} \right] \\
 &= \frac{2}{s-3i}
 \end{aligned}$$

solve by using conjugate, therefore

$$\begin{aligned}
 &= \frac{2}{s-3i} \cdot \left( \frac{s+3i}{s+3i} \right) \\
 &= \frac{s+3i}{s^2+3^2} \\
 &= \frac{s}{s^2+9} + \frac{3i}{s^2+9}
 \end{aligned}$$

As  $\cos 3t$  is the real part, thus

$$F(s) = \frac{s}{s^2+9}$$





By using the Table of Laplace Transform, find the Laplace Transform for the following functions.

Refer to the Laplace Transform table for each term

$$\begin{aligned}
 1. \quad f(t) &= 3t^2(e^{2t} - t)^2 \\
 &= 3t^2(e^{4t} - 2te^{2t} + t^2) \\
 &= 3t^2e^{4t} - 6t^3e^{2t} + 3t^4 \\
 \therefore F(s) &= 3 \cdot \frac{2!}{(s-4)^{2+1}} - 6 \cdot \frac{3!}{(s-2)^{3+1}} + 3 \cdot \frac{4!}{s^{4+1}} \\
 &= 3 \cdot \frac{2}{(s-4)^3} - 6 \cdot \frac{6}{(s-2)^4} + 3 \cdot \frac{24}{s^5} \\
 &= \frac{6}{(s-4)^3} - \frac{36}{(s-2)^4} + \frac{72}{s^5}
 \end{aligned}$$

$$\begin{aligned}
 2. \quad F(s) &= 3L_t \cos 2t - 5L_t \sin 4t + Lt^3 e^{4t} \\
 \therefore F(s) &= 3 \cdot \frac{s^2 - 2^2}{(s^2 + 2^2)^2} - 5 \cdot \frac{2(4)s}{(s^2 + 4^2)^2} + \frac{3!}{(s-4)^{3+1}} \\
 &= \frac{3(s^2 - 4)}{(s^2 + 4)^2} - \frac{40s}{(s^2 + 16)^2} + \frac{6}{(s-4)^4}
 \end{aligned}$$

$$\begin{aligned}
 3. \quad f(t) &= \frac{2t \tan 2t}{\cos 2t} + e^{-3t} \sinh 5t + \frac{t^4}{2} \\
 f(t) &= \frac{2 \cdot \frac{\sin 2t}{\cos 2t}}{\cos 2t} + e^{-3t} \sinh 5t + \frac{1}{2} t^4 \\
 &= 2 \sin 2t + e^{-3t} \sinh 5t + \frac{1}{2} t^4 \\
 F(s) &= 2L \sin 2t + L e^{-3t} \sinh 5t + L \frac{1}{2} t^4 \\
 \therefore F(s) &= 3 \cdot \frac{2}{s^2 + 2^2} + \frac{5}{(s+3)^2 - 5^2} + \frac{1}{2} \cdot \frac{4!}{s^{4+1}} \\
 &= \frac{6}{s^2 + 4} + \frac{5}{(s+3)^2 - 25} + \frac{1}{2} \cdot \frac{24}{s^5} \\
 &= \frac{6}{s^2 + 4} + \frac{5}{s^2 + 6s - 16} + \frac{12}{s^5}
 \end{aligned}$$

$$\begin{aligned}
 4. \quad f(t) &= \frac{9t^4 + 6t^3 + t^2}{e^{5t}} \\
 f(t) &= \frac{9t^4 + 6t^3 + t^2}{e^{5t}} \\
 &= 9t^4 e^{-5t} + 6t^3 e^{-5t} + t^2 e^{-5t} \\
 F(s) &= 9Lt^4 e^{-5t} + 6Lt^3 e^{-5t} + Lt^2 e^{-5t} \\
 \therefore F(s) &= 9 \cdot \frac{4!}{(s - (-5))^{4+1}} + 6 \cdot \frac{3!}{(s - (-5))^{3+1}} + \frac{2!}{(s - (-5))^{2+1}} \\
 &= \frac{216}{(s+5)^5} + \frac{36}{(s+5)^4} + \frac{2}{(s+5)^3}
 \end{aligned}$$





Find the Laplace Transform for the following functions by using first shifting theorem method.

$$1. \quad f(t) = (e^{3t} + t^2)^2 \\ = e^{6t} + 2e^{3t}t^2 + t^4$$

$$\mathcal{L}e^{at}f(t) = \mathcal{L}2e^{3t}t^2$$

$$a = 3$$

$$F(s - a) = F(s - 3)$$

$f(t) \rightarrow$  Refer Laplace Transform Table

$$F(s) = 2 \cdot \frac{3!}{s^{3+1}} \\ = \frac{12}{s^4}$$

$$\text{Thus, } F(s - a) = F(s - 3)$$

$$\therefore F(s) = \frac{12}{(s - 3)^4}$$

$$3. \quad f(t) = 3e^t + e^{4t}\cosh 2t \\ \mathcal{L}e^{at}f(t) = \mathcal{L}e^{4t}\cosh 2t$$

$$a = 4$$

$$F(s - a) = F(s - 4)$$

$f(t) \rightarrow$  Refer Laplace Transform Table

$$F(s) = 3\mathcal{L}e^t + \mathcal{L}e^{4t}\cosh 2t$$

$$= 3 \cdot \frac{1}{s - 1} + \frac{s}{s^2 - 2^2} \\ = \frac{3}{s - 1} + \frac{s}{s^2 - 4}$$

$$\text{Thus, } F(s - a) = F(s - 4)$$

$$\therefore F(s) = \frac{3}{s - 1} + \frac{s}{(s - 4)^2 - 4} \\ = \frac{3}{s - 1} + \frac{s}{s^2 - 8s + 16 - 4} \\ = \frac{3}{s - 1} + \frac{s}{s^2 - 8s + 12}$$

$$2. \quad f(t) = \frac{t^3 + 2t^2 + t}{e^{3t}} \\ f(t) = e^{-3t}(t^3 + 2t^2 + t) \\ = e^{-3t}t^3 + 2e^{-3t}t^2 + e^{-3t}t$$

$$\mathcal{L}e^{at}f(t) \\ = \mathcal{L}e^{-3t}t^3 + 2\mathcal{L}e^{-3t}t^2 + \mathcal{L}e^{-3t}t$$

$$a = -3$$

$$F(s - a) = F(s - (-3)) \\ = F(s + 3)$$

$f(t) \rightarrow$  Refer Laplace Transform Table

$$F(s) = \frac{3!}{s^{3+1}} + 2 \cdot \frac{2!}{s^{2+1}} + \frac{1}{s^2} \\ = \frac{6}{s^4} + \frac{4}{s^3} + \frac{1}{s^2}$$

$$\text{Thus, } F(s - a) = F(s + 3)$$

$$\therefore F(s) = \frac{6}{(s + 3)^4} + \frac{4}{(s + 3)^3} + \frac{1}{(s + 3)^2}$$

$$4. \quad f(t) = \frac{e^{2t}\sinh\sqrt{5}t}{3} \\ \mathcal{L}e^{at}f(t) = \frac{1}{3}\mathcal{L}e^{2t}\sinh\sqrt{5}t$$

$$a = 2$$

$$F(s - a) = F(s - 2)$$

$f(t) \rightarrow$  Refer Laplace Transform Table

$$F(s) = \frac{1}{3}\mathcal{L}e^{2t}\sinh\sqrt{5}t \\ = \frac{1}{3} \cdot \frac{\sqrt{5}}{s^2 - (\sqrt{5})^2} \\ = \frac{\sqrt{5}}{3(s^2 - 5)}$$

$$\text{Thus, } F(s - a) = F(s - 2)$$

$$\therefore F(s) = \frac{\sqrt{5}}{3((s - 2)^2 - 5)} \\ = \frac{\sqrt{5}}{3(s^2 - 4s + 4 - 5)} \\ = \frac{\sqrt{5}}{3(s^2 - 4s - 1)}$$





By using the Multiplication of  $t^n$  method find the Laplace Transform for the following functions:

$$\begin{aligned} 1. \quad f(t) &= e^{4t}(3t + \cos 2t) \\ &= 3te^{4t} + e^{4t} \cos 2t \end{aligned}$$

$$\begin{aligned} \mathcal{L}(3te^{4t}) \\ n = 1 \quad \text{and} \quad f(t) &= e^{4t} \\ f(t) &\rightarrow \text{Refer Laplace Transform Table} \\ F(s) &= \frac{1}{s-4} \\ &= (s-4)^{-1} \\ F'(s) &= -(s-4)^{-2}(1) \\ &= \frac{-1}{(s-4)^2} \end{aligned}$$

Substitute into the formula:

$$\begin{aligned} \mathcal{L}t^n f(t) &= (-1)^n F^n(s) \\ \therefore \mathcal{L}3te^{4t} &= 3(-1) \frac{-1}{(s-4)^2} = \frac{3}{(s-4)^2} \end{aligned}$$

$$\therefore F(s) = \frac{3}{(s-4)^2} + \frac{s-4}{(s-4)^2 + 4}$$

$$3. \quad f(t) = \frac{3e^{\sqrt{5}t}t^2}{2}$$

$$\begin{aligned} n = 2 \quad \text{and} \quad f(t) &= \frac{3e^{\sqrt{5}t}}{2} \\ f(t) &\rightarrow \text{Refer Laplace Transform Table} \\ F(s) &= \frac{3}{2} \cdot \frac{1}{s-\sqrt{5}} \\ &= \frac{3}{2}(s-\sqrt{5})^{-1} \\ F'(s) &= -\frac{3}{2}(s-\sqrt{5})^{-2} \\ &= \frac{-3}{2(s-\sqrt{5})^2} \end{aligned}$$

Substitute into the formula:

$$\begin{aligned} \mathcal{L}t^n f(t) &= (-1)^n F^n(s) \\ \therefore \mathcal{L}\left(\frac{3e^{\sqrt{5}t}t^2}{2}\right) &= (-1)^2 \frac{3}{2(s-\sqrt{5})^2} \\ &= \frac{3}{2(s-\sqrt{5})^2} \end{aligned}$$

$$2. \quad \mathcal{L}(t \sin \pi t)$$

$$n = 1 \quad \text{and} \quad f(t) = \sin \pi t$$

$f(t) \rightarrow$  Refer Laplace Transform Table

$$\begin{aligned} F(s) &= \frac{\pi}{s^2 + \pi^2} \\ &= \pi(s^2 + \pi^2)^{-1} \\ F'(s) &= -\pi(s^2 + \pi^2)^{-2}(2s) \\ &= \frac{-2\pi s}{(s^2 + \pi^2)^2} \end{aligned}$$

Substitute into the formula:

$$\begin{aligned} \mathcal{L}t^n f(t) &= (-1)^n F^n(s) \\ \therefore \mathcal{L}(t \sin \pi t) &= (-1) \frac{-2\pi s}{(s^2 + \pi^2)^2} \\ &= \frac{-2\pi s}{(s^2 + \pi^2)^2} \end{aligned}$$

$$4. \quad f(t) = t^2(e^t + 1)^2$$

$$\begin{aligned} f(t) &= t^2(e^{2t} + 2e^t + 1) \\ &= t^2e^{2t} + 2t^2e^t + t^2 \end{aligned}$$

$$n = 2$$

$f(t) \rightarrow$  Refer Laplace Transform Table

$$\begin{aligned} F(s) &= \frac{1}{s-2} + \frac{1}{s-1} + \frac{1}{s} \\ &= (s-2)^{-1} + (s-1)^{-1} + s^{-1} \end{aligned}$$

$$F'(s) = -(s-2)^{-2} - (s-1)^{-2} - s^{-2}$$

$$\begin{aligned} F''(s) &= 2(s-2)^{-3} + 2(s-1)^{-3} + 2s^{-3} \\ &= 2 \left[ \frac{1}{(s-2)^3} + \frac{1}{(s-1)^3} + \frac{1}{2s^3} \right] \end{aligned}$$

Substitute into the formula:

$$\begin{aligned} \mathcal{L}t^n f(t) &= (-1)^n F^n(s) \\ \therefore \mathcal{L}(t^2e^{2t} + 2t^2e^t + t^2) &= (-1)^2 2 \left[ \frac{1}{(s-2)^3} + \frac{1}{(s-1)^3} + \frac{1}{2s^3} \right] \\ &= 2 \left[ \frac{1}{(s-2)^3} + \frac{1}{(s-1)^3} + \frac{1}{2s^3} \right] \end{aligned}$$





Apply Inverse Laplace Transform using Laplace Transform Table.

$$\begin{aligned}
 1. \quad F(s) &= \frac{11}{s^4} + \frac{3}{s-4} - \frac{2s-6}{s^2+49} \\
 &= \frac{11}{s^4} + \frac{3}{s-4} - \left[ \frac{2s}{s^2+49} - \frac{6}{s^2+49} \right] \\
 f(t) &= \frac{11}{6} \mathcal{L}^{-1} \frac{3!}{s^{3+1}} + 3 \mathcal{L}^{-1} \frac{1}{s-4} - 2 \mathcal{L}^{-1} \frac{s}{s^2+7^2} + \frac{6}{7} \mathcal{L}^{-1} \frac{7}{s^2+7^2} \\
 &= \frac{11}{6} t^3 + 3e^{4t} - 2\cos 7t + \frac{6}{7} \sin 7t
 \end{aligned}$$

$$\begin{aligned}
 2. \quad f(t) &= \mathcal{L}^{-1} \frac{4}{s^2} + \mathcal{L}^{-1} \frac{2s-5}{s^2-25} \\
 &= \mathcal{L}^{-1} \frac{4}{s^2} + \mathcal{L}^{-1} \frac{2s}{s^2-25} - \mathcal{L}^{-1} \frac{5}{s^2-25} \\
 &= \mathcal{L}^{-1} \frac{4}{s^2} + 2 \mathcal{L}^{-1} \frac{s}{s^2-5^2} - \mathcal{L}^{-1} \frac{5}{s^2-5^2} \\
 &= 4t^2 + 2 \cosh 5t - \sinh 5t
 \end{aligned}$$

$$\begin{aligned}
 3. \quad f(t) &= \mathcal{L}^{-1} \frac{10}{2s^2-8} - \mathcal{L}^{-1} \frac{2}{s-11} \\
 &= \mathcal{L}^{-1} \frac{10}{2(s^2-4)} - \mathcal{L}^{-1} \frac{2}{s-11} \\
 &= \mathcal{L}^{-1} \frac{5}{s^2-4} - \mathcal{L}^{-1} \frac{2}{s-11} \\
 &= \frac{5}{2} \mathcal{L}^{-1} \frac{2}{s^2-2^2} - 2 \mathcal{L}^{-1} \frac{1}{s-11} \\
 &= \frac{5}{2} \sinh 2t - 2e^{11t}
 \end{aligned}$$

$$\begin{aligned}
 4. \quad F(s) &= \frac{10}{(s+5)^2-16} - \frac{4}{(s-3)^5} \\
 f(t) &= 10 \mathcal{L}^{-1} \frac{4}{(s+5)^2-4^2} - \frac{1}{6} \mathcal{L}^{-1} \frac{4!}{(s-3)^{4+1}} \\
 &= 10e^{-5t} \sinh 4t - \frac{1}{6} t^4 e^{3t}
 \end{aligned}$$





Apply Inverse Laplace Transform using Completing the Square Method.

$$\begin{aligned}
 1. \quad f(t) &= \mathcal{L}^{-1} \frac{6}{s^2 + 4s + 13} \\
 &= \mathcal{L}^{-1} \frac{6}{(s+2)^2 - 9} \\
 &= 2 \mathcal{L}^{-1} \frac{3}{(s+2)^2 - 3^2} \\
 &= 2e^{-2t} \sinh 3t
 \end{aligned}$$

$$\begin{aligned}
 2. \quad F(s) &= \frac{5}{s^2 - 6s + 10} \\
 f(t) &= \mathcal{L}^{-1} \frac{5}{(s-3)^2 - 1} \\
 &= 5 \mathcal{L}^{-1} \frac{1}{(s-3)^2 - 1^2} \\
 &= 5e^{3t} \sinh t
 \end{aligned}$$

$$\begin{aligned}
 3. \quad f(t) &= \mathcal{L}^{-1} \frac{2s}{s^2 - 6s + 18} \\
 &= \mathcal{L}^{-1} \frac{2(s-3) + 3}{(s-3)^2 + 9} \\
 &= \mathcal{L}^{-1} \frac{2(s-3)}{(s-3)^2 + 3^2} + \mathcal{L}^{-1} \frac{3}{(s-3)^2 + 3^2} \\
 &= 2 \mathcal{L}^{-1} \frac{(s-3)}{(s-3)^2 + 3^2} + \mathcal{L}^{-1} \frac{3}{(s-3)^2 + 3^2} \\
 &= 2e^{3t} \cos 3t + e^{3t} \sin 3t
 \end{aligned}$$

$$\begin{aligned}
 4. \quad F(s) &= \frac{s+6}{s^2 + 4s - 12} \\
 f(t) &= \mathcal{L}^{-1} \frac{(s+2) + 4}{(s+2)^2 - 16} \\
 &= \mathcal{L}^{-1} \frac{(s+2)}{(s+2)^2 - 4^2} + \mathcal{L}^{-1} \frac{4}{(s+2)^2 - 4^2} \\
 &= e^{-2t} \cosh 4t + e^{-2t} \sinh 4t
 \end{aligned}$$





Apply Inverse Laplace Transform using Partial Fraction Method (Linear Factor).

$$1. \quad F(s) = \frac{3s}{(s+2)(s-1)}$$

$$\frac{3s}{(s+2)(s-1)} = \frac{A}{s+2} + \frac{B}{s-1}$$

$$3s = A(s-1) + B(s+2)$$

$$s-1=0, \quad s=1 \qquad s+2=0, \quad s=-2$$

$$3(1) = B(3) \qquad 3(-2) = A(-3)$$

$$3 = B(3) \qquad -6 = A(-3)$$

$$B = 1 \qquad A = 2$$

$$F(s) = \frac{2}{s+2} + \frac{1}{s-1}$$

$$f(t) = 2\mathcal{L}^{-1}\frac{1}{s+2} + \mathcal{L}^{-1}\frac{1}{s-1}$$

$$= 2e^{-2t} + e^t$$

$$2. \quad F(s) = \frac{s-3}{s^2-6s+5} = \frac{s-3}{(s-5)(s-1)}$$

$$\frac{s-3}{(s-5)(s-1)} = \frac{A}{s-5} + \frac{B}{s-1}$$

$$s-3 = A(s-1) + B(s-5)$$

$$s-1=0, \quad s=1 \qquad s-5=0, \quad s=5$$

$$1-3 = B(-4) \qquad 5-3 = A(4)$$

$$-2 = B(-4) \qquad 2 = A(4)$$

$$B = \frac{1}{2} \qquad A = \frac{1}{2}$$

$$F(s) = \frac{\frac{1}{2}}{s-5} + \frac{\frac{1}{2}}{s-1}$$

$$f(t) = \frac{1}{2}\mathcal{L}^{-1}\frac{1}{s-5} + \frac{1}{2}\mathcal{L}^{-1}\frac{1}{s-1}$$

$$= \frac{1}{2}e^{5t} + \frac{1}{2}e^t$$

$$3. \quad F(s) = \frac{s+3}{s^2-1} = \frac{s+3}{(s+1)(s-1)}$$

$$\frac{s+3}{(s+1)(s-1)} = \frac{A}{s+1} + \frac{B}{s-1}$$

$$s+3 = A(s-1) + B(s+1)$$

$$s-1=0, \quad s=1 \qquad s+1=0, \quad s=-1$$

$$1+3 = B(2) \qquad -1+3 = A(-2)$$

$$4 = B(2) \qquad 2 = A(-2)$$

$$B = 2 \qquad A = -1$$

$$F(s) = \frac{-1}{s+1} + \frac{2}{s-1}$$

$$f(t) = -\mathcal{L}^{-1}\frac{1}{s+1} + \mathcal{L}^{-1}\frac{2}{s-1}$$

$$= -e^{-t} + 2e^t$$

4.

$$F(s) = \frac{4 + 2s - s^2}{s(s-2)^2}$$

$$\frac{4 + 2s - s^2}{s(s-2)^2} = \frac{A}{s} + \frac{B}{s-2} + \frac{C}{(s-2)^2}$$

$$4 + 2s - s^2 = A(s-2)^2 + Bs(s-2) + Cs$$

$$\text{assume } s = 0$$

$$4 + 2(0) - 0^2 = A(-2)^2$$

$$4 = A(4)$$

$$A = 1$$

$$s - 2 = 0, \quad s = 2$$

$$4 + 2(2) - 2^2 = C(2)$$

$$6 = C(2)$$

$$C = 3$$

$$\text{assume } s = 1$$

$$4 + 2(1) - 1^2 = 1(1)^2 + B(1)(-1) + 3(1)$$

$$5 = 1 - B + 3$$

$$B = -1$$

$$F(s) = \frac{1}{s} + \frac{-1}{s-2} + \frac{3}{(s-2)^2}$$

$$f(t) = \mathcal{L}^{-1} \frac{1}{s} - \mathcal{L}^{-1} \frac{1}{s-2} + 3 \mathcal{L}^{-1} \frac{1}{(s-2)^2}$$

$$= 1 - e^{2t} + 3te^{2t}$$

5.

$$F(s) = \frac{9s^2 + 5s - 1}{(s-2)(s+1)^2}$$

$$\frac{9s^2 + 5s - 1}{(s-2)(s+1)^2} = \frac{A}{s-2} + \frac{B}{s+1} + \frac{C}{(s+1)^2}$$

$$9s^2 + 5s - 1 = A(s+1)^2 + B(s+1)(s-2) + C(s-2)$$

$$s + 1 = 0, \quad s = -1$$

$$9(-1)^2 + 5(-1) - 1 = C(-3)$$

$$3 = C(-3)$$

$$C = -1$$

$$s - 2 = 0, \quad s = 2$$

$$9(2)^2 + 5(2) - 1 = A(3)^2$$

$$45 = A(9)$$

$$A = 5$$

$$\text{assume } s = 0$$

$$9(0)^2 + 5(0) - 1 = 5(1)^2 + B(-2)(1) - 1(-2)$$

$$-1 = 5 - 2B + 2$$

$$B = 4$$

$$F(s) = \frac{5}{s-2} + \frac{4}{s+1} + \frac{-1}{(s+1)^2}$$

$$f(t) = 5 \mathcal{L}^{-1} \frac{1}{s-2} + 4 \mathcal{L}^{-1} \frac{1}{s+1} - \mathcal{L}^{-1} \frac{1}{(s+1)^2}$$

$$= 5e^{2t} + 4e^{-t} - te^{-t}$$

6.

$$F(s) = \frac{12s^2 + 24s + 9}{s^3 + 3s^2} = \frac{12s^2 + 24s + 9}{s^2(s+3)}$$

$$\frac{12s^2 + 24s + 9}{s^2(s+3)} = \frac{A}{s} + \frac{B}{s^2} + \frac{C}{s+3}$$

$$12s^2 + 24s + 9 = As(s+3) + B(s+3) + Cs^2$$

$$\text{assume } s = 0$$

$$12(0)^2 + 24(0) + 9 = B(3)$$

$$9 = B(3)$$

$$B = 3$$

$$s + 3 = 0, \quad s = -3$$

$$12(-3)^2 + 24(-3) + 9 = C(-3)^2$$

$$45 = C(9)$$

$$C = 5$$

$$\text{assume } s = 1$$

$$12(1)^2 + 24(1) + 9 = A(1)(4) + 3(4) + 5(1)$$

$$45 = 4A + 12 + 5$$

$$A = 7$$

$$F(s) = \frac{7}{s} + \frac{3}{s^2} + \frac{5}{s+3}$$

$$f(t) = \mathcal{L}^{-1} \frac{7}{s} + \mathcal{L}^{-1} \frac{3}{s^2} + 5 \mathcal{L}^{-1} \frac{1}{s+3}$$

$$= 7 + 3t + 5e^{-3t}$$

7.

$$F(s) = \frac{2s - 1}{s(s^2 + 4)}$$

$$\frac{2s - 1}{s(s^2 + 4)} = \frac{A}{s} + \frac{Bs + C}{s^2 + 4}$$

$$\begin{aligned} 2s - 1 &= A(s^2 + 4) + (Bs + C)s \\ &= As^2 + 4A + Bs^2 + Cs \\ &= (A + B)s^2 + Cs + 4A \end{aligned}$$

Co-eff  $s^0$ 

$$4A = -1$$

$$A = -\frac{1}{4}$$

Co-eff  $s^1$ 

$$C = 2$$

Co-eff  $s^2$ 

$$A + B = 0$$

$$-\frac{1}{4} + B = 0$$

$$B = \frac{1}{4}$$

$$F(s) = \frac{-\frac{1}{4}}{s} + \frac{\frac{1}{4}s + 2}{s^2 + 4}$$

$$= -\frac{1}{4s} + \frac{\frac{1}{4}s}{s^2 + 4} + \frac{2}{s^2 + 4}$$

$$f(t) = -\frac{1}{4} \mathcal{L}^{-1} \frac{1}{s} + \frac{1}{4} \mathcal{L}^{-1} \frac{s}{s^2 + 2^2} + \mathcal{L}^{-1} \frac{2}{s^2 + 2^2}$$

$$= -\frac{1}{4} + \frac{1}{4} \cos 2t + \sin 2t$$

8.

$$F(s) = \frac{5s^2 - 6s + 7}{(s^2 + 1)(s - 1)}$$

$$\frac{5s^2 - 6s + 7}{(s^2 + 1)(s - 1)} = \frac{As + B}{s^2 + 1} + \frac{C}{s - 1}$$

$$\begin{aligned} 5s^2 - 6s + 7 &= (As + B)(s - 1) + C(s^2 + 1) \\ &= As^2 - As + Bs - B + Cs^2 + C \\ &= (A + C)s^2 + (B - A)s + (C - B) \end{aligned}$$

$$s - 1 = 0, \quad s = 1$$

$$5(1)^2 - 6(1) + 7 = C(1^2 + 1)$$

$$6 = C(2)$$

$$C = 3$$

Co-eff  $s^2$ 

$$A + C = 5$$

$$A + 3 = 5$$

$$A = 2$$

Co-eff  $s^1$ 

$$B - A = -6$$

$$B - 2 = -6$$

$$B = -4$$

$$F(s) = \frac{2s + (-4)}{s^2 + 1} + \frac{3}{s - 1}$$

$$= \frac{2s}{s^2 + 1} - \frac{4}{s^2 + 1} + \frac{3}{s - 1}$$

$$f(t) = 2 \mathcal{L}^{-1} \frac{s}{s^2 + 1^2} - 4 \mathcal{L}^{-1} \frac{1}{s^2 + 1^2} + 3 \mathcal{L}^{-1} \frac{1}{s - 1}$$

$$= 2 \cos t - 4 \sin t + 3e^t$$

9.

$$F(s) = \frac{2s - 3}{(s + 2)(s^2 + 6s + 10)}$$

$$\frac{2s - 3}{(s + 2)(s^2 + 6s + 10)} = \frac{A}{s + 2} + \frac{Bs + C}{s^2 + 6s + 10}$$

$$2s - 3 = A(s^2 + 6s + 10) + (Bs + C)(s + 2)$$

$$= As^2 + 6As + 10A + Bs^2 + 2Bs + Cs + 2C$$

$$= (A + B)s^2 + (6A + 2B + C)s + (10A + 2C)$$

$$s + 2 = 0, \quad s = -2$$

$$2(-2) - 3 = A((-2)^2 + 6(-2) + 10)$$

$$-7 = A(2)$$

$$A = -\frac{7}{2}$$

Co-eff  $s^2$ 

$$A + B = 0$$

$$-\frac{7}{2} + B = 5$$

$$B = \frac{7}{2}$$

Co-eff  $s^0$ 

$$10A + 2C = -3$$

$$10\left(-\frac{7}{2}\right) + 2C = -3$$

$$2C = 32$$

$$C = 16$$

$$F(s) = \frac{-\frac{7}{2}}{s + 2} + \frac{\frac{7}{2}s + 16}{s^2 + 6s + 10}$$

$$= \frac{-\frac{7}{2}}{s + 2} + \frac{1}{2} \left[ \frac{7s + 32}{s^2 + 6s + 10} \right]$$

$$f(t) = -\frac{7}{2} \mathcal{L}^{-1} \frac{1}{s + 2} + \frac{1}{2} \mathcal{L}^{-1} \left[ \frac{7(s + 3) + 11}{(s + 3)^2 + 1} \right]$$

$$= -\frac{7}{2} \mathcal{L}^{-1} \frac{1}{s + 2} + \frac{1}{2} \mathcal{L}^{-1} \frac{7(s + 3)}{(s + 3)^2 + 1} + \frac{1}{2} \mathcal{L}^{-1} \frac{11}{(s + 3)^2 + 1}$$

$$= -\frac{7}{2} \mathcal{L}^{-1} \frac{1}{s + 2} + \frac{7}{2} \mathcal{L}^{-1} \frac{(s + 3)}{(s + 3)^2 + 1^2} + \frac{11}{2} \mathcal{L}^{-1} \frac{1}{(s + 3)^2 + 1^2}$$

$$= -\frac{7}{2} e^{-2t} + \frac{7}{2} e^{-3t} \cos t + \frac{11}{2} e^{-3t} \sin t$$





By applying the inverse Laplace Transform, solve the differential equation for the followings.

1.  $2y' + y = 4e^{2t}$ , given  $y(0) = 3$

$$2(sY(s) - y(0)) + Y(s) = \frac{4}{s-2}$$

$$2sY(s) - 2y(0) + Y(s) = \frac{4}{s-2}$$

$$2sY(s) - 2(3) + Y(s) = \frac{4}{s-2}$$

$$2sY(s) - 6 + Y(s) = \frac{4}{s-2}$$

$$2sY(s) + Y(s) = \frac{4}{s-2} + 6$$

$$Y(s)(2s + 1) = \frac{4 + 6s - 12}{s-2}$$

$$Y(s) = \frac{6s - 8}{(s-2)(2s+1)}$$

$$\frac{6s - 8}{(s-2)(2s+1)} = \frac{A}{s-2} + \frac{B}{2s+1}$$

$$6s - 8 = A(2s + 1) + B(s - 2)$$

$$s - 2 = 0, \quad s = 2$$

$$6(2) - 8 = A(5)$$

$$4 = A(5)$$

$$A = \frac{4}{5}$$

$$2s + 1 = 0, \quad s = -\frac{1}{2}$$

$$6\left(-\frac{1}{2}\right) - 8 = B\left(-\frac{5}{2}\right)$$

$$-11 = B\left(-\frac{5}{2}\right)$$

$$B = \frac{22}{5}$$

$$\frac{6s - 8}{(s-2)(2s+1)} = \frac{4}{5} \frac{1}{s-2} + \frac{22}{5} \frac{1}{2s+1}$$

$$\therefore y(t) = \frac{4}{5} \mathcal{L}^{-1} \frac{1}{s-2} + \frac{22}{5} \mathcal{L}^{-1} \frac{1}{2s+1}$$

$$= \frac{4}{5} \mathcal{L}^{-1} \frac{1}{s-2} + \frac{22}{5} \mathcal{L}^{-1} \frac{1}{2\left(s + \frac{1}{2}\right)}$$

$$= \frac{4}{5} \mathcal{L}^{-1} \frac{1}{s-2} + \frac{11}{5} \mathcal{L}^{-1} \frac{1}{\left(s + \frac{1}{2}\right)}$$

$$= \frac{4}{5} e^{2t} + \frac{11}{5} e^{-\frac{1}{2}t}$$

2.  $y'(t) - y(t) = \sin 2t$ , given  $y(0) = 0$

$$sY(s) - y(0) + Y(s) = \frac{1}{s^2 + 4}$$

$$sY(s) - 0 + Y(s) = \frac{1}{s^2 + 4}$$

$$Y(s)(s + 1) = \frac{1}{s^2 + 4}$$

$$Y(s) = \frac{1}{(s^2 + 4)(s + 1)}$$

$$\frac{1}{(s^2 + 4)(s + 1)} = \frac{As + B}{s^2 + 4} + \frac{C}{s + 1}$$

$$\begin{aligned} 1 &= (As + B)(s + 1) + C(s^2 + 4) \\ &= As^2 - As + Bs - B + Cs^2 + 4C \\ &= (A + C)s^2 + (B - A)s + 4C \end{aligned}$$

coefficients, $s^0$	coefficients, $s^2$	coefficients, $s^1$
$4C = 1$	$A + C = 0$	$B - A = 0$
$C = \frac{1}{4}$	$A + \frac{1}{4} = 0$	$B - \left(-\frac{1}{4}\right) = 0$
	$A = -\frac{1}{4}$	$B = -\frac{1}{4}$

$$\frac{1}{(s^2 + 4)(s + 1)} = \frac{-\frac{1}{4}s - \frac{1}{4}}{s^2 + 4} + \frac{\frac{1}{4}}{s + 1}$$

$$\begin{aligned} \therefore y(t) &= -\frac{1}{4} \mathcal{L}^{-1} \frac{s}{s^2 + 4} - \frac{1}{4} \mathcal{L}^{-1} \frac{1}{s^2 + 4} + \frac{1}{4} \mathcal{L}^{-1} \frac{1}{s + 1} \\ &= -\frac{1}{4} \mathcal{L}^{-1} \frac{s}{s^2 + 2^2} - \frac{1}{8} \mathcal{L}^{-1} \frac{2}{s^2 + 2^2} + \frac{1}{4} \mathcal{L}^{-1} \frac{1}{s + 1} \\ &= -\frac{1}{4} \cos 2t - \frac{1}{4} \sin 2t + \frac{1}{4} e^{-t} \end{aligned}$$

3.  $y'' - 3y' + 2y = 6$ , given  $y(0) = 2, y'(0) = 6$

$$s^2Y(s) - sy(0) - y'(0) - 3(sY(s) - y(0)) + 2Y(s) = 6$$

$$s^2Y(s) - sy(0) - y'(0) - 3sY(s) + 3y(0) + 2Y(s) = 6$$

$$s^2Y(s) - s(2) - 6 - 3sY(s) + 3(2) + 2Y(s) = 6$$

$$s^2Y(s) - 3sY(s) + 2Y(s) = 6 + 2s$$

$$Y(s)(s^2 - 3s + 2) = 6 + 2s$$

$$\begin{aligned} Y(s) &= \frac{6 + 2s}{(s^2 - 3s + 2)} \\ &= \frac{6 + 2s}{(s - 2)(s - 1)} \end{aligned}$$

$$\frac{6 + 2s}{(s - 2)(s - 1)} = \frac{A}{s - 2} + \frac{B}{s - 1}$$

$$6 + 2s = A(s - 1) + B(s - 2)$$

$$s - 2 = 0, \quad s = 2 \qquad s - 1 = 0, \quad s = 1$$

$$6 + 2(2) = B(1) \qquad 6 + 2(1) = A(-1)$$

$$10 = B(1) \qquad 8 = A(-1)$$

$$B = 10 \qquad A = -8$$

$$\frac{6 + 2s}{(s - 2)(s - 1)} = \frac{-8}{s - 2} + \frac{10}{s - 1}$$

$$\therefore y(t) = -8 \mathcal{L}^{-1} \frac{1}{s - 2} + 10 \mathcal{L}^{-1} \frac{1}{s - 1}$$

$$= -8e^{2t} + 10e^{-t}$$

$$\begin{aligned}
4. \quad y''(t) + 2y'(t) + 10y(t) &= 5, \text{ given } y(0) = 0, y'(0) = 1 \\
s^2Y(s) - sy(0) - y'(0) + 2(sY(s) - y(0)) + 10Y(s) &= \frac{5}{s} \\
s^2Y(s) - sy(0) - y'(0) + 2sY(s) - 2y(0) + 10Y(s) &= \frac{5}{s} \\
s^2Y(s) - s(0) - 1 + 2sY(s) - 2(0) + 10Y(s) &= \frac{5}{s} \\
s^2Y(s) + 2sY(s) + 10Y(s) &= \frac{5}{s} + 1 \\
Y(s)(s^2 + 2s + 10) &= \frac{5 + s}{s} \\
Y(s) &= \frac{5 + s}{s(s^2 + 2s + 10)}
\end{aligned}$$

$$\begin{aligned}
\frac{5 + s}{s(s^2 + 2s + 10)} &= \frac{A}{s} + \frac{Bs + C}{s^2 + 2s + 10} \\
5 + s &= A(s^2 + 2s + 10) + (Bs + C)s \\
&= As^2 + 2As + 10A + Bs^2 + Cs \\
&= (A + B)s^2 + (2A + C)s + 10A
\end{aligned}$$

coefficients,  $s^0$

$$10A = 5$$

$$A = \frac{1}{5}$$

coefficients,  $s^2$

$$A + B = 0$$

$$\frac{1}{5} + B = 0$$

$$B = -\frac{1}{5}$$

coefficients,  $s^1$

$$2A + C = 1$$

$$2\left(\frac{1}{5}\right) + C = 1$$

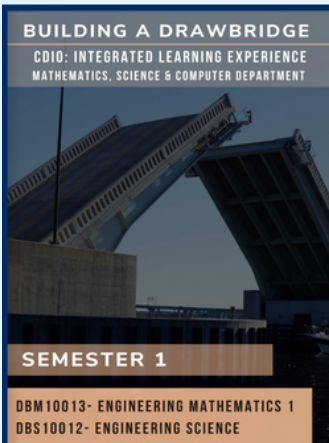
$$C = \frac{3}{5}$$

$$\begin{aligned}
\frac{5 + s}{s(s^2 + 2s + 10)} &= \frac{1}{5s} + \frac{-\frac{1}{5}s + \frac{3}{5}}{s^2 + 2s + 10} \\
\therefore y(t) &= \frac{1}{5}\mathcal{L}^{-1}\frac{1}{s} + \frac{1}{5}\mathcal{L}^{-1}\frac{4 - (s + 1)}{(s + 1)^2 + 9} \\
&= \frac{1}{5}\mathcal{L}^{-1}\frac{1}{s} + \frac{1}{5}\mathcal{L}^{-1}\frac{4}{(s + 1)^2 + 3^2} + \frac{1}{5}\mathcal{L}^{-1}\frac{(s + 1)}{(s + 1)^2 + 3^2} \\
&= \frac{1}{5} + \frac{1}{5}e^{-t}\sin 3t + \frac{1}{5}e^{-t}\cos 3t
\end{aligned}$$



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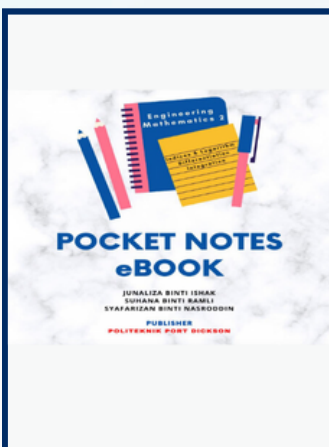
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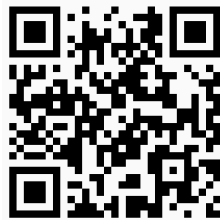
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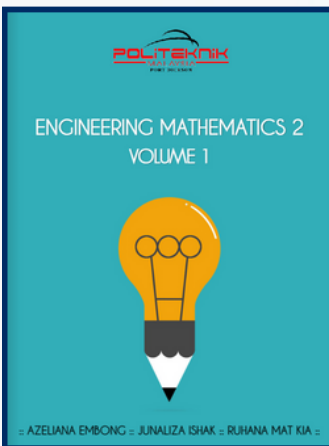
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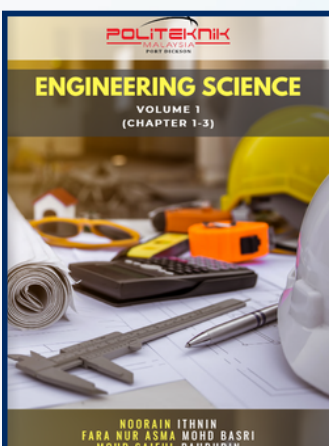
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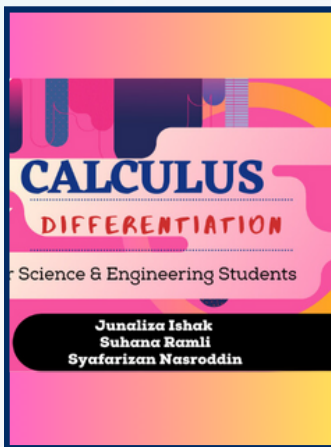
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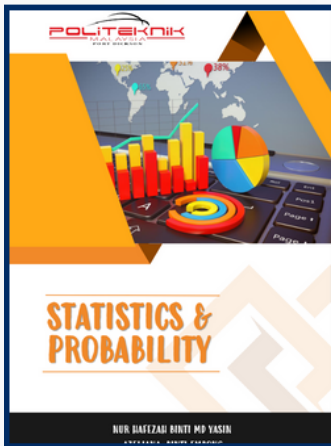
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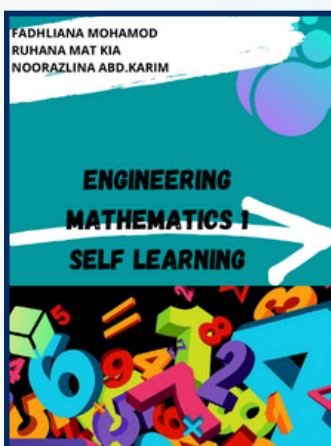


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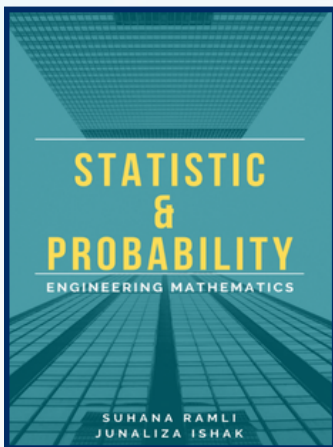
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