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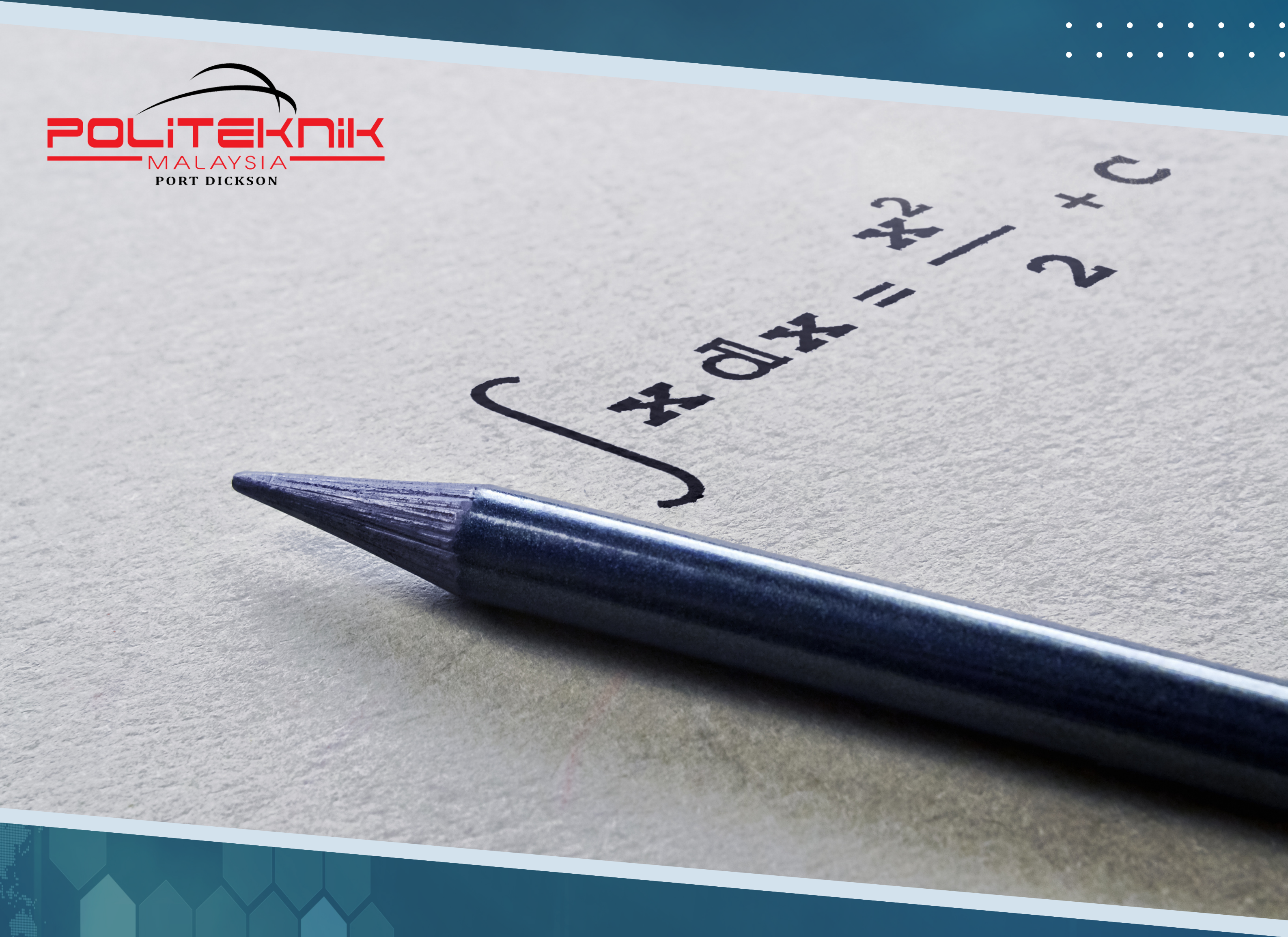

INTEGRATION

POLYTECHNIC EDITION

Suhana Ramli
Junaliza Ishak
Syafarizan Nasroddin

INTEGRATION

Polytechnic Edition

POLITEKNIK MALAYSIA
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PREFACE

Grateful to Allah because with His permission, the eBook Integration Polytechnic Edition was able to be published as scheduled. This eBook is written by lecturers who have been teaching in Engineering Mathematics for more than 15 years. This eBook can be used by all institutions of higher learning such as Polytechnics and Colleges as well as private and public universities. The purpose for this eBook was written is to make it easier for students to gain knowledge and review the topic of differentiation and integration.

There are dozens of examples in various forms of questions being included in this eBook with detailed steps of solution in order to make it easier for students to quickly understand the method of its solution. In addition, students will also be able to improve and strengthen their understanding through the included practice questions. The authors hope that this eBook can benefit all students as well as educators around the world in the field of Engineering Mathematics.

Thank You So Much.

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We hereby declare that this module is our original work. To the best of our knowledge it contains no materials previously written or published by another person. However, if there is any, due acknowledgement and credit are mentioned accordingly in the e-book.

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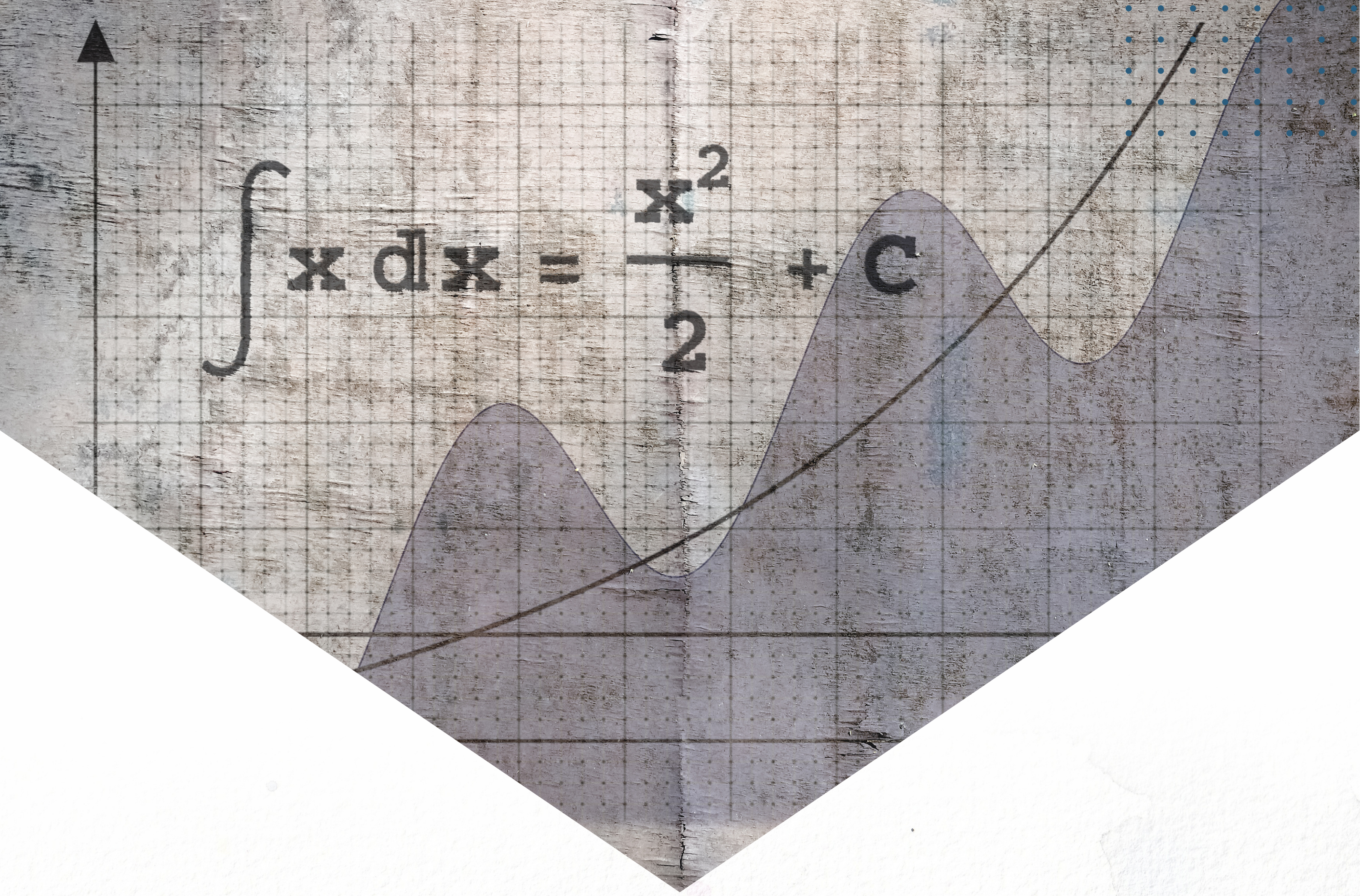


TOPIC

Basic of Integration

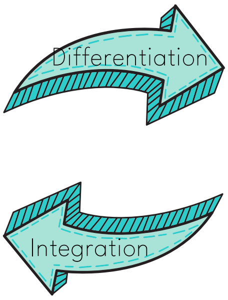
At the end of this chapter, you should be able to:

- Explain the indefinite integrals
- Calculate definite integrals
- Apply integrals of trigonometric function, reciprocal function and exponential function

INTRODUCTION

Integration is the inverse or reverse process of differentiation.
The process of obtaining y from $\frac{dy}{dx}$ is known as integration.

$$y = \int f(x)$$



$$\frac{dy}{dx} = \int f'(x)$$

EXAMPLE

Differentiation	**Integration**
$y = f(x) = x^3$	$\int dy = \int 3x^2\,dx$
$\frac{dy}{dx} = f'(x) = 3x^2$	$y = \frac{3x^3}{3} + c$
	$= x^3 + c$

Symbol of integration

$$\int \ldots\, dx$$

Integration of y with respect to x is

$$\int f(x)\, dx$$

- There are **no limits** of integration in indefinite integral.
- In **indefinite integral**, the constant c must be stated.
- In **definite integrals** this c has some value and can be determined by some function value (*the values of limits: upper limits - lower limits*)

Indefinite integral

$$\int f(x)\,dx = F(x) + c$$

REMEMBER!

where c is a constant and must be stated

Definite integral

$$\int_a^b f(x)\,dx = F(b) - F(a)$$

Important

a) Must have **POWER**

$$\int \sqrt[5]{x^3}\ dx \;\rightarrow\; \int x^{\frac{3}{5}}\ dx$$

b) Bring the power **UP**

$$\int \frac{2}{x^4}\ dx \;\rightarrow\; \int 2x^{-4}\ dx$$

c) Expand

$$\int (x+2)(1-x)\ dx \;\rightarrow\; \int 2 - x - x^2\ dx$$

$$\int (x-3)^2\ dx \;\rightarrow\; \int x^2 - 6x + 9\ dx$$

d) Separate

$$\int \frac{2x^5 - x^2 + 5}{x^2}\ dx \;\rightarrow\; \int \frac{2x^5}{x^2} - \frac{x^2}{x^2} + \frac{5}{x^2}\ dx$$

$$= \int 2x^3 - 1 + 5x^{-2}\ dx$$

INTEGRATION OF ALGEBRAIC FUNCTIONS

a) Constant Function

$$\int a\,dx$$
$$= ax + c$$

b) Constant Multiple Rule

$$\int ax\,dx, \qquad n \neq -1$$
$$= \frac{ax^{n+1}}{n+1} + c$$

c) Power Function Rule

$$\int x^n\,dx, \qquad n \neq -1$$
$$= \frac{x^{n+1}}{n+1} + c$$

d) Addition & Subtraction

$$\int f(x) \pm g(x)\,dx$$
$$= \int f(x)\,dx \pm \int g(x)\,dx$$

e) Formula Method/ Composite Function

$$\int (ax+b)^n\,dx, \qquad n \neq -1$$
$$= \frac{(ax+b)^{n+1}}{(n+1)\left(\frac{d}{dx}(ax+b)\right)} + c$$

f) Substitution Method

$$\int f[g(x)] \times g'(x)\,dx = f(u)\cdot du,$$

where $u = g(x)$

$$\frac{du}{dx} = g'(x)$$
$$du = g'(x)dx$$

Important

How to Integrate...

$$\int x^n \; dx$$

$$= \frac{x^{n+1}}{n+1} + c$$

1. POWER plus 1
2. OVER the new power
3. ‘ $+c$ ’

NOTES

The ‘$+c$’ is just how we write “plus an unknown constant”

Example 1

a) Constant Function

Find the integral of each of the following:

a) $\int 5 \; dx$

$= 5x + c$

b) $\int \pi \; dz$

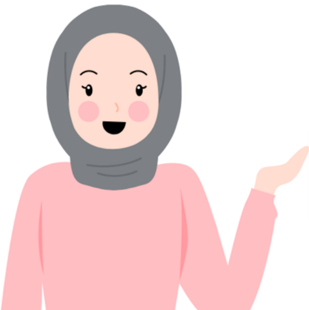
$= \pi z + c$

c) $\int \frac{1}{5} \; dt$

$= \frac{1}{5} t + c$

d) $\int \sqrt[3]{5^2} \; dm$

$= \sqrt[3]{5^2} m + c$



$\int a \; dx = ax + c$ where a is constant (number)

b) Constant Multiple Rule

Find the integral of each of the following:

a) $\int 7x \; dx$

$= \frac{7x^{1+1}}{1+1} + c$

$= \frac{7x^2}{2} + c$

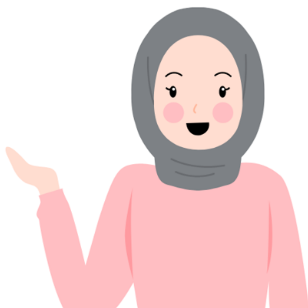
b) $\int \frac{2}{3} z \; dz$

$= \frac{2}{3} \cdot \frac{z^{1+1}}{1+1} + c$

$= \frac{2}{3} \cdot \frac{z^2}{2} + c$

$= \frac{z^2}{3} + c$

$$\int ax \, dx = \frac{ax^{n+1}}{n+1} + c, n \neq -1$$



Example 2

c) Power Function Rule

Find the integral of each of the following:

a) $\int x^3 \ dx$

$= \frac{x^{3+1}}{3+1} + c$

$= \frac{x^4}{4} + c$

b) $\int 4z^2 \ dz$

$= \frac{4z^{2+1}}{2+1} + c$

$= \frac{4z^3}{3} + c$

c) $\int \frac{1}{k^5} \ dk$

change form

$= \int k^{-5} \ dk$

$= \frac{k^{-5+1}}{-5+1} + c$

$= \frac{k^{-4}}{-4} + c$

$= \frac{1}{-4k^4} + c$

d) $\int \frac{dm}{\sqrt{m}}$

change form

$= \int \frac{1}{m^{\frac{1}{2}}} \ d$

change form

$= \int m^{-\frac{1}{2}} \ dm$

$= \frac{m^{-\frac{1}{2}+1}}{-\frac{1}{2}+1} + c$

$= \frac{m^{\frac{1}{2}}}{-\frac{1}{2}} + c$

$= -2m^{\frac{1}{2}} + c$ @ $2\sqrt{m} +$

e) $\int 3\sqrt{x} \ dx$

change form

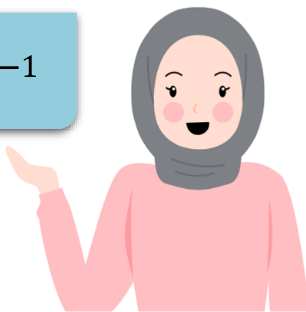
$= \int 3x^{\frac{1}{2}} \ dx$

$= \frac{3x^{\frac{1}{2}+1}}{\frac{1}{2}+1} + c$

$= \frac{3x^{\frac{3}{2}}}{\frac{3}{2}} + c$

$= 2x^{\frac{3}{2}} + c$@ $2\sqrt{x^3} + c$

$$\int x^n \ dx = \frac{x^{n+1}}{n+1} + c, n \neq -1$$



Example 3

d) Addition & Subtraction

Find the integral of each of the following:

a. $\int 4x^2 + 2\ dx$

$= \int 4x^2\ dx + \int 2\ dx$

$= \dfrac{4x^3}{3} + 2x + c$

$= \dfrac{4x^3}{3} + 2x + c$

b. $\int \left(3 + \dfrac{2}{x^2} - \dfrac{4}{x^3}\right) dx$ — change form

$= \int 3 + 2x^{-2} - 4x^{-3}\ dx$

$= \int 3x\ dx + \int 2x^{-2}\ dx - \int 4x^{-3}\ dx$

$= 3x + \dfrac{2x^{-1}}{(-1)} - \dfrac{4x^{-2}}{(-2)} + c$

$= 3x - \dfrac{2}{x} + \dfrac{4}{2x^2} + c$

$= 3x - \dfrac{2}{x} + \dfrac{2}{x^2} + c$

c. $\int x\left(2x^2 - 3\right) dx$ — expand

$= \int 2x^3 - 3x\, dx$

$= \int 2x^3 dx - \int 3x\ dx$

$= \dfrac{2x^4}{4} - \dfrac{3x^2}{2} + c$

$= \dfrac{x^4}{2} - \dfrac{3x^2}{2} + c$

$= \dfrac{x^4 - 3x^2}{2} + c$

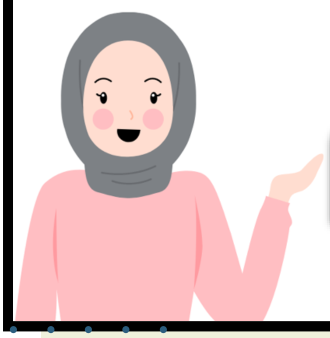
d. $\int (2x+1)(x-2) dx$ — expand

$= \int 2x^2 - 4x + x - 2\ dx$

$= \int 2x^2 - 3x - 2\ dx$

$= \int 2x^2 dx - \int 3x\, dx - \int 2\, dx$

$= \dfrac{2x^3}{3} - \dfrac{3x^2}{2} - 2x + c$



$$\int f(x) \pm g(x)\, dx = \int f(x)\, dx \pm \int g(x)\, dx$$

Example 4

Find the integral of each of the following:

a) $\int \left(\frac{x^2+2}{x^2} \right)^2 dx$

separate

$= \int \left(\frac{x^2}{x^2} + \frac{2}{x^2} \right)^2 dx$

$= \int \left(1 + 2x^{-2}\right)^2 dx$

$= \int \left(1 + 2x^{-2}\right)\left(1 + 2x^{-2}\right) dx$

$= \int 1 + 4x^{-2} + 4x^{-4} \, dx$

$= x + \frac{4x^{-1}}{(-1)} + \frac{4x^{-3}}{(-3)} + c$

$= x - \frac{4}{x} - \frac{4}{3x^3} + c$

b) $\int \frac{(3+x)(2-x)}{x^4} dx$

expand & separate

$= \int \frac{6-x-x^2}{x^4} dx$

$= \int \frac{6}{x^4} dx + \int \frac{-x}{x^4} dx - \int \frac{-x^2}{x^4} dx$

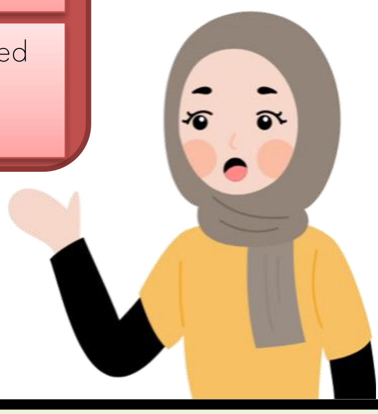
$= \int 6x^{-4} \, dx - \int x^{-3} \, dx - \int x^{-2} \, dx$

$= \frac{6x^{-3}}{(-3)} - \frac{x^{-2}}{(-2)} - \frac{x^{-1}}{(-1)} + c$

$= \frac{-2}{x^3} + \frac{1}{2x^2} + \frac{1}{x} + c$

DON'T FORGET

- In integration, there is no product rule or quotient rule
- The functions need to be expanded or separated then simplified and lastly integrate term by term



Exercise

Find the integral of each of the following:

a) $\int dx$

b) $\int 7x^2\, dx$

c) $\int \frac{3}{2}t^4\, dt$

d) $\int \frac{5}{x^3}\, dx$

e) $\int -\frac{1}{3y^5}\, dy$

f) $\int (5x^3 + 2x^2 - 9)\, dx$

g) $\int \left(4 + \frac{1}{x^2} - \frac{3}{x^4}\right) dx$

h) $\int x^2(x-3)\, dx$

i) $\int (3-5z)^2\, dz$

j) $\int \frac{x^3 + x^2 + 10}{x^2}\, dx$

k) $\int \left(\frac{x^6+2}{x^3}\right)^2 dx$

l) $\int \left(1+\frac{2}{u}\right)\left(1-\frac{2}{u}\right) du$

ANSWER

a) $x + c$

b) $\frac{7x^3}{3} + c$

c) $\frac{3t^5}{10} + c$

d) $-\frac{5}{2x^2} + c$

e) $\frac{1}{12y^4} + c$

f) $\frac{5x^4}{4} + \frac{2x^3}{3} - 9x + c$

g) $4x - \frac{1}{x} + \frac{1}{x^3} + c$

h) $\frac{x^4}{4} - x^3 + c$

i) $9z - 15z^2 + \frac{25z^3}{3} + c$

j) $\frac{x^2}{2} + x - \frac{10}{x} + c$

k) $\frac{x^7}{7} + 4x - \frac{4}{5x^5} + c$

l) $u + \frac{4}{u} + c$

INTEGRATION OF ALGEBRAIC FUNCTIONS

e) Formula Method/Composite Function

$$\int (ax+b)^n\,dx, \qquad n \neq -1$$

$$= \frac{(ax+b)^{n+1}}{\left(\frac{d}{dx}(ax+b)\right)(n+1)} + c$$

REMEMBER!

this FORMULA can ONLY be used when POWER of x is 1 !

Important

How to Integrate composite function…

$$\int (ax+b)^n\,dx$$

$$= \frac{(ax+b)^{n+1}}{\left(\frac{d}{dx}(ax+b)\right)(n+1)} + c$$

1. **POWER plus 1**
2. **OVER** $\rightarrow$ $\left(\frac{d}{dx}(ax+b)\right)$ **x (the new power)**
3. '$+c$'

f) Substitution Method

$$\int f[g(x)] \times g'(x)\,dx = f(u)\cdot du,$$

$$\text{where } u = g(x)$$

$$\frac{du}{dx} = g'(x)$$

$$du = g'(x)\,dx$$

REMEMBER

Substitution Method:

Find $u \rightarrow$ priority u

i. Inside bracket ➔ $\int 5x(2x^2+5)^3 dx$

ii. Denominator ➔ $\int \frac{2x}{(1-x^2)} dx$

iii. Highest power ➔ $\int \sin 3x \cos^4 3x dx$

iv. Power of exponent ➔ $\int 5x\, e^{3x^2} dx$

Important

How to Integrate by using substitution method:

1. Identify 'u'
2. Find $\frac{du}{dx}$, then make dx as subject
3. Substitute 'u' and dx into the original question
4. Simplify, then integrate
5. Substitute back u=$g(x)$ into the final answer

Example 5

Solve the integral $\int (2x-3)^4 dx$

Solution :

which METHOD to be used is based on the question given

REMEMBER!

METHOD 1 : By using **Substitution Method**

Step 1 : Identify 'u' → must be in the bracket & substitute $u = g(x)$

$$u = 2x - 3$$

Step 2 : Find $\frac{du}{dx}$ → differentiate u, then find dx

$$\frac{du}{dx} = 2$$

$$dx = \frac{du}{2}$$

Step 3 : Substitute $g(x) = u$ and $dx = \frac{du}{g'(x)}$ into the original question

$$= \int u^4 \frac{du}{2}$$

$$= \frac{1}{2}\int u^4 \, du$$

Put the coefficient outside $\int$

Step 4 : Integrate

$$= \frac{1}{2} \cdot \frac{u^5}{5} + c$$

$$= \frac{u^5}{10} + c$$

Step 5 : Substitute back $u = g(x)$ into the final answer

$$= \frac{(2x-3)^5}{10} + c$$

METHOD 2 : By using **Formula Method/Composite Function**

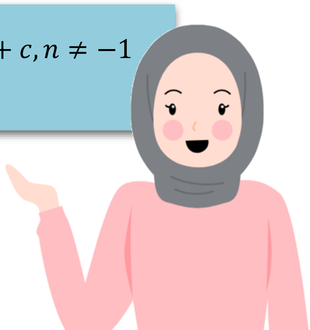
$$\int (ax+b)^n \, dx, = \frac{(ax+b)^{n+1}}{\left(\frac{d}{dx}(ax+b)\right)(n+1)} + c, n \neq -1$$

Step 1 : Integrate

$$= \frac{(2x-3)^{4+1}}{2\,(4+1)} + c$$

Step 2 : Simplify

$$= \frac{(2x-3)^5}{10} + c$$



Example 6

Solve the integral $\int 5(5z+2)^5 dx$

Solution :

METHOD 1 : By using Substitution Method

Step 1 : Identify 'u' → must be in the bracket & substitute $u = g(z)$

$u = 5z + 2$

Step 2 : Find $\frac{du}{dz}$ → differentiate u, then find dz

$\frac{du}{dz} = 5$

$dz = \frac{du}{5}$

Step 3 : Substitute $g(z) = u$ and $dz = \frac{du}{g\prime(z)}$ into the original question

$$= \int 5u^5 \frac{du}{5}$$

$$= \int u^5 \, du$$

Step 4 : Integrate

$$= \frac{u^6}{6} + c$$

Step 5 : Substitute back $u = g(z)$ into the final answer

$$= \frac{(5z+2)^6}{6} + c$$

METHOD 2 : By using Formula Method/Composite Function

Step 1 : Integrate

$$= \frac{5(5z+2)^{5+1}}{5\,(5+1)} + c$$

Step 2 : Simplify

$$= \frac{(5z+2)^6}{6} + c$$

Example 7

Solve the integral $\int x(3-2x^2)^3\,dx$

DON'T FORGET

ONLY use Substitution Method if the variable in the BRACKET of the function has POWER

Solution :

METHOD: By using **Substitution Method**

Step 1 : Identify 'u' → must be in the bracket & substitute $u = g(x)$

$$u = 3 - 2x^2$$

Step 2 : Find $\frac{du}{dx}$ → differentiate u, then find dx

$$\frac{du}{dx} = -4x$$

$$dx = \frac{du}{-4x}$$

Step 3 : Substitute $g(x) = u$ and $dx = \frac{du}{g'(x)}$ into the original question

$$= \int x\,u^3 \frac{du}{-4x}$$

$$= -\frac{1}{4}\int u^3\,du$$

Put the coefficient outside $\int$

Step 4 : Integrate

$$= -\frac{1}{4}\cdot\frac{u^4}{4} + c$$

$$= \frac{u^4}{16} + c$$

Step 5 : Substitute back $u = g(x)$ into the final answer

$$= \frac{(3-2x^2)^4}{16} + c$$

NOTES

A constant factor in an integral can be moved outside the integral sign:

$$\int k\,f(x)\,dx = k\int f(x)\,dx$$

Example 8

Solve the integral $\int 2p(p-7)^5\,dp$

Solution :

DON'T FORGET

ONLY use Substitution Method if there is a variable outside the bracket and the variable in the bracket of the function has NO POWER

Step 1 : Identify 'u' → must be in the bracket & substitute $u = g(x)$

$$u = p - 7$$

Step 2 : Find $\frac{du}{dp}$ → differentiate u, then find dp

$$\frac{du}{dp} = 1$$

$$dp = du$$

Step 3 : Substitute $g(p) = u$ and $dp = \frac{du}{g'(p)}$ into the original question

$$= \int 2p \cdot u^7\ du$$

There are 2 variables

Make p as subject matter:
$u = p - 7$
$p = u + 7$

$$= 2\int (u+7)\,u^7\ du$$

$$= 2\int u^8 + 7u^7\ du$$

Step 4 : Integrate

$$= 2\left[\frac{u^9}{9} + \frac{7u^8}{8}\right] + c$$

$$= \frac{2u^9}{9} + \frac{7u^8}{4} + c$$

REMEMBER!

If there are 2 variables, NEED to:
- Remain the variable u
- Change the other variable to u → $\int f(u)\ du$

Step 5 : Substitute back $u = g(p)$ into the final answer

$$= \frac{2(p-7)^9}{9} + \frac{7(p-7)^8}{4} + c$$

Exercise

1. Find the integral of each of the following by using Formula Method

a) $\int (-3t+5)^3 \, dt$

b) $\int 3\left(\frac{x}{2}-4\right)^3 \, dx$

c) $\int 4(6x-4)^{-2} dx$

d) $\int \frac{5}{(4-3x)^3} \, dx$

e) $\int 2(7-3x)^{-6} dx$

f) $\int \frac{1}{\sqrt{x+6}} \, dx$

2. Find the integral of each of the following by using Substitution Method

a) $\int z^2\sqrt{(z^3+7)^3} \, dz$

b) $\int (1-2x)(3x-3x^2)^{\frac{1}{2}} \, dx$

c) $\int (3x^2+4x)(x^3+2x^2-5)^2 dx$

d) $\int \frac{1}{(3x+2)^4} \, dx$

e) $\int \frac{-3}{4(3y+1)^6} \, dy$

f) $\int \frac{1}{\sqrt[3]{9-2z}} \, dz$

ANSWER : by using Formula Method

a) $\frac{[-3t+5^3]^4}{-12} + c$

b) $\frac{3\left[\frac{x}{2}-4\right]^4}{2} + c$

c) $-\frac{2}{3[6x+4]} + c$

d) $\frac{5}{6[4-3x]^2} + c$

e) $\frac{2}{15[7-3x]^5} + c$

f) $2\sqrt{x+6} + c$

ANSWER : by using Substitution Method

a) $\frac{2}{15}[z^3+7]^{\frac{5}{2}} + c$

b) $\frac{2}{9}[3x-3x^2]^{\frac{3}{2}} + c$

c) $\frac{1}{3}[x^3+2x^2-5]^3 + c$

d) $-\frac{1}{9[3x+2]^3} + c$

e) $\frac{1}{20[3y+1]^5} + c$

f) $-\frac{3}{4}[9-2z]^{\frac{2}{3}} + c$

DEFINITE INTEGRALS

Definite integral is integrals with the limits of integration. It has **start** and **end** values which also known an interval $[a, b]$.

a is the lower limit of integration and
b is the upper limit of integration

$$\int_a^b f(x)\ dx$$

the integral from a to b of $f(x)$ with respect to x

If a function $f(x)$ is continuous on the interval $[a\ , b]$, which means

$$\int_a^b f(x)\, dx = F[b] - F[a]$$

Properties of Definite Integral

REMEMBER!

$\int f(x)\ dx$ from $x = a$ (lower limit) to $x = b$ (upper limit)

a. $\displaystyle\int_a^b f(x)\, dx = F(b) - F(a)$

b. $\displaystyle\int_a^b kf(x)\, dx = k\int_a^b f(x)\, dx;\ \ k \text{ is a constant}$

c. $\displaystyle\int_a^b [f_1(x) \pm f_2(x)]dx = \int_a^b f_1(x)\ dx \pm \int_a^b f_2(x)\ dx$

d. $\displaystyle\int_a^b f(x)\, dx = -\int_b^a f(x)\, dx$

e. $\displaystyle\int_a^c f(x)\, dx = \int_a^b f(x)\, dx + \int_b^c f(x)\, dx$

Example

1. Evaluate the definite integrals $\int_1^2 3x^2 - 3 \; dx$

Solution :

Step 1 : Integrate the term

$$= \left[\frac{3x^3}{3} - 3x\right]_1^2$$

upper limit value

lower limit value

$$= [x^3 - 3x]_1^2$$

Step 2 : Substitute limit where the upper value minus the lower value

$$= [(2)^3 - 3(2)] - [(1)^3 - 3(1)]$$

Step 3 : Solve/Simplify

$$= [2] - [-2]$$

$$= 4$$

REMEMBER

The final answer will be a number

2. Evaluate the definite integrals $\int_2^4 5 + \frac{1}{x^2} - 2x - \sqrt[3]{x} \; dx$

Solution :

Step 1 : Change form

$$= \int_2^4 5 + x^{-2} - 2x - x^{\frac{1}{3}} \; dx$$

Step 2 : Integrate the term

$$= \left[5x + \frac{x^{-1}}{-1} - \frac{2x^2}{2} - \frac{x^{\frac{4}{3}}}{\frac{4}{3}}\right]_2^4$$

$$= \left[5x - \frac{1}{x} - x^2 - \frac{3\sqrt[3]{x^4}}{4}\right]_2^4$$

Step 3 : Substitute limit where the upper value minus the lower value

$$= \left[5(4) - \frac{1}{(4)} - (4)^2 - \frac{3\sqrt[3]{(4)^4}}{4}\right] - \left[5(2) - \frac{1}{(2)} - (2)^2 - \frac{3\sqrt[3]{(2)^4}}{4}\right]$$

Step 4 : Solve/Simplify

$$= [-1.0122] - [3.6101]$$

$$= -4.6223$$

Example

3. Evaluate the definite integrals $\int_{-1}^{2}(2x+1)(2x-1)\,dx$

Solution :

Step 1 : Change form

$$=\int_{-1}^{2}4x^2-1\,dx$$ ← $(2x+1)(2x-1)$ => expand

Step 2 : Integrate the term

$$=\left[\frac{4x^3}{3}-x\right]_{-1}^{2}$$

Step 3 : Substitute limit where the upper value minus the lower value

$$=\left[\frac{4(2)^3}{3}-2\right]-\left[\frac{4(-1)^3}{3}-(-1)\right]$$

Step 4 : Solve/Simplify

$$=\frac{26}{3}-\left[-\frac{1}{3}\right]$$

$$=9$$

4. Evaluate the definite integrals $\int_{2}^{4}\sqrt[3]{(3x-5)^4}\,dx$

Solution :

Step 1 : Change form

$$=\int_{2}^{4}(3x-5)^{\frac{4}{3}}\,dx$$

Step 2 : Integrate the term

$$=\left[\frac{(3x-5)^{\frac{7}{3}}}{3\frac{7}{3}}\right]_{2}^{4}$$

Step 3 : Substitute limit where the upper value minus the lower value

$$=\left[\frac{(3(4)-5)^{\frac{7}{3}}}{7}\right]-\left[\frac{(3(2)-5)^{\frac{7}{3}}}{7}\right]$$

Step 4 : Solve/Simplify

$$=[13.39]-[0.14]$$

$$=13.25$$

Example

5. Evaluate the definite integrals $\int_0^3 \frac{5x^2-x^4}{3x^2}\,dx$

Solution :

Step 1 : Change form

$$\int_0^3 \frac{5x^2 - x^4}{3x^2}\,dx$$

separate

$$= \int_0^3 \frac{5x^2}{3x^2} - \frac{x^4}{3x^2}\,dx$$

$$= \int_0^3 \frac{5}{3} - \frac{x^2}{3}\,dx$$

Step 2 : Integrate the term

$$= \left[\frac{5}{3}x - \frac{x^3}{(3)3}\right]_0^3$$

$$= \left[\frac{5}{3}x - \frac{x^3}{9}\right]_0^3$$

Step 3 : Substitute limit where the upper value minus the lower value

$$= \left[\frac{5}{3}(3) - \frac{(3)^3}{9}\right] - \left[\frac{5}{3}(0) - \frac{(0)^3}{9}\right]$$

Step 4 : Solve/Simplify

$$= 2 - [0]$$

$$= 2$$

Example

6. Evaluate the definite integrals $\int_0^1 8x\,(2x^2-3)^5 dx$

Solution :

METHOD: By using **Substitution Method**

Step 1 : Identify 'u' → must be in the bracket & substitute $u = g(x)$

$u = 2x^2 - 3$

Step 2 : Find $\frac{du}{dx}$ → differentiate u, then find dx

$\frac{du}{dx} = 4x$

$dx = \frac{du}{4x}$

Step 3 : Substitute $g(x) = u$ and $dx = \frac{du}{g'(x)}$ into the original question

$$= \int_0^1 8x\, u^5 \frac{du}{4x}$$

$$= 2\int_0^1 u^5\, du$$

Put the coefficient outside $\int$

Step 4 : Integrate

$$= 2\left[\frac{u^6}{6}\right]_0^1$$

Step 5 : Substitute back $u = g(x)$ into the final answer

$$= \left[\frac{(2x^2-3)^6}{3}\right]_0^1$$

Step 6 : Substitute limit where the upper value minus the lower value

$$= \left[\frac{(2(1)^2-3)^6}{3}\right] - \left[\frac{(2(0)^2-3)^6}{3}\right]$$

Step 7 : Solve/Simplify

$$= \frac{1}{3} - 243$$

$$= -\frac{728}{3}$$

Example

7. Given , $\int_{-2}^{3} f(x)\,dx = 7$, find the values of the following

a) $\int_{3}^{-2} f(x)\,dx$

b) $\int_{-2}^{3} \frac{1}{2} f(x)\,dx$

c) $\int_{-2}^{2} f(x)\,dx + \int_{2}^{3} f(x)\,dx$

d) $\int_{-2}^{3} [x^2 - 2f(x)]\,dx$

Solution :

a) $\int_{3}^{-2} f(x)\,dx$

$= -\int_{-2}^{3} f(x)\,dx$

$= -7$

Use:
$\int_{a}^{b} f(x)\,dx = -\int_{b}^{a} f(x)\,dx$

b) $\int_{-2}^{3} \frac{1}{2} f(x)\,dx$

$= \frac{1}{2}\int_{-2}^{3} f(x)\,dx$

$= \frac{1}{2}(7)$

$= \frac{7}{2}$

c) $\int_{-2}^{2} f(x)\,dx + \int_{2}^{3} f(x)\,dx$

$= \int_{-2}^{3} f(x)\,dx$

$= 7$

Use:
$\int_{a}^{c} f(x)\,dx = \int_{a}^{b} f(x)\,dx + \int_{b}^{c} f(x)\,dx$

d) $\int_{-2}^{3} [x^2 - 2f(x)]\,dx$

$= \int_{-2}^{3} x^2\,dx - 2\int_{-2}^{3} f(x)\,dx$

$= \left[\frac{x^3}{3}\right]_{-2}^{3} - 2(7)$

$= \left[\frac{(3)^3}{3} - \frac{(-2)^3}{3}\right] - 14$

$= \left[\frac{(3)^3}{3} - \frac{(-2)^3}{3}\right] - 14$

$= \left[9 + \frac{8}{3}\right] - 14$

$= \frac{35}{3} - 14$

$= -\frac{7}{3}$

Use:
$\int_{a}^{c} f(x)\,dx = \int_{a}^{b} f(x)\,dx + \int_{b}^{c} f(x)\,dx$

Exercise

Evaluate each of the following:

a) $\int_{1}^{2} 4w(2w^2 - 2)^2 dw$

b) $\int_{\frac{\pi}{2}}^{\pi} \frac{z}{(1 - z^2)^3} dz$

c) $\int_{-1}^{2} (2t - 2t^5)\, dt$

d) $\int_{-4}^{-1} \frac{16}{x^5}\, dx$

e) Given $\int_1^3 g(h)\, dh = 9$ and $\int_3^5 g(h) = 2$. Find

i. $\int_{5}^{3} g(h)\, dh$

ii. $\int_{1}^{5} g(h)\, dh$

iii. $2\int_{1}^{3} g(h)\, dh$

iv. $\int_{1}^{3} [h + g(h)]\, dh$

ANSWER

a) 72 b) -0.112 c) -18 d) -3.984

e) i) -2 e) ii) 11 e) iii) 18 e) iv) 14

TRIGONOMETRIC FUNCTION

Trigonometric integral will take us to look at how to integrate certain combinations of products and powers of trigonometric functions.

Let's us scrutinize at these **Trigonometry Identities**:

Reciprocal Identities

1. $\tan x = \frac{\sin x}{\cos x}$
2. $\operatorname{cosec} x = \frac{1}{\sin x}$
3. $\sec x = \frac{1}{\cos x}$
4. $\cot x = \frac{1}{\tan x} = \frac{\cos x}{\sin x}$

Double Angle Identities

1. $\sin 2\theta = 2 \sin\theta \cos\theta$
2. $\cos 2\theta = \cos^2\theta - \sin^2\theta$
3. $\cos 2\theta = 2\cos^2\theta - 1$
4. $\cos 2\theta = 1 - 2\sin^2\theta$
5. $\tan 2\theta = \frac{2\tan\theta}{1-\tan^2\theta}$

Integrals of Trigonometric Function

1. $\int \sin x \, dx = -\cos x + c$
2. $\int \cos x \, dx = \sin x + c$
3. $\int \sec^2 x \, dx = \tan x + c$
4. $\int \sin(ax+b) \, dx = -\frac{1}{a}\cos(ax+b) + c$
5. $\int \cos(ax+b) \, dx = \frac{1}{a}\sin(ax+b) + c$
6. $\int \sec^2(ax+b) \, dx = \frac{1}{a}\tan(ax+b) + c$

Trigonometric Identities

1. $\cos^2 x + \sin^2 x = 1$
2. $1 + \tan^2 x = \sec^2 x$
3. $\cot^2 x + 1 = \operatorname{cosec}^2 x$

REMEMBER

Integration for Trigonometric Function:
- ✓ function **CANNOT** have power except $\sec^2 x$
- ✓ Solve using *Trigonometric Identity* if the function HAS a power

Example 1

Find each of the following indefinite trigonometric function:

a) $\int sin\ 2x\ \ dx$

b) $\int 7\ cos\ 5x\ \ dx$

c) $\int 2 + cos\ (5x - 2)\ dx$

d) $\int 5sec^2 3x - sin\frac{2}{3}x\ dx$

Solution :

Step 1 : Integrate → refer to the formula Integrals of Trigonometric Function

a) $\int sin\ 2x\ \ dx$

$= -\frac{1}{2}\ cos\ 2x + c$

b) $\int 7\ cos\ 5x\ \ dx$

$= 7 \cdot \frac{1}{5}\ sin\ 5x + c$

$= \frac{7}{5}\ sin\ 5x + c$

c) $\int 2 + cos\ (5x - 2)\ dx$

$= 2x + \frac{1}{5}\ sin\ (5x - 2) + c$

d) $\int 5\ sec^2 3x - sin\frac{2}{3}x\ dx$

$= 5 \cdot \frac{1}{3}\ tan\ 3x - \left[-\frac{1}{\frac{2}{3}} cos\frac{2}{3}x \right] + c$

$= \frac{5}{3}\ tan\ 3x + \frac{3}{2} cos\frac{2}{3}x + c$

Example 2

Solve the integral $\int x \cos x^2 \, dx$

Solution :

METHOD: By using **Substitution Method**

Step 1 : Identify 'u' → must be the highest power & substitute $u = g(x)$

$$u = x^2$$

Step 2 : Find $\frac{du}{dx}$ → differentiate u, then find dx

$$\frac{du}{dx} = 2x$$

$$dx = \frac{du}{2x}$$

Step 3 : Substitute $g(x) = u$ and $dx = \frac{du}{g'(x)}$ into the original question

$$= \int x \cos u \, \frac{du}{2x}$$

$$= \frac{1}{2} \int \cos u \, du$$

Step 4 : Integrate

$$= \frac{1}{2} \cdot \sin u + c$$

$$= \frac{\sin u}{2} + c$$

Step 5 : Substitute back $u = g(x)$ into the final answer

$$= \frac{\sin x^2}{2} + c$$

Example 3

NOTES
Trigonometric function has POWER

Solve the integral $\int \sin 4x \cos^5 4x \ dx$

Solution :

METHOD: By using **Substitution Method**

Step 1 : Identify 'u' $\rightarrow$ must be the highest power & substitute $u = g(x)$

$$u = \cos 4x$$

Step 2 : Find $\frac{du}{dx}$ $\rightarrow$ differentiate u, then find dx

$$\frac{du}{dx} = -4 \sin 4x$$

$$dx = \frac{du}{-4 \sin 4x}$$

Step 3 : Substitute $g(x) = u$ and $dx = \frac{du}{g'(x)}$ into the original question

$$= \int \sin 4x \cdot u^5 \frac{du}{-4 \sin 4x}$$

$$= -\frac{1}{4} \int u^5 \, du$$

Step 4 : Integrate

$$= -\frac{1}{4} \cdot \frac{u^6}{6} + c$$

$$= -\frac{u^6}{24} + c$$

Step 5 : Substitute back $u = g(x)$ into the final answer

$$= -\frac{(\cos 4x)^6}{24} + c$$

REMEMBER
Can also be written:
$$= -\frac{\cos^6 4x}{24} + c$$

Example 4

NOTES

Trigonometric function has POWER

Solve the integral $\int sin^2 3x \; dx$

Solution :

Step 1 : Refer to Double Angle Identities

$$cos\ 2\theta = 1 - 2x sin^2\theta$$

$$sin^2\theta = \frac{1}{2}(1 - cos\ 2\theta)$$

$$\theta = 3x$$

$$\therefore 2\theta = 6x$$

$$sin^2\ 3x = \frac{1}{2}(1 - cos\ 6x)$$

Step 2 : Substitute $sin^2\ 3x$ into the original question

$$= \int \frac{1}{2}(1 - cos\ 6x)\ dx$$

$$= \frac{1}{2}\int (1 - cos\ 6x)\ dx$$

Step 3 : Integrate

$$= \frac{1}{2}\left[x - \frac{1}{6} sin\ 6x\right] + c$$

$$= \frac{x}{2} - \frac{1}{12} sin\ 6x\ + c$$

Example 5

Solve the integral $\int cos^3 2x\ dx$

NOTES
Trigonometric function has POWER

Solution :

Step 1 : Break the question

$$\int cos^3 2x\ dx$$

$$= \int cos\ 2x\ cos^2 2x\ dx$$

Step 2 : Refer to Trigonometric Identity

$$cos^2\theta + sin^2\theta = 1$$

$$cos^2\theta = 1 - sin^2\theta$$

$$\theta = 2x$$

$$cos^2\ 2x = 1 - sin^2\ 2x$$

Step 2 : Substitute $cos^2\ 2x$ into the original question

$$= \int cos\ 2x\ (1 - sin^2 2x\ )\ dx$$

Step 3 : By using Substitution Method

$$u = sin\ 2x$$

$$\frac{du}{dx} = 2\ cos\ 2x$$

$$dx = \frac{du}{2\ cos\ 2x}$$

$$= \int cos\ 2x\ (1 - u^2)\ \frac{du}{2\ cos\ 2x}$$

$$= \frac{1}{2}\int 1 - u^2\ du$$

$$= \frac{1}{2}\left[u - \frac{u^3}{3}\right] + c$$

$$= \frac{1}{2}\left[sin2x\ - \frac{(sin2x)^3}{3}\right] + c$$

REMEMBER
Can also be written:
$$-\frac{sin^3 2x}{3} + c$$

Example 6

Evaluate the definite integral $\int_1^3 sec^2 5x \ dx$

Solution :

Step 1 : Integrate → refer to the formula Integrals of Trigonometric Function

$$= \left[\frac{1}{5} tan \ 5x\right]_1^3$$

Step 2 : Substitute limit where the upper value minus the lower value

$$= \left[\frac{1}{5} tan \ 5(3)\right] - \left[\frac{1}{5} tan \ 5(1)\right]$$

Step 4 : Solve/Simplify

$$= 0 - [-0.6761]$$
$$= 0.6761$$

REMEMBER

Use Mode Radian in calculator

Example 7

Evaluate the definite integral $\int_1^{\pi} cos \ (2x + 1) \ dx$

Solution :

Step 1 : Integrate → refer to the formula Integrals of Trigonometric Function

$$= \left[\frac{1}{2} sin \ (2x + 1)\right]_1^{\pi}$$

Step 2 : Substitute limit where the upper value minus the lower value

$$= \left[\frac{1}{2} sin \ (2(\pi) + 1)\right] - \left[\frac{1}{2} sin \ (2(1) + 1)\right]$$

Step 4 : Solve/Simplify

$$= 0.4207 - [0.0706]$$
$$= 0.3501$$

Tips

For definite trigonometry function, the calculator mode should be changes to radian mode if the limit used in radian units and vice versa

Exercise

Find each of the following indefinite and definite integrals for trigonometry function

a) $\int (2\sin 3x + sec^2 4x)\ dx$

b) $\int 9\,cos(10x - 3)\ dx$

c) $\int 2\,sin^2 x\,cos\,x\,dx$

d) $\int \frac{2}{cos^2 2x}\,dx$

e) $\int (2\theta + 2)\,sec^2(\theta^2 + 2\theta)\,d\theta$

f) $\int cos^3 2x\,dx$

g) $\int_{\frac{\pi}{2}}^{\pi} \cos(3x - 2)\,dx$

h) $\int_{20^0}^{60^0} \sin\frac{x}{3}\,dx$

i) $\int_{1}^{\pi} \sin(2x + 1)\,dx$

j) $\int_{50^0}^{90^0} 2\,sec^2 4x\,dx$

ANSWER

a) $-\frac{2\cos 3x}{3} + \frac{\tan 4x}{4} + c$ b) $\frac{9\sin(10x-3)}{10} + c$ c) $\frac{2}{3} sin^3 x + c$ d) $\tan 2x + c$ e) $\frac{\sin 2x}{2} - \frac{\sin(2x)^3}{6} + c$

f) $\tan(\theta^2 + 2\theta) + c$ g) 0.164 h) 0.16 i) -0.765 j) -0.182

RECIPROCAL FUNCTION

Reciprocal integral refer to the integral of multiplicative inverse of a variable which equals to sum of natural logarithm of variable and constant of integration. The multiplicative inverse or reciprocal of variable x is expressed as $\frac{1}{x}$ in mathematical form.

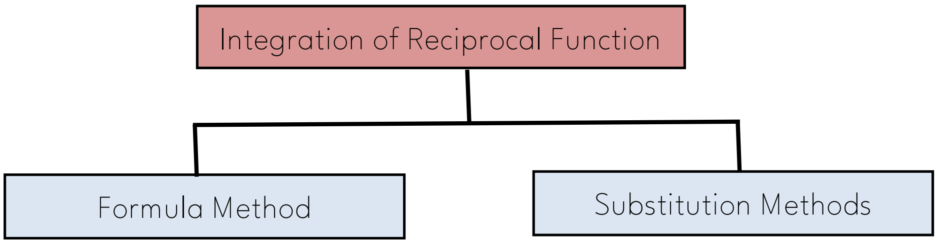


Figure 1 : Methods To Solve Integration of Reciprocal Function

Let's us scrutinize at these **Integral of Reciprocal Function by Formula Methods**:

a. $\int \frac{1}{x}\, dx = \ln|x| + c$

b. $\int \frac{1}{ax+b}\, dx = \frac{1}{a} \ln|ax+b| + c$

REMEMBER

How to Integrate by using substitution method:

1. Identify 'u'
2. Find du/dx , then make dx as subject
3. Substitute 'u' and dx into the original question
4. Simplify, then integrate
5. Substitute back u=$g(x)$ into the final answer

Example 1

Find each of the following indefinite and definite reciprocal function

a) $\int \frac{1}{2x} dx$

b) $\int \frac{1}{4-x} dx$

c) $\int \frac{2}{(2x+5)} dx$

d) $\int_1^3 \frac{5}{(2x-1)} dx$

Solution :

Step 1 : Integrate → refer to the formula Integrals of Reciprocal Function

a) $\int \frac{1}{2x} dx$

$= \frac{1}{2} \int \frac{1}{x} dx$ ← Put the coefficient outside $\int$

$= \frac{1}{2} ln\ x + c$

b) $\int \frac{1}{4-x} dx$

$= -\frac{1}{1} \cdot ln\ |4-x| + c$

$= -ln\ |4-x| + c$

c) $\int \frac{2}{(2x+5)} dx$

$= 2 \int \frac{1}{(2x+5)} dx$

$= 2 \left[\frac{1}{2} \cdot ln\ |2x+5| \right] + c$

$= ln\ |2x+5| + c$

d) $\int_1^3 \frac{5}{(2x-1)} dx$

$= 5 \int_1^3 \frac{1}{(2x-1)} dx$

$= 5 \cdot \frac{1}{2} [ln(2x-1)]_1^3$

$= \frac{5}{2} [ln(2(3)-1) - ln(2(1)-1)]$

$= 4.024$

REMEMBER

$\int \frac{1}{ax+b} dx = \frac{1}{a} ln(ax+b) + c,$

where a is from $\frac{d}{dx}(ax+b)$

Example 2

Solve the integral $\int \frac{5x}{(x^2+3)} dx$

Solution :

METHOD: By using **Substitution Method**

Step 1 : Identify 'u' → must be the denominator & substitute $u = g(x)$

$$u = x^2 + 3$$

Step 2 : Find $\frac{du}{dx}$ → differentiate u, then find dx

$$\frac{du}{dx} = 2x$$

$$dx = \frac{du}{2x}$$

Step 3 : Substitute $g(x) = u$ and $dx = \frac{du}{g'(x)}$ into the original question

$$= \int \frac{5x}{u} \frac{du}{2x}$$

$$= \frac{5}{2} \int \frac{1}{u} du$$

Step 4 : Integrate

$$= \frac{5}{2}[ln\, u] + c$$

Step 5 : Substitute back $u = g(x)$ into the final answer

$$= \frac{5}{2}[ln(x^2 + 3)] + c$$

Example 3

Solve the integral $\int tan\ 2x\ dx$

> **NOTES**
> CANNOT integrate
> $tan\ x$

Solution :

METHOD: By using **Substitution Method**

Step 1 : Refer to Reciprocal Identities

$$\int tan\ 2x\ dx$$

$$= \int \frac{sin\ 2x}{cos\ 2x}\ dx$$

change form

Step 2 : Identify 'u' → must be the denominator & substitute $u = g(x)$

$$u = cos\ 2x$$

Step 2 : Find $\frac{du}{dx}$ → differentiate u, then find dx

$$\frac{du}{dx} = -2\ sin\ 2x$$

$$dx = \frac{du}{-2\ sin\ 2x}$$

Step 3 : Substitute $g(x) = u$ and $dx = \frac{du}{g\prime(x)}$ into the original question

$$= \int \frac{sin\ 2x}{u} \frac{du}{-2\ sin\ 2x}$$

$$= -\frac{1}{2}\int \frac{1}{u}\ du$$

Step 4 : Integrate

$$= -\frac{1}{2}[ln\ u] + c$$

Step 5 : Substitute back $u = g(x)$ into the final answer

$$= -\frac{1}{2}[ln(cos\ 2x\ )] + c$$

Example 4

Solve the integral $\int_0^1 \frac{8x}{(2x^2+3)} dx$

Solution :

METHOD: By using **Substitution Method**

Step 1 : Identify 'u' → must be the denominator & substitute $u = g(x)$

$$u = 2x^2 + 3$$

Step 2 : Find $\frac{du}{dx}$ → differentiate u, then find dx

$$\frac{du}{dx} = 4x$$

$$dx = \frac{du}{4x}$$

Step 3 : Substitute $g(x) = u$ and $dx = \frac{du}{g\prime(x)}$ into the original question

$$= \int_0^1 \frac{8x}{u} \cdot \frac{du}{4x}$$

$$= 2\int_0^1 \frac{1}{u} du$$

Step 4 : Integrate

$$= 2\,[ln\, u]_0^1$$

Step 5 : Substitute back $u = g(x)$ into the final answer

$$= 2\,[ln\,(2x^2 + 3)]_0^1$$

Step 6 : Substitute limit where the upper value minus the lower value

$$= 2[ln(2(1)^2 + 3) - ln(2(0)^2 + 3)]$$

Step 7 : Solve/Simplify

$$= 2(1.6094 - 1.0986)$$

$$= 1.0216$$

EXPONENTIAL FUNCTION

An exponential function is a Mathematical function in the form $f(x) = a^x$, where "x" is a variable and "a" is a constant which is called the base of the function and it should be greater than 0. The most commonly used exponential function base is the transcendental number *e*, which is approximately equal to 2.71828

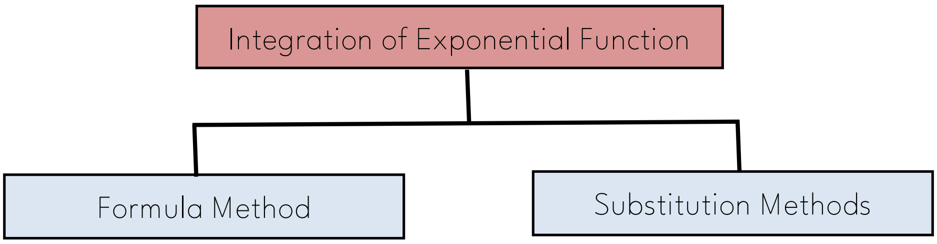


Figure 2 : Methods To Solve Integration of Exponential Function

Let's us scrutinize at these

Integral of Exponential Function:

Formula Method

a. $\int e^x \, dx = e^x + c$

b. $\int e^{ax+b} \, dx = \frac{1}{a} e^{ax+b} + c$

How to Integrate by using

Substitution Method:

1. Identify 'u'
2. Find du/dx , then make dx as subject
3. Substitute 'u' and dx into the original question
4. Simplify, then integrate
5. Substitute back $u=g(x)$ into the final answer

Important

Remember the Law of Exponential...

a) $e^m \times e^n = e^{m+n}$

b) $e^m \div e^n = e^{m-n}$

c) $(e^m)^n = e^{mn}$

d) $m^{-n} = \frac{1}{m^n}$

e) $\sqrt[n]{e^m} = e^{\frac{m}{n}}$

f) $e^0 = 1$

Example 1

Find each of the following indefinite and indefinite exponential function

a) $\int e^{-7x}\ dx$

b) $\int e^{x}(1-e^{x})\ dx$

c) $\int 5e^{2x-3}+\frac{1}{e^{4x}}\ dx$

d) $\int \frac{3e^{4x}-e^{x}}{e^{3x}}\ dx$

e) $\int_{1}^{2} 2e^{4x-1}\, dx$

Solution :

Step 1 : Integrate → refer to the formula Integrals of Exponential Function

a) $\int e^{-7x}\ dx$

$= \frac{1}{-7}\cdot e^{-7x}+c$

$= \frac{e^{-7x}}{-7}+c$

b) $\int e^{x}(1-e^{x})\ dx$ (expand)

$= \int e^{x}-e^{2x}\ dx$

$= e^{x}-\frac{1}{2}e^{2x}+c$

c) $\int 5e^{2x-3}+\frac{1}{e^{4x}}\ dx$ (change form)

$= \int 5e^{2x-3}+e^{-4x}dx$

$= 5\cdot\frac{1}{2}e^{2x-3}+\left(\frac{1}{-4}\right)e^{-4x}+c$

$= \frac{5}{2}e^{2x-3}-\frac{1}{4e^{4x}}+c$

d) $\int \frac{3e^{4x}-e^{x}}{e^{3x}}\ dx$ (separate)

$= \int \frac{3e^{4x}}{e^{3x}}-\frac{e^{x}}{e^{3x}}dx$

$= \int 3e^{x}-e^{-2x}dx$

$= 3\cdot e^{x}-\left(\frac{1}{-2}\right)e^{-2x}+c$

$= 3e^{x}+\frac{1}{2e^{2x}}+c$

e) $\int_{1}^{2} 2e^{4x-1}\, dx$

$= 2\int_{1}^{2} e^{4x-1}\, dx$

$= 2\left[\frac{1}{4}e^{4x-1}\right]_{1}^{2}$

$= 2\left[\frac{1}{4}e^{4(2)-1}-\frac{1}{4}e^{4(1)-1}\right]_{1}^{2}$

$= 538.27$

Example 2

Solve the integral $\int e^x(1+2e^x)^{-2}\ dx$

Solution :

METHOD: By using **Substitution Method**

Step 1 : Identify 'u' → must be in the bracket & substitute $u = g(x)$

$$u = 1 + 2e^x$$

Step 2 : Find $\frac{du}{dx}$ → differentiate u, then find dx

$$\frac{du}{dx} = 2e^x$$

$$dx = \frac{du}{2e^x}$$

Step 3 : Substitute $g(x) = u$ and $dx = \frac{du}{g'(x)}$ into the original question

$$= \int e^x \cdot u^{-2}\ \frac{du}{2e^x}$$

$$= \frac{1}{2}\int u^{-2}\ du$$

Step 4 : Integrate

$$= \frac{1}{2} \cdot \frac{u^{-1}}{-1} + c$$

$$= -\frac{1}{2u} + c$$

Step 5 : Substitute back $u = g(x)$ into the final answer

$$= -\frac{1}{2(1+2e^x)} + c$$

Example 3

Solve the integral $\int_0^1 x^2 e^{1-x^3}\, dx$

Solution :

METHOD: By using **Substitution Method**

Step 1 : Identify 'u' → must be power of exponent & substitute $u = g(x)$

$u = 1 - x^3$

Step 2 : Find $\frac{du}{dx}$ → differentiate u, then find dx

$$\frac{du}{dx} = -3x^2$$
$$dx = \frac{du}{-3x^2}$$

Step 3 : Substitute $g(x) = u$ and $dx = \frac{du}{g'(x)}$ into the original question

$$= \int_0^1 x^2\, e^u \frac{du}{-3x^2}$$

$$= \frac{1}{-3}\int_0^1 e^u\, du$$

Step 4 : Integrate

$$= \frac{1}{-3}[e^u]_0^1$$

Step 5 : Substitute back $u = g(x)$ into the final answer

$$= \frac{1}{-3}\left[e^{1-x^3}\right]_0^1$$

Step 6 : Substitute limit where the upper value minus the lower value

$$= \frac{1}{-3}\left[e^{1-(1)^3} - e^{1-(0)^3}\right]$$

Step 7 : Solve/Simplify

$$= \frac{1}{-3}(1 - 2.7183)$$

$$= 0.5728$$

Example 4

Solve the integral $\int_0^2 e^{2x}(\sin e^{2x})\, dx$

Solution :

METHOD: By using **Substitution Method**

Step 1 : Identify 'u' & substitute $u = g(x)$

$$u = e^{2x}$$

Step 2 : Find $\frac{du}{dx}$ → differentiate u, then find dx

$$\frac{du}{dx} = 2e^{2x}$$

$$dx = \frac{du}{2e^{2x}}$$

Step 3 : Substitute $g(x) = u$ and $dx = \frac{du}{g'(x)}$ into the original question

$$= \int_0^2 e^{2x}(\sin u)\frac{du}{2e^{2x}}$$

$$= \frac{1}{2}\int_0^2 \sin u \, du$$

Step 4 : Integrate

$$= \frac{1}{2}[-\cos u]_0^2$$

Step 5 : Substitute back $u = g(x)$ into the final answer

$$= -\frac{1}{2}[\cos e^{2x}]_0^2$$

Step 6 : Substitute limit where the upper value minus the lower value

$$= -\frac{1}{2}\left[\cos e^{2(2)} - \cos e^{2(0)}\right]$$

> **NOTES**
> Use mode radian for trigonometry (calculator)

Step 7 : Solve/Simplify

$$= -\frac{1}{2}(-0.371 - 0.540)$$

$$= 0.4555$$

Exercise

Find each of the following indefinite and definite integrals for reciprocal and exponential function

a) $\int \frac{5z}{10z^2}\, dz$

b) $\int \frac{-3}{9t+3}\, dt$

c) $\int \frac{1}{2(1-3x)}\, dx$

d) $\int \frac{2x-1}{2x^2-2x+5}\, dx$

e) $\int \frac{\sin 2x}{\cos 2x}\, dx$

f) $\int 8\cot(2x)\, dx$

g) $\int_2^4 \left(\frac{1}{2x+4}\right) dx$

h) $\int_0^1 \left(\frac{2}{3x+2}\right) dx$

i) $\int \frac{1}{2e^{\frac{x}{2}}}\, dx$

j) $\int \frac{e^x}{\sqrt{1+e^x}}\, dx$

k) $\int z e^{z^2} dz$

l) $\int e^{2x}\left(\frac{e^{2x}}{4}+5\right)^3 dx$

m) $\int e^{\cos x} \sin x\ dx$

n) $\int \frac{e^{3x}}{e^{3x}+4}\, dx$

o) $\int_1^3 e^{2x+1} dx$

p) $\int_0^1 36x^2\, e^{(2x^3+3)}\, dx$

q) $\int_0^1 e^{5x} \cos e^{5x}\, dx$

r) $\int_1^2 9 + e^{2x} - \frac{5}{x^3}\, dx$

ANSWER

a) $\frac{1}{2}\ln|z| + c$

b) $-\frac{1}{3}\ln|9t+3| + c$

c) $-\frac{1}{6}\ln|1-3x| + c$

d) $\frac{1}{2}\ln|2x^2-2x+5| + c$

e) $-\frac{1}{2}\ln|\cos 2x| + c$

f) $4\ln|\sin 2x| + c$

g) 0.2027

h) 0.610

i) $-\frac{1}{e^{\frac{x}{2}}} + c$

j) $2\sqrt{1+e^x} + c$

k) $\frac{e^{z^2}}{2} + c$

l) $\frac{\left[\frac{e^{2x}}{4}+5\right]^4}{2} + c$

m) $-e^{\cos x} + c$

n) $\frac{1}{3}\ln|e^{3x}+4| + c$

o) 538.27

p) 769.965

q) -0.305

r) 30.729

TOPIC

Techniques of Integration

At the end of this chapter, you should be able to:

- Solve integration by parts
- Solve integration of partial fraction

INTEGRATION BY PARTS

Integration by Parts is used to integrate the product of two or more functions with one that can be easily integrated, and one that can be easily differentiated. Sometimes, we have to use Integration by Parts twice.

Guidelines for choosing 'u'

This can be remembered using the rule **LIATE** where the following function comes first in the following order is assume it as u.

- L - Logarithmic, example $\ln 5x, \log 3x$ etc
- I - Inverse Trigonometric, example $\sin^{-1} 4x$ etc
- A - Algebraic, example $4x^2, 2x + 7$ etc
- T - Trigonometric, example $\sin 6x$ etc
- E - Exponential, example e^{3x} etc

Table 1 : How To Identify 'u' and 'dv'

Functions	HOW TO IDENTIFY 'u' AND 'dv'	
	u	dv
$\int x \cos x \, dx$	x	$\cos x \, dx$
$\int x^2 \sin 2x \, dx$	x^2	$\sin 2x \, dx$
$\int x^2 \ln x \, dx$	$\ln x$	$x^2 \, dx$
$\int 2x e^{3x} \, dx$	$2x$	$e^{3x} \, dx$

NOTES

For a combination of trigonometry and exponential, any can be selected as 'u'

INTEGRATION BY PARTS

Apply the Method of Integration by Parts to Solve Related Problem Involving Two Functions

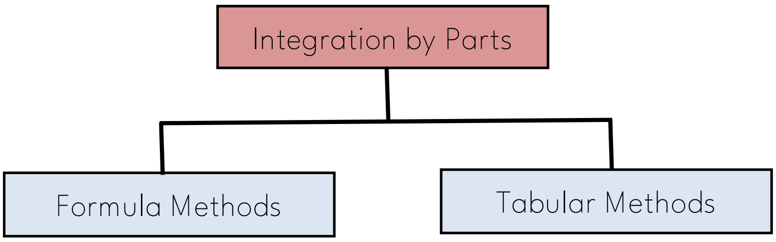


Figure 3 : Methods To Solve Integration by Parts

a) Integration by Parts Formula Method

For an integral, $\int f(x)dx$ can be written as $u\,dv$, then

$$\int u\,dv = uv - \int v\,du$$

** The key to Integration by Parts is to identify the part of the integrand as "u" and the rest part as "dv".

How to Integrate by using formula method ?

Step 1 : Identify 'u' (check the priority) and dv

Step 2 : Find $\frac{du}{dx}$, then make du as subject matter

Step 3 : From dv, find $v = \int dv$

Step 4 : Integrate by using the formula integration by part

b) Tabular Method for Integration by Parts

The tabular integration method can be applied to any function which is the product of two expressions, where one of the expressions can be differentiated until it gets zero, and another expression can be integrated simultaneously multiple times

How to Integrate by using tabular method ?

Step 1 : Identify 'u' (check the priority) and dv

Step 2 : Fill in the table as below

Sign	u (Differentiate)	dv (Integrate)
+	$u = ?$	$dv = ?$
-	$\frac{du}{dx} = ?$	
+		
	Stop Here until we got ZERO	**STOP HERE**

NOTES

For Tabular Method, it is only suitable if the chosen 'u' can be zero if differentiated

Step 3 : Write the final answer according to the signs and arrows

COMMON MISTAKES

- ✓ the mistake of choosing 'u' according to priority
- ✓ using the differentiation method when doing integration

Example 1

Solve the integral $\int xsin3x\, dx$

Solution :

METHOD 1 : By using **Integration by Part Formula**

Step 1 : Identify 'u' (check the priority) and dv

$u = x$ and $dv = sin\ 3x\ dx$

Step 2 : Find $\frac{du}{dx}$, then find du

$$u = x$$

$$\frac{du}{dx} = 1$$

$$du = dx$$

Step 3 : From dv, find $v = \int dv$

$$dv = sin\ 3x\ dx$$

$$v = \int sin\ 3x dx$$

$$= -\frac{cos\ 3x}{3}$$

Step 4 : Integrate by using the formula integration by part

$$\int u\, dv = uv - \int v\, du$$

$$\int x\ sin3x\ dx = x\left(-\frac{cos\ 3x}{3}\right) - \int -\frac{cos\ 3x}{3} dx$$

$$= x\left(-\frac{cos\ 3x}{3}\right) + \frac{1}{3}\int cos\ 3x\ dx$$

$$= -\frac{x\ cos\ 3x}{3} + \frac{sin\ 3x}{9} + c$$

Example 1

METHOD 2 : By using **Tabular Method**

Step 1 : Identify 'u' (check the priority) and dv

$$u = x \text{ and } dv = \sin 3x\, dx$$

Step 2 : Fill in the table as below

Sign	u (Differentiate)	dv (Integrate)
+	$u = x$	$dv = \sin 3x dx$
-	1	$-\frac{\cos 3x}{3}$
+	0	$-\frac{\sin 3x}{9}$
	Stop Here until we got ZERO	**STOP HERE**

Step 3 : write the final answer according to the signs and arrows

$$\int x \sin 3x\, dx = x\left(-\frac{\cos 3x}{3}\right) - \left(-\frac{\sin 3x}{9}\right) + c$$

$$= -\frac{x \cos 3x}{3} + \frac{\sin 3x}{9} + c$$

WARNING

Be careful with positive and negative signs

Example 2

Solve the integral $\int 6x^2 e^{4x-3} dx$

Solution :

METHOD 1 : By using **Integration by Part Formula**

Step 1 : Identify 'u' (check the priority) and dv

$$u = 6x^2 \quad \text{and} \quad dv = e^{4x-3}$$

Step 2 : Find $\frac{du}{dx}$, then find du

$$\frac{du}{dx} = 12x$$

$$du = 12x\,dx$$

Step 3 : From dv, find $v = \int dv$

$$dv = e^{4x-3}$$

$$v = \int e^{4x-3}$$

$$= \frac{e^{4x-3}}{4}$$

Step 4 : Integrate by using the formula integration by part

$$\int u dv = uv - \int v du$$

$$\int 6x^2\, e^{4x-3} dx = 6x^2 \left(\frac{e^{4x-3}}{4}\right) - \int \frac{e^{4x-3}}{4} (12x\, dx)$$

$$= \frac{3x^2\, e^{4x-3}}{2} - \int 3x\, e^{4x-3} dx$$

Integrate By Part again (Repeat step 1 – 4)

Example 2

Step 5 : Repeat step 1 – 4 to integrate $\int 3x\, e^{4x-3} dx$

$$u = 3x \qquad dv = e^{4x-3}$$

$$\frac{du}{dx} = 3 \qquad v = \int e^{4x-3}$$

$$du = 3dx \qquad = \frac{e^{4x-3}}{4}$$

$$\int 6x^2 e^{4x-3} dx = \frac{3x^2 e^{4x-3}}{2} - \left(3x\left(\frac{e^{4x-3}}{4}\right) - \int \frac{e^{4x-3}}{4} \cdot 3dx\right)$$

$\int udv = uv - \int vdu$

Copy from step 4

$$= \frac{3x^2 e^{4x-3}}{2} - \frac{3}{4} x e^{4x-3} + \frac{3}{4} \int e^{4x-3}\, dx$$

$$= \frac{3x^2 e^{4x-3}}{2} - \frac{3}{4} x e^{4x-3} + \frac{3}{4}\left(\frac{1}{4} e^{4x-3}\right) + c$$

$$= \frac{3x^2 e^{4x-3}}{2} - \frac{3}{4} x e^{4x-3} + \frac{3}{16} e^{4x-3} + c$$

$$= \frac{3e^{4x-3}}{2}\left(x^2 - \frac{x}{2} + \frac{1}{8}\right) + c$$

NOTES

$\int 6x^2 e^{4x-3} dx \rightarrow$ Because x^2, therefore we need to integrate by part 2 times & using formula (the last integration)

Example 2

METHOD 2 : By using **Tabular Method**

Step 1 : Identify 'u' (check the priority) and dv

$$u = 6x^2 \text{ and } dv = e^{4x-3}dx$$

Step 2 : Fill in the table as below

Sign	u (Differentiate)	dv (Integrate)
+	$u = 6x^2$	$dv = e^{4x-3}dx$
-	$12x$	$\frac{e^{4x-3}}{4}$
+	12	$\frac{e^{4x-3}}{16}$
-	0	$\frac{e^{4x-3}}{64}$
	Stop Here until we got ZERO	**STOP HERE**

Step 3 : write the final answer according to the signs and arrows

$$\int 6x^2\, e^{4x-3}dx = 6x^2\left(\frac{e^{4x-3}}{4}\right) - 12x\left(\frac{e^{4x-3}}{16}\right) + 12\left(\frac{e^{4x-3}}{64}\right) + c$$

$$= \frac{3x^2e^{4x-3}}{2} - \frac{3xe^{4x-3}}{4} + \frac{3e^{4x-3}}{16} + c$$

$$= \frac{3e^{4x-3}}{2}\left(x^2 - \frac{x}{2} + \frac{1}{8}\right) + c$$

Example 3

Solve the integral $\int lnx^2dx$

Solution :

By using **Integration by Part Formula**

Step 1 : Identify 'u' (check the priority) and dv

$u = ln\,x^2$ and $dv = 1dx$

Step 2 : Find $\frac{du}{dx}$, then find du

$$u = ln\,x^2$$
$$\frac{du}{dx} = \frac{2}{x^2}(2x)$$
$$= \frac{2}{x}$$
$$du = \frac{2}{x}dx$$

Step 3 : From dv, find $v = \int dv$

$$dv = 1dx$$
$$dv = \int 1dx$$
$$= x$$

Step 4 : Integrate by using the formula integration by part

$$\int u\,dv = uv - \int v\,du$$

$$\int lnx^2\,dx = lnx^2.x - \int x \cdot \frac{2}{x}dx$$
$$= x\,lnx^2 - \int 2dx$$
$$= x\,lnx^2 - 2x + c$$

Example 4

Solve the integral $\int e^{x} \sin 5x\, dx$

Solution :

By using **Integration by Part Formula**

Step 1 : Identify 'u' (check the priority) and dv

$$u = \sin 5x \quad \text{and} \quad dv = e^{x} dx$$

Step 2 : Find $\frac{du}{dx}$, then find du

$$u = \sin 5x$$

$$\frac{du}{dx} = 5 \cos 5x$$

$$du = 5 \cos 5x \; dx$$

Step 3 : From dv, find $v = \int dv$

$$dv = e^{x} dx$$

$$v = \int e^{x} dx$$

$$= e^{x}$$

Step 4 : Integrate by using the formula integration by part

$$\int u dv = uv - \int v du$$

$$\int e^{x} \sin 5x\, dx = \sin 5x . e^{x} - \int e^{x} . 5 \cos 5x dx$$

$$= e^{x} \sin 5x - \int 5e^{x} \cos 5x dx$$

Integrate By Part again (Repeat step 1 – 4)

Example 4

Step 5 : Repeat step 1 – 4 to integrate $\int 5e^{x} cos5xdx$

$u = cos5x \qquad\qquad dv = e^{x}$

$\frac{du}{dx} = -5\, sin5x \qquad\qquad v = \int e^{x}dx$

$du = -5\, sin\, 5x\, dx \qquad\qquad = e^{x}$

$$\int e^{x}\, sin\, 5x\, dx = e^{x}sin5x - \left(cos5x \cdot e^{x} - \int e^{x} \cdot -5sin5xdx\right)$$

Copy from step 4

$\int udv = uv - \int vdu$

$$= e^{x}sin5x - e^{x}cos5x - 5\int e^{x}sin5xdx$$

We notify that $e^{x}sin5x\, dx$ is same with the question given. In this case we need to assume $I = \int e^{x}\, sin\, 5x\, dx$ for easier calculation.

$$I = e^{x}sin5x - e^{x}cos5x - 5I$$

$$6I = e^{x}\, cos5\, x - e^{x}sin5x$$

$$I = \frac{e^{x}\, cos5\, x - e^{x}sin5x}{6} + c$$

Exercise

Find each of the followings integrals:

a) $\int 2xsin3xdx$

b) $\int x^2cos\ xdx$

c) $\int 4xe^{7x-2}dx$

d) $\int ln|4x|dx$

e) $\int \frac{3}{x^2}ln|x|dx$

f) $\int 5xsec^2 2xdx$

g) $\int e^{2x}cos\ xdx$

h) $\int_1^2 xln|x^2|dx$

i) $\int_0^2 2xcos4xdx$

j) $\int_1^3 \frac{4x}{e^{3x}}dx$

Answer

a) $\frac{2}{3}\left[-xcos3x + \frac{1}{3}sin3x + c\right]$

b) $x^2sinx + 2xcosx - 2sinx + c$

c) $\frac{4e^{7x-2}}{49}[7x - 1] + c$

d) $x[ln|4x| - 1] + c$

e) $-\frac{3}{x}[ln|x| + 1] + c$

f) $\frac{5}{2}\left[xtan(2x) + \frac{1}{2}ln|\cos(2x)|\right] + c$

g) $\frac{2e^{2x}}{5}\left[cos(x) + \frac{1}{2}sinx\right] + c$

h) 1.2726

i) 0.8462

j) 0.0880

INTEGRATION OF PARTIAL FRACTION

Integration by partial fractions is a method used to decompose and then integrate a rational fraction integrand that has complex terms in the denominator. The steps needed to decompose an algebraic fraction into its partial fractions result from a consideration of the reverse process – addition (or subtraction).

The list of formulas used to decompose the given proper rational functions is given below.

No.	Denominator containing...	Expression	Form of Partial Fraction
1.	Linear Factor	$\frac{f(x)}{(x+a)(x+b)}$	$\frac{A}{x+a}+\frac{B}{x+b}$
2.	Repeated linear factors	$\frac{f(x)}{(x+a)^n}$	$\frac{A}{x+a}+\frac{B}{(x+a)^2}+\ldots+\frac{}{(x+a)^n}$

How to Integrate by Partial Fractions ?

Step 1 : Factorize the polynomials in the denominator (if needed)

Step 2 : Decompose the integrand using a suitable expression

Step 3 : Divide the integration into parts and integrate the individual functions

COMMON MISTAKES

- ✓ do not factor (if needed)
- ✓ just do decompose of partial fraction without integrate

Example 1

Solve each of the integral below:

a) $\dfrac{7x+3}{(x+5)(x-3)}$

b) $\displaystyle\int \frac{9x-1}{4x^2-11x-3}\,dx$

Solution :

a) $\dfrac{7x+3}{(x+5)(x-3)}$

Step 1 : Decompose the integrand using a suitable expression

$$\frac{7x+3}{(x+5)(x-3)} = \frac{A}{(x+5)} + \frac{B}{(x-3)}$$

Multiply by denominator $(x+5)(x-3)$

$$\left[\frac{7x+3}{(x+5)(x-3)} = \frac{A}{(x+5)} + \frac{B}{(x-3)}\right]\mathbf{(x+5)(x-3)}$$

$$\frac{(7x+3)\mathbf{(x+5)(x-3)}}{(x+5)(x-3)} = \frac{A\mathbf{(x+5)(x-3)}}{(x+5)} + \frac{B\mathbf{(x+5)(x-3)}}{(x-3)}$$

Simplify

Then we get

$$7x+3 = A(x-3) + B(x+5)$$

Let $x+5=0 \Rightarrow x=-5$, then constant of A can be determined.

$$7(-5)+3 = A(-5-3) + B(0)$$
$$-32 = -8A$$
$$A = 4$$

Let $x-3=0 \Rightarrow x=3$, then constant of B can be determined.

$$7(3)+3 = A(0) + B(3+5)$$
$$24 = 8B$$
$$B = 3$$

Example 1

Therefore,

$$\frac{7x+3}{(x+5)(x-3)} = \frac{4}{(x+5)} + \frac{3}{(x-3)}$$

Step 2 : Divide the integration into parts and integrate the individual functions

$$\int \frac{7x+3}{(x+5)(x-3)}dx = \int \frac{4}{x+5}dx + \int \frac{3}{x-3}dx$$

$$= 4ln|x+5| + 3ln|x-3| + c$$

b) $\int \frac{9x-1}{4x^2-11x-3}dx$

Solution :

Step 1 : Factorize the polynomials in the denominator

$$\int \frac{9x-1}{4x^2-11x-3}dx = \int \frac{9x-1}{(4x+1)(x-3)}dx$$

Step 2 : Decompose the integrand using a suitable expression

$$\frac{9x-1}{(4x+1)(x-3)} = \frac{A}{(4x+1)} + \frac{B}{(x-3)}$$

Multiply by denominator $(4x+1)(x-3)$

$$\left[\frac{9x-1}{(4x+1)(x-3)} = \frac{A}{(4x+1)} + \frac{B}{(x-3)}\right]\mathbf{(4x+1)(x-3)}$$

$$\frac{(9x-1)\mathbf{(4x+1)(x-3)}}{(4x+1)(x-3)} = \frac{A\mathbf{(4x+1)(x-3)}}{(4x+1)} + \frac{B\mathbf{(4x+1)(x-3)}}{(x-3)}$$

Simplify

Example 1

Then we get

$$9x - 1 = A(x-3) + B(4x+1)$$

Let $4x + 1 = 0 \Rightarrow x = -\frac{1}{4}$, then constant of A can be determined.

$$9\left(-\frac{1}{4}\right) - 1 = A\left(-\frac{1}{4} - 3\right) + B(0)$$

$$-\frac{13}{4} = -\frac{13}{4}A$$

$$A = 1$$

Let $x - 3 = 0 \Rightarrow x = 3$, then constant of B can be determined.

$$9(3) - 1 = A(0) + B(4(3) + 1)$$

$$26 = 13B$$

$$B = 2$$

Therefore,

$$\frac{9x-1}{(4x+1)(x-3)} = \frac{1}{(4x+1)} + \frac{2}{(x-3)}$$

Step 3 : Divide the integration into parts and integrate the individual functions

$$\int \frac{9x-1}{4x^2 - 11x - 3}dx = \int \frac{1}{4x+1}dx + \int \frac{2}{x-3}dx$$

$$= \frac{1}{4}ln|4x+1| + 2ln|x-3| + c$$

Example 2

Solve each of the integral below: (Repeated Linear Factor)

a) $\dfrac{x^2+10x+25}{(x+4)^3}$ b) $\displaystyle\int \frac{7x^3+7x^2-16x+16}{x^2(x^2-16)}dx$

Solution :

a) $\dfrac{x^2+10x+25}{(x+4)^3}$

Step 1 : Decompose the integrand using a suitable expression

$$\frac{x^2+10x+25}{(x+4)^3}=\frac{A}{(x+4)}+\frac{B}{(x+4)^2}+\frac{C}{(x+4)^3}$$

Multiply by denominator $(x+4)^3$

$$\left[\frac{x^2+10x+25}{(x+4)^3}=\frac{A}{(x+4)}+\frac{B}{(x+4)^2}+\frac{C}{(x+4)^3}\right](\boldsymbol{x+4})^3$$

$$\frac{(x^2+10x+25)(\boldsymbol{x+4})^3}{(x+4)^3}=\frac{A(\boldsymbol{x+4})^3}{(x+4)}+\frac{B(\boldsymbol{x+4})^3}{(x+4)^2}+\frac{C(\boldsymbol{x+4})^3}{(x+4)^3}$$

Simplify

Then we get

$$x^2+10x+25=A(x+4)^2+B(x+4)+C$$

Let $x+4=0 \Rightarrow x=-4$, then constant of C can be determined.

$$(-4)^2+10(-4)+25=A(0)+B(0)+C$$
$$C = 1$$

Expand : $x^2+10x+25=A(x+4)^2+B(x+4)+C$

$$x^2+10x+25=A(x^2+8x+16)+B(x+4)+C$$

Equating coeffiecient of x^2 to find A:

$$1=A$$

Example 2

Equating coeffiecient of x or constant to find B:

Coefficient x,

$$10 = 8A + B$$

$$10 = 8(1) + B$$

$$B = 2$$

OR

Constant,

$$25 = 16A + 4B + C$$

$$25 = 16(1) + 4B + 1$$

$$B = 2$$

Therefore,

$$\frac{x^2 + 10x + 25}{(x+4)^3} = \frac{1}{(x+4)} + \frac{2}{(x+4)^2} + \frac{1}{(x+4)^3}$$

Step 2 : Divide the integration into parts and integrate the individual functions

$$\int \frac{x^2 + 10x + 25}{(x+4)^3} = \int \frac{1}{(x+4)} dx + \int \frac{2}{(x+4)^2} dx + \int \frac{1}{(x+4)^3} dx$$

$$= ln|x+4| + \int 2(x+4)^{-2} dx + \int (x+4)^{-3} dx$$

$$= ln|x+4| - \frac{2}{x+4} - \frac{1}{2(x+4)^2} + c$$

Composite function :

$$\int (ax+b)^n dx = \frac{(ax+b)^{n+1}}{a(n+1)} + c$$

Example 2

b) $\int \frac{7x^3 + 7x^2 - 16x + 16}{x^2(x^2 - 16)} dx$

Solution :

Step 1 : Change the denominator to linear form

$$\frac{7x^3 + 7x^2 - 16x + 16}{x^2(x^2 - 16)} = \frac{7x^3 + 7x^2 - 16x + 16}{x^2(x-4)(x+4)}$$

Step 2 : Decompose the integrand using a suitable expression

$$\frac{7x^3 + 7x^2 - 16x + 16}{x^2(x-4)(x+4)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{(x-4)} + \frac{D}{x+4}$$

Multiply by denominator $x^2(x-4)(x+4)$

$$\left[\frac{7x^3 + 7x^2 - 16x + 16}{x^2(x-4)(x+4)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{(x-4)} + \frac{D}{x+4}\right]\boldsymbol{x^2(x-4)(x+4)}$$

$$\frac{(7x^3 + 7x^2 - 16x + 16)\boldsymbol{x^2(x-4)(x+4)}}{x^2(x-4)(x+4)}$$

$$= \frac{A\boldsymbol{x^2(x-4)(x+4)}}{x} + \frac{B\boldsymbol{x^2(x-4)(x+4)}}{x^2} + \frac{C\boldsymbol{x^2(x-4)(x+4)}}{(x-4)}$$
$$+ \frac{D\boldsymbol{x^2(x-4)(x+4)}}{x+4}$$

Then we get

$$7x^3 + 7x^2 - 16x + 16$$
$$= Ax(x-4)(x+4) + B(x-4)(x+4) + Cx^2(x+4) + Dx^2(x-4)$$

Let $x = 0$, then constant of B can be determined

$$16 = B(0-4)(0+4)$$
$$16 = -16B$$
$$B = -1$$

Example 2

Let $x - 4 = 0 \Rightarrow x = 4$, then constant of C can be determined

$$7(4)^3 + 7(4)^2 - 16(4) + 16 = C(4)^2(4+4)$$
$$512 = C(128)$$
$$C = 4$$

Let $x + 4 = 0 \Rightarrow x = -4$, then constant of D can be determined

$$7(-4)^3 + 7(-4)^2 - 16(-4) + 16 = D(-4)^2(-4-4)$$
$$-256 = -128D$$
$$D = 2$$

Expand

$$7x^3 + 7x^2 - 16x + 16 = Ax(x-4)(x+4) + B(x-4)(x+4) + Cx^2(x+4) + Dx^2(x-4)$$
$$7x^3 + 7x^2 - 16x + 16 = Ax(x^2-16) + B(x^2-16) + Cx^2(x+4) + Dx^2(x-4)$$
$$7x^3 + 7x^2 - 16x + 16 = A(x^3-16x) + B(x^2-16) + C(x^3+4x^2) + D(x^3-4x^2)$$

Equating coeffiecient of x^3 or x to find A:

Coefficient x^3 ,

$$7 = A + C + D$$
$$7 = A + 4 + 2$$
$$A = 1$$

OR

Coefficient x

$$-16 = -16A$$
$$A = 1$$

Therefore,

$$\frac{7x^3 + 7x^2 - 16x + 16}{x^2(x-4)(x+4)} = \frac{1}{x} - \frac{1}{x^2} + \frac{4}{(x-4)} + \frac{2}{x+4}$$

Step 3 : Divide the integration into parts and integrate the individual functions

$$\int \frac{7x^3 + 7x^2 - 16x + 16}{x^2(x-4)(x+4)} dx = \int \frac{1}{x} dx - \int \frac{1}{x^2} dx + \int \frac{4}{(x-4)} dx + \int \frac{2}{x+4} dx$$
$$= ln|x| + \frac{1}{x} + 4\, ln|x-4| + 2\, ln|x+4| + c$$

Exercise

Integrate each of the followings:

a) $\displaystyle\int \frac{7x-2}{2x(x-2)}dx$

b) $\displaystyle\int \frac{5x+3}{(2x+1)(x+1)}dx$

c) $\displaystyle\int \frac{8x-6}{x^2-9}dx$

d) $\displaystyle\int \frac{-11}{2x^2+5x-12}dx$

e) $\displaystyle\int \frac{5x-7}{x^2+2x-15}dx$

f) $\displaystyle\int \frac{22x^2+16x-2}{3x(x+2)(x-1)}dx$

g) $\displaystyle\int \frac{2x^2+3x+2}{x^2(x+2)}dx$

h) $\displaystyle\int \frac{2x^2+10x-2}{(x+2)^2(x-5)}dx$

i) $\displaystyle\int \frac{3x^2+15x+14}{(x-1)(x+3)^2}dx$

j) $\displaystyle\int \frac{2x^3+4x^2+3x-2}{x^3(x+2)}dx$

k) $\displaystyle\int_1^5 \frac{5x+19}{(x+2)(x+5)}dx$

l) $\displaystyle\int_3^5 \frac{7x+1}{2x^2-3x-2}dx$

Answer

a) $\frac{1}{2}ln|x| + 3ln|x-2| + c$

b) $2\,ln|x+1| + \frac{1}{2}ln|2x+1| + c$

c) $3ln|x-3| + 5ln|x+3| + c$

d) $ln|x+4| - ln|2x-3| + c$

e) $4ln|x+5| + ln|x-3| + c$

f) $3ln|x+2| + 4ln|x-1| + \frac{1}{3}ln|x| + c$

g) $ln|x| - \frac{1}{x} + ln|x+2| + c$

h) $\frac{2}{x+5} + ln|x-5| + c$

i) $2ln|x-1| + ln|x+3| - \frac{1}{x+3} + c$

j) $ln|x| - \frac{2}{x} + \frac{2}{x^2} + ln|x+2| + c$

k) 3.564

l) 3.522

TOPIC

Application of Integrations

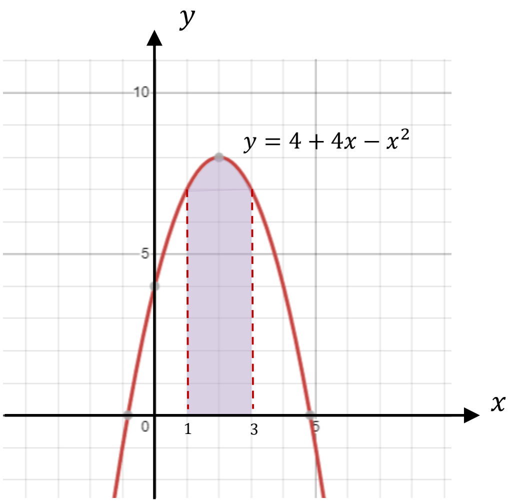
At the end of this chapter, you should be able to:

- Apply the techniques of integration

AREA

Area Under A Curve

x-axis

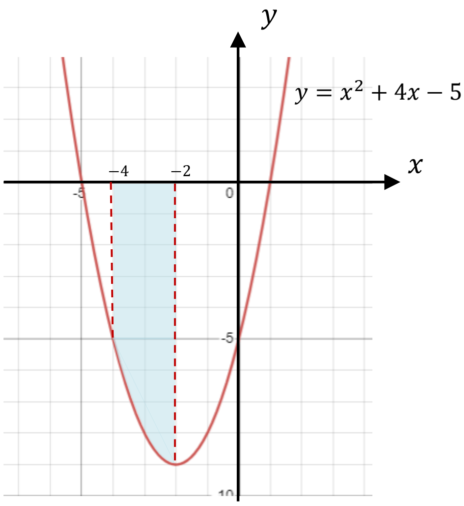


above x-axis

$$A = \int_a^b y \, dx$$

upper limit value

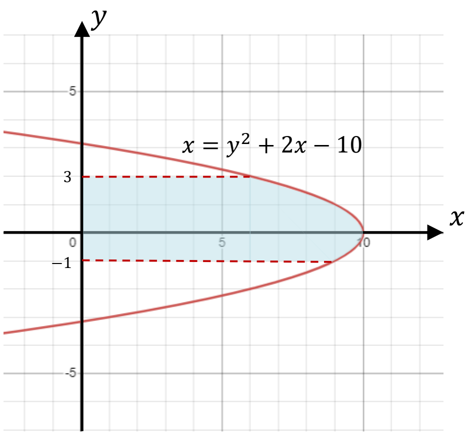
lower limit value



below x-axis

$$A = \left| \int_a^b y \, dx \right|$$

y-axis

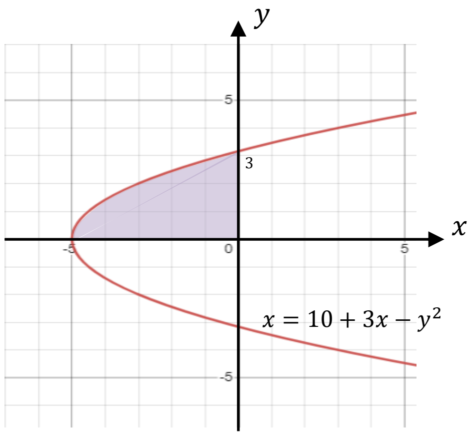


right y-axis

$$A = \int_a^b x \, dy$$

upper limit value

lower limit value

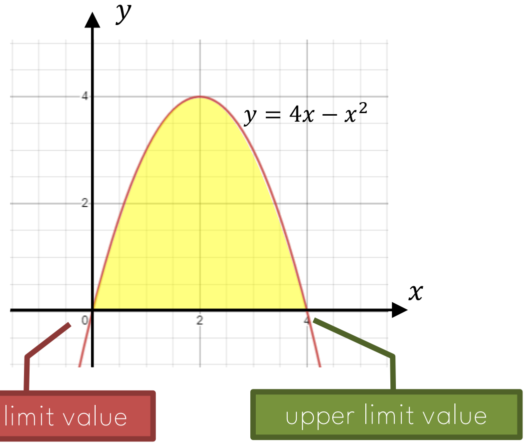


left y-axis

$$A = \left| \int_a^b x \, dy \right|$$

Example 1

Find the area of the region bounded by the curve along the x-axis.



Solution :

Step 1 : Substitute the equation and the limit into the formula

$$A = \int_0^4 4x - x^2 \, dx$$

Write the LIMIT
Write the equation y

Step 2 : Integrate

$$= \left[\frac{4x^2}{2} - \frac{x^3}{3}\right]_0^4$$

$$= \left[2x^2 - \frac{x^3}{3}\right]_0^4$$

Step 3 : Substitute value of the limit into the equation

$$= \left[2(4)^2 - \frac{(4)^3}{3}\right] - \left[2(0)^2 - \frac{(0)^3}{3}\right]$$

UPPER limit – LOWER limit

Step 4 : Solve

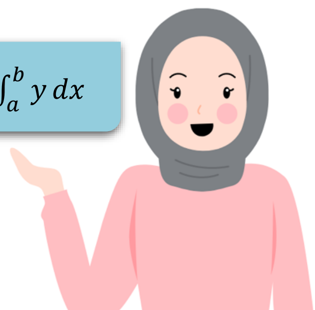
$$= \frac{32}{3} - 0$$

$$= \frac{32}{3}$$

Step 5 : Write the answer

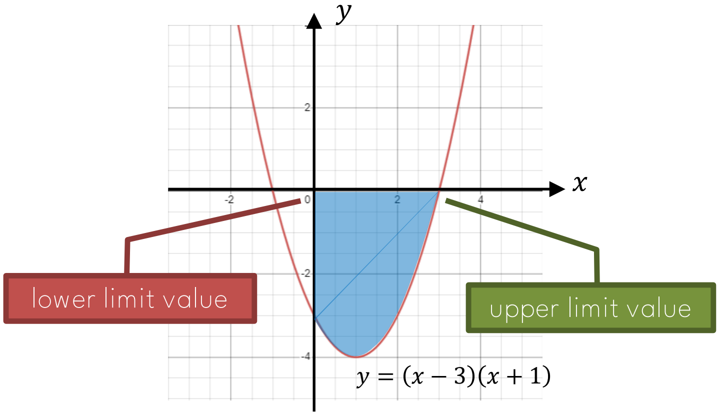
$$\therefore Area\ is\ \frac{32}{3} unit^2$$

Formula Area: $A = \int_a^b y \, dx$



Example 2

Find the area of the region bounded by the curve $y = (x-3)(x+1)$ between $x = 0$ and $x = 3$.



Solution :

REMINDER : EXPAND the equation given

$$y = (x-3)(x+1)$$
$$= x^2 - 2x - 3$$

Step 1 : Substitute the equation and the limit into the formula

$$A = \left| \int_0^3 x^2 - 2x - 3 \, dx \right|$$

Write the LIMIT
Write the equation y

Step 2 : Integrate

$$= \left| \left[\frac{x^3}{3} - \frac{2x^2}{2} - 3x \right]_0^3 \right|$$

$$= \left| \left[\frac{x^3}{3} - 2x^2 - 3x \right]_0^3 \right|$$

Step 3 : Substitute value of the limit into the equation

$$= \left| \left[\frac{3^3}{3} - 2(3)^2 - 3(3) \right] \right| - \left| \left[\frac{0^3}{3} - 2(0)^2 - 3(0) \right] \right|$$

UPPER limit – LOWER limit

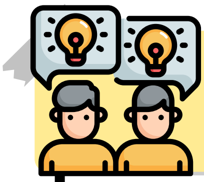
Step 4 : Solve

$$= |-18 - 0|$$
$$= |-18|$$

Formula Area: $A = \int_a^b y \, dx$

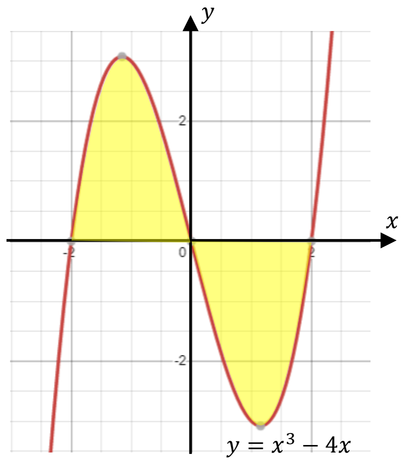
Step 5 : Write the answer

$\therefore Area\ is\ 18\ unit^2$



Example 3

Find the area of the region bounded by the curve along the x-axis.



Solution :

Step 1 : Substitute the equation and the limit into the formula

$$A = \left|\int_{-2}^{0} x^3 - 4x\ dx\right| + \int_{0}^{2} x^3 - 4x\ dx$$

Write the LIMIT
Write the equation y

Step 2 : Integrate

$$= \left|\left[\frac{x^4}{4} - \frac{4x^2}{2}\right]_{-2}^{0}\right| + \left[\frac{x^4}{4} - \frac{4x^2}{2}\right]_{0}^{2}$$

$$= \left|\left[\frac{x^4}{4} - 2x^2\right]_{-2}^{0}\right| + \left|\frac{x^4}{4} - 2x^2\right|_{0}^{2}$$

Step 3 : Substitute value of the limit into the equation

$$= \left|\left[\left(\frac{0^4}{4} - 2(0)^2\right) - \left(\frac{(-2)^4}{4} - 2(-2)^2\right)\right]\right| + \left[\left(\frac{2^4}{4} - 2(2)^2\right) - \left(\frac{(0)^4}{4} - 2(0)^2\right)\right]$$

UPPER limit – LOWER limit

Step 4 : Solve

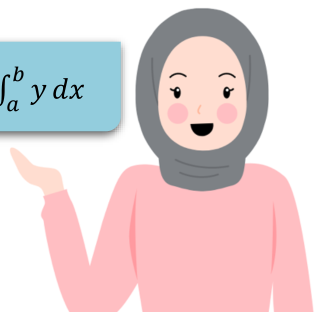
$= |-4| - (-4)$

$= 8$

Step 5 : Write the answer

$\therefore$ *Area is* $8\ unit^2$

Formula Area: $A = \int_a^b y\,dx$



Exercise

Find the followings area of the region bounded by the curve along the x-axis:

a) $y = x^2 + x - 6$ for $x = -2$ & $x = 1$

b) $y = 8 + 2x - x^2$ for $x = 0$ & $x = 3$

c) $x^2 = 5y$ for $x = -5$ & $x = 3$

d) $5x + y = x^2$ for $x = 0$ & $x = 5$

e) $y = (x + 2)^2 - 2$ for $x = -3$ & $x = -1$

f) $y = (1 + x)(3 - x)$ for $x = -1$ & $x = 2$

g) $y + 1 = -x^2 - 4x$ for $x = -3$ & $x = -1$

h) $y = x^2 + 4$ for $x = -2$ & $x = 2$

i) $y = 2x^2 - 10x + 3$ for $x = 1$ & $x = 4$

j) $y = -x^2 + 2x + 2$ for $x = 0$ & $x = 2$

k) $y = x^3 - 3x^2 - x + 3$ for $x = -1, x = 1$ & $x = 4$

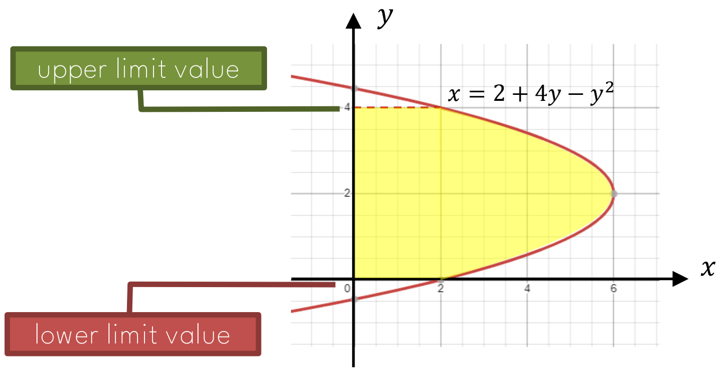
l) $y = (x + 3)(x + 2)(x - 1)$ for $x = -1, x = 1$ & $x = 4$

Answer

a)	$\frac{28}{3}$	b)	36	c)	$\frac{152}{15}$	d)	$\frac{125}{6}$	e)	$\frac{10}{3}$	f)	$\frac{32}{3}$
g)	$\frac{16}{3}$	h)	$\frac{64}{3}$	i)	24	j)	$\frac{16}{3}$	k)	8	l)	$\frac{71}{6}$

Example 1

Find the followings area of the region bounded by the curve $x = 2 + 4y - y^2$ between $y = 0$ and $y = 4$.



Solution :

Step 1 : Substitute the equation and the limit into the formula

$$A = \int_0^4 2 + 4y - y^2 \, dy$$

Write the LIMIT
Write the equation x

Step 2 : Integrate

$$= \left[2y + \frac{4y^2}{2} - \frac{y^3}{3}\right]_0^4$$

$$= \left[2y + 2y^2 - \frac{y^3}{3}\right]_0^4$$

Step 3 : Substitute value of the limit into the equation

$$= \left[2(4) + 2(4)^2 - \frac{4^3}{3}\right] - \left[2(0) + 2(0)^2 - \frac{0^3}{3}\right]$$

UPPER limit – LOWER limit

Step 4 : Solve

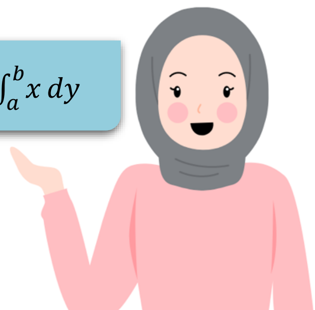
$$= \left[\frac{56}{3}\right] - 0$$

$$= \frac{56}{3}$$

Step 5 : Write the answer

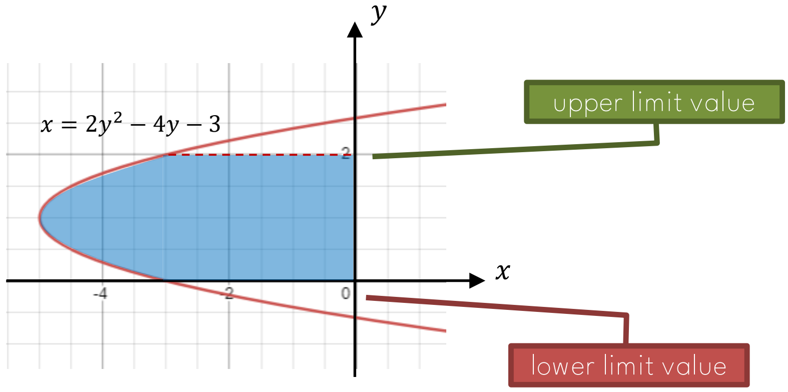
$$\therefore Area\ is\ \frac{56}{3} unit^2$$

Formula Area: $A = \int_a^b x \, dy$



Example 2

Find the followings area of the region bounded by the curve $x = 2y^2 - 4y - 3$ between $y = 0$ and $y = 4$. :



Solution :

Step 1 : Substitute the equation and the limit into the formula

$$A = \left| \int_0^2 2y^2 - 4y - 3\, dy \right|$$

Write the LIMIT
Write the equation x

Step 2 : Integrate

$$= \left| \left[\frac{2y^3}{3} - \frac{4y^2}{2} - 3y \right]_0^2 \right|$$

$$= \left| \left[\frac{2y^3}{3} - 2y^2 - 3y \right]_0^2 \right|$$

Step 3 : Substitute value of the limit into the equation

$$= \left| \left[\frac{2(2)^3}{3} - 2(2)^2 - 3(2) \right] - \left[\frac{2(0)^3}{3} - 2(0)^2 - 3(0) \right] \right|$$

UPPER limit – LOWER limit

Step 4 : Solve

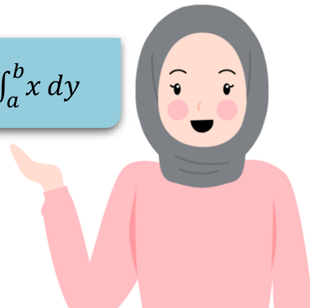
$$= \left| \left[-\frac{26}{3} \right] - 0 \right|$$

$$= \left| -\frac{26}{3} \right|$$

Step 5 : Write the answer

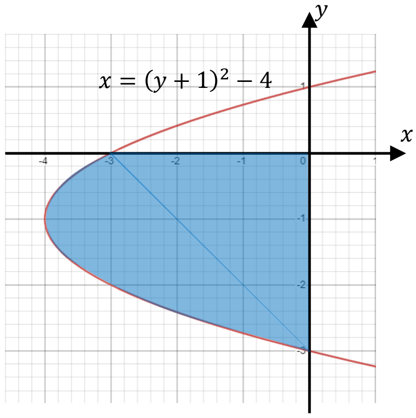
$$\therefore Area\ is\ \frac{26}{3} unit^2$$

Formula Area: $A = \int_a^b x\, dy$



Example 3

Find the followings area of the region bounded by the curve along the y-axis:



Solution :

REMINDER : EXPAND the equation given

$$y = (y+1)^2 - 4$$
$$= y^2 + 2y - 3$$

Step 1 : Substitute the equation and the limit into the formula

$$A = \int_{-3}^{0} y^2 + 2y - 3 \; dy$$

Write the LIMIT
Write the equation y

Step 2 : Integrate

$$= \left[\frac{y^3}{3} + \frac{2y^2}{2} - 3y\right]_{-3}^{0}$$

$$= \left[\frac{y^3}{3} + y^2 - 3y\right]_{-3}^{0}$$

Step 3 : Substitute value of the limit into the equation

$$= \left[\frac{(0)^3}{3} + (0)^2 - 3(0)\right] - \left[\frac{(-3)^3}{3} + (-3)^2 - 3(-3)\right]$$

UPPER limit – LOWER limit

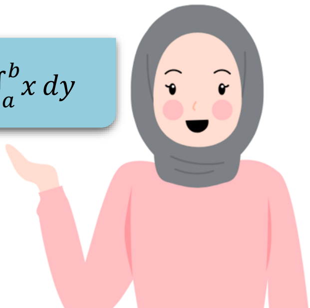
Step 4 : Solve

$$= 0 - 9$$
$$= -9$$

Step 5 : Write the answer

$\therefore$ *Area is* 9 *unit*^{2}

Formula Area: $A = \int_a^b x \, dy$



Exercise

Find the followings area of the region bounded by the curve along the y-axis:

a) $x = y^2 + y - 12$ for $y = -2$ & $y = 2$

b) $x - 2y = 8 - y^2$ for $y = -1$ & $y = 3$

c) $x + 4y = y^2$ for $y = 0$ & $y = 4$

d) $y^2 = 3x$ for $y = 0$ & $y = 3$

e) $x = (y + 3)^2 - 9$ for $y = -6$ & $y = -1$

f) $x = (2 + y)(3 - y)$ for $y = -2$ & $y = 3$

g) $x = -y^2 - 8y - 7$ for $y = -5$ & $y = -2$

h) $x - y^2 = -4$ for $y = -1$ & $y = 2$

i) $x = 2y^2 - 10y + 1$ for $y = 1$ & $y = 4$

j) $x = -2y^2 + 2y + 12$ for $y = -1$ & $y = 2$

k) $x = y^2 - 5y - 6$ for $y = 0$ & $y = 6$

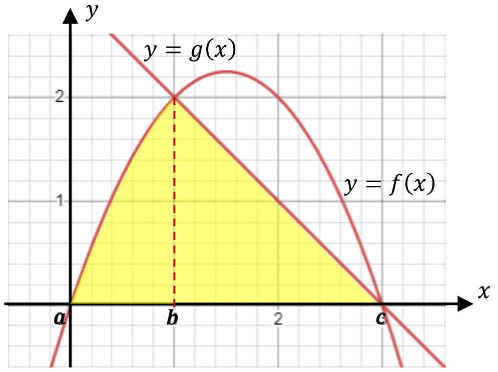
l) $x = (y - 1)(y + 2)(y + 3)$ for $y = -3, y = -2$ & $y = 1$

Answer

a)	47.8091	b)	33.7367	c)	19.5032	d)	12	e)	39.0718	f)	23.7831
g)	43.1696	h)	12.6677	i)	15.8669	j)	14.6873	k)	53.6405	l)	$\frac{179}{6}$

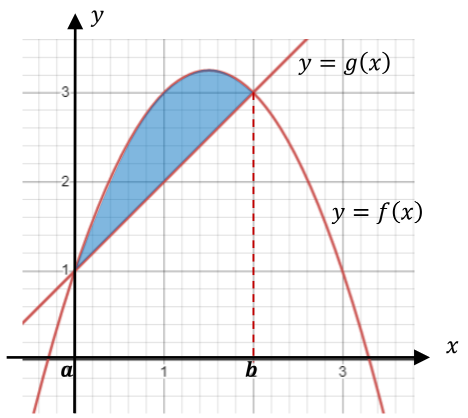
Area Under A Curve & A Straight Line

x-axis



$$A = \int_a^b f(x)\,dx + \int_b^c g(x)\,dx$$

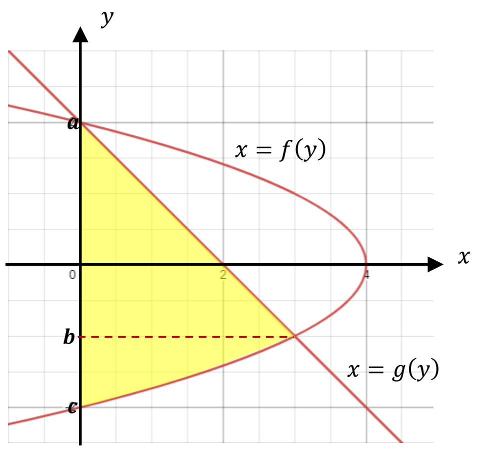
PLUS ➔ shaded region near the x-axis



$$A = \int_a^b f(x)\,dx - \int_a^b g(x)\,dx$$

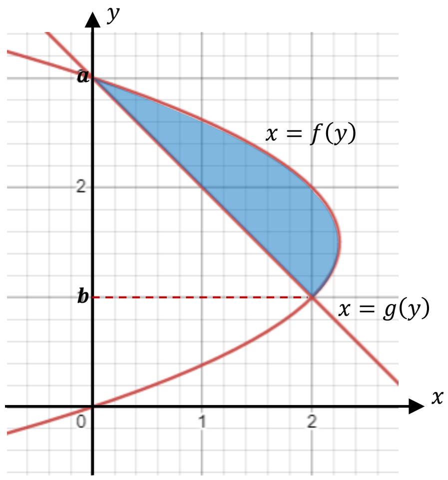
MINUS ➔ shaded region far from x-axis
UPPER line – LOWER line

y-axis



$$A = \int_a^b g(y)\,dy + \int_b^c f(y)\,dy$$

PLUS ➔ shaded region near the y-axis

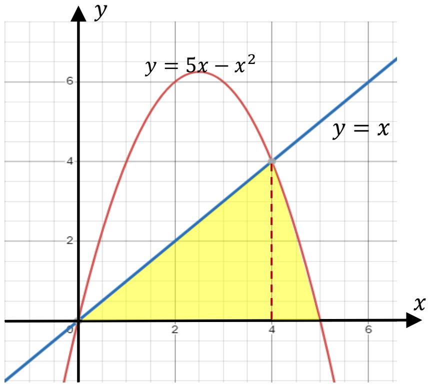


$$A = \int_a^b f(y)\,dy - \int_a^b g(y)\,dy$$

MINUS ➔ shaded region far from y-axis
UPPER line – LOWER line

Example 1

Find the followings area of the region bounded by the curve along the x-axis and the straight line:



DON'T FORGET

PLUS ➔ shaded region near the x-axis

Solution :

Step 1 : Identify the shaded region (near or far from x-axis)

$$A = \int_0^4 x\,dx + \int_4^5 5x - x^2\,dx$$

NOTES

You also CAN USE the Triangle formula instead of $A = \int_0^4 x\,dx$

$\therefore \left[\frac{1}{2} \times 4 \times 4\right] + \int_4^5 5x - x^2\,dx$

Step 2 : Integrate

$$= \left[\frac{x^2}{2}\right]_0^4 + \left[\frac{5x^2}{2} - \frac{x^3}{3}\right]_4^5$$

Step 3 : Substitute value of the limit into the equation

$$= \left[\frac{4^2}{2} - \frac{0^2}{2}\right] + \left[\left(\frac{5(5)^2}{2} - \frac{(5)^3}{3}\right) - \left(\frac{5(4)^2}{2} - \frac{(4)^3}{3}\right)\right]$$

Step 4 : Solve

$$= 8 + \left(\frac{125}{6} - \frac{56}{3}\right)$$

$$= \frac{61}{6}$$

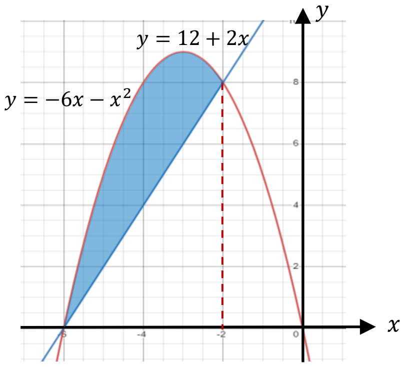
Step 5 : Write the answer

$$\therefore Area\ is\ \frac{61}{6} unit^2$$

Formula Area: $A = \int_a^b y\,dx$

Example 2

Find the followings area of the region bounded by the curve along the x-axis and the straight line:



Solution :

DON'T FORGET

MINUS ➔ shaded region far from x-axis

Step 1 : Identify the shaded region (near or far from x-axis)

$$A = \int_{-2}^{-6} -6x - x^2\,dx - \int_{-2}^{-6} 12 + 2x\,dx$$

$$= \int_{-2}^{-6} -6x - x^2 - (12 + 2x)dx$$

$$= \int_{-2}^{-6} -x^2 - 8x - 12\,dx$$

NOTES

You also CAN USE the Triangle formula instead of $A = \int_{-2}^{-6} 12 + 2x\,dx$

$\therefore A = \int_{-2}^{-6} -6x - x^2\,dx - \left[\frac{1}{2} \times 4 \times 8\right]$

Step 2 : Integrate

$$= \left[-\frac{x^3}{3} - \frac{8x^2}{2} - 12x\right]_{-2}^{-6}$$

$$= \left[-\frac{x^3}{3} - 4x^2 - 12x\right]_{-2}^{-6}$$

Step 3 : Substitute value of the limit into the equation

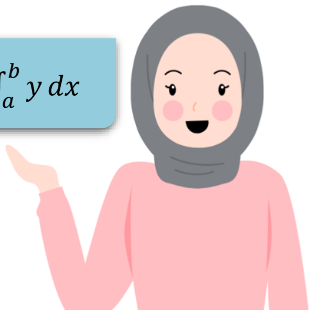
$$= \left[-\frac{(-6)^3}{3} - 4(-6)^2 - 12(-6)\right] - \left[-\frac{(-2)^3}{3} - 4(-2)^2 - 12(-2)\right]$$

Step 4 : Solve

$$= 0 - \left(\frac{32}{3}\right)$$

$$= -\frac{32}{3}$$

Formula Area: $A = \int_a^b y\,dx$

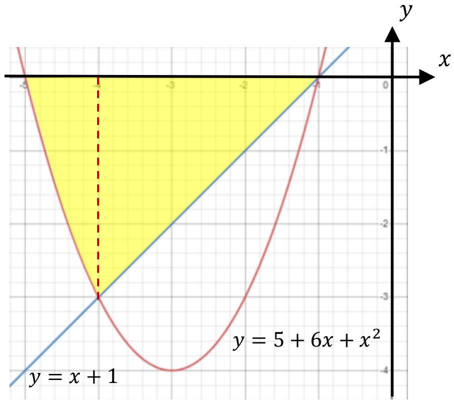


Step 5 : Write the answer

$\therefore Area\ is \frac{32}{3} unit^2$

Example 3

Find the followings area of the region bounded by the curve along the x-axis and the straight line:



Solution :

Step 1 : Identify the shaded region (near or far from x-axis)

$$A = \int_{-4}^{-1} x + 1\,dx + \int_{-5}^{-4} 5 + 6x + x^2\,dx$$

DON'T FORGET

PLUS ➔ shaded region near the x-axis

Step 2 : Integrate

$$= \left[\frac{x^2}{2} + x\right]_{-4}^{-1} + \left[5x + \frac{6x^2}{2} - \frac{x^3}{3}\right]_{-4}^{-5}$$

$$= \left[\frac{x^2}{2} + x\right]_{-4}^{-1} + \left[5x + 3x^2 - \frac{x^3}{3}\right]_{-4}^{-5}$$

Step 3 : Substitute value of the limit into the equation

$$= \left[\left(\frac{(-1)^2}{2} + (-1)\right) - \left(\frac{(-4)^2}{2} + (-4)\right)\right]$$

$$+ \left[\left(5(-5) + 3(-5)^2 - \frac{(-5)^3}{3}\right) - \left(5(-4) + 3(-4)^2 - \frac{(-4)^3}{3}\right)\right]$$

Step 4 : Solve

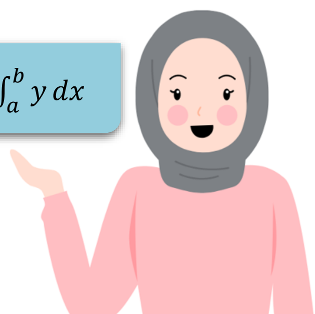
$$= -54 - \left(-\frac{49}{6}\right)$$

$$= -\frac{275}{6}$$

Step 5 : Write the answer

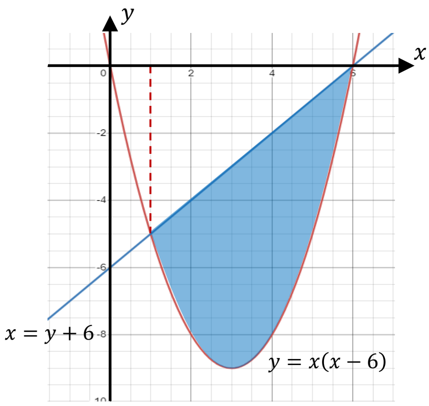
$$\therefore Area\ is\ \frac{275}{6}\,unit^2$$

Formula Area: $A = \int_a^b y\,dx$



Example 4

Find the followings area of the region bounded by the curve along the x-axis and the straight line:



Solution :

Step 1 : Identify the shaded region (near or far from x-axis)

$$A = \int_1^6 x^2 - 6x\,dx - \int_1^6 6 - x\,dx$$

$$= \int_1^6 x^2 - 6x - (6 - x)dx$$

$$= \int_1^6 x^2 - 6x - 6 + x\,dx$$

$$= \int_1^6 x^2 - 5x - 6\ dx$$

DON'T FORGET

MINUS ➔ shaded region far from x-axis

Step 2 : Integrate

$$= \left[\frac{x^3}{3} - \frac{5x^2}{2} - 6x\right]_1^6$$

Step 3 : Substitute value of the limit into the equation

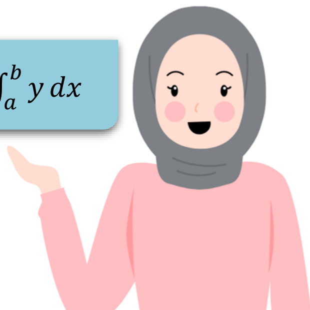
$$= \left[\frac{(6)^3}{3} - \frac{5(6)^2}{2} - 6(6)\right] - \left[\frac{(1)^3}{3} - \frac{5(1)^2}{2} - 6(1)\right]$$

Step 4 : Solve

$$= -54 - \left(-\frac{49}{6}\right)$$

$$= -\frac{275}{6}$$

Formula Area: $A = \int_a^b y\,dx$

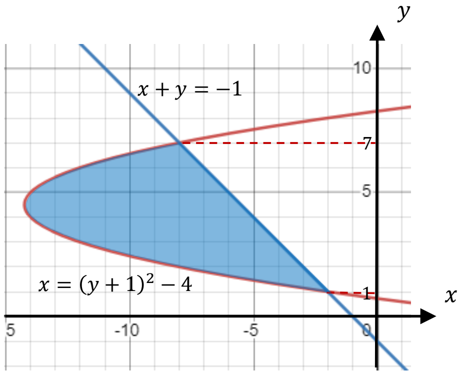


Step 5 : Write the answer

$$\therefore Area\ is\ \frac{275}{6}\ unit^2$$

Example 5

Find the followings area of the region bounded by the curve along the y-axis and the straight line:



Solution :

Step 1 : Identify the shaded region (near or far from x-axis)

$$A = \int_1^7 y^2 + 2y - 3\,dy - \int_1^7 -y - 1\,dy$$

$$= \int_1^7 y^2 + 2y - 3 - (-y-1)dx$$

$$= \int_1^7 y^2 + 3y - 2\,dy$$

DON'T FORGET

MINUS → shaded region far from x-axis

Step 2 : Integrate

$$= \left[\frac{y^3}{3} - \frac{3y^2}{2} - 2y\right]_1^7$$

Step 3 : Substitute value of the limit into the equation

$$= \left[\frac{(7)^3}{3} - \frac{3(7)^2}{2} - 2(7)\right] - \left[\frac{(1)^3}{3} - \frac{3(1)^2}{2} - 2(1)\right]$$

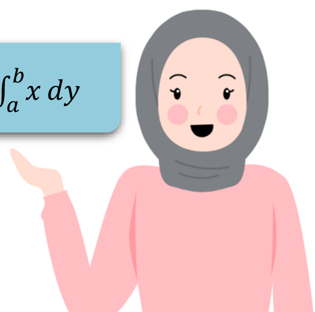
UPPER limit – LOWER limit

Step 4 : Solve

$$= \frac{1043}{6} - \left(-\frac{1}{6}\right)$$

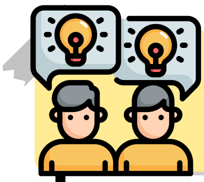
$$= \frac{1044}{6}$$

Formula Area: $A = \int_a^b x\,dy$



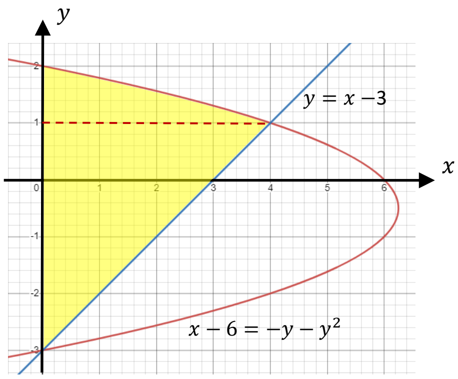
Step 5 : Write the answer

$$\therefore Area\ is\ \frac{1044}{6}\ unit^2$$



Example 6

Find the followings area of the region bounded by the curve along the y-axis and the straight line:



Solution :

Step 1 : Identify the shaded region (near or far from y-axis)

$$A = \int_{-3}^{1} y + 3\,dy + \int_{1}^{2} 6 - y - y^2\,dy$$

DON'T FORGET

PLUS ➔ shaded region near the x-axis

Step 2 : Integrate

$$= \left[\frac{y^2}{2} + 3y\right]_{-3}^{1} + \left[6y - \frac{y^2}{2} - \frac{y^3}{3}\right]_{1}^{2}$$

Step 3 : Substitute value of the limit into the equation

$$= \left[\left(\frac{(1)^2}{2} + 3(1)\right) - \left(\frac{(-3)^2}{2} + 3(-3)\right)\right] + \left[\left(6(2) - \frac{(2)^2}{2} - \frac{(2)^3}{3}\right) - \left(6(1) - \frac{(1)^2}{2} - \frac{(1)^3}{3}\right)\right]$$

UPPER limit – LOWER limit

Step 4 : Solve

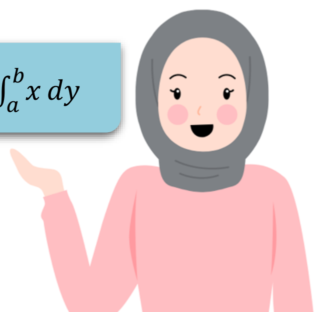
$$= \left(\frac{7}{2} - \left(-\frac{9}{2}\right)\right) + \left(\frac{22}{3} - \frac{31}{6}\right)$$

$$= \frac{61}{6}$$

Step 5 : Write the answer

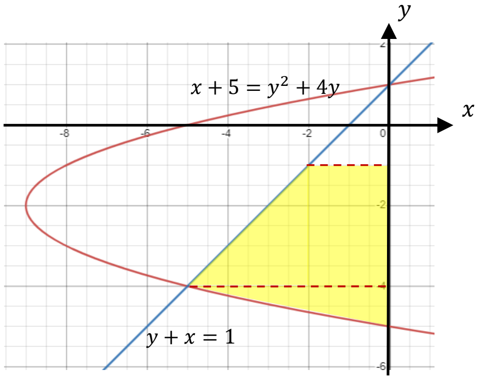
$\therefore Area\ is\ \frac{61}{6}\ unit^2$

Formula Area: $A = \int_a^b x\,dy$



Example 7

Find the followings area of the region bounded by the curve along the y-axis and the straight line:



Solution :

Step 1 : Identify the shaded region (near or far from y-axis)

$$A = \int_{-5}^{-4} 1 - y \, dy + \int_{-4}^{-1} y^2 + 4y - 5 \, dy$$

DON'T FORGET

PLUS ➔ shaded region near the x-axis

Step 2 : Integrate

$$= \left[y - \frac{y^2}{2}\right]_{-5}^{-4} + \left[\frac{y^3}{3} + \frac{4y^2}{2} - 5y\right]_{-4}^{-1}$$

$$= \left[y - \frac{y^2}{2}\right]_{-5}^{-4} + \left[\frac{y^3}{3} + 2y^2 - 5y\right]_{-4}^{-1}$$

Step 3 : Substitute value of the limit into the equation

$$= \left[\left((-4) - \frac{(-4)^2}{2}\right) - \left((-5) - \frac{(-5)^2}{2}\right)\right] + \left[\frac{(-1)^3}{3} + 2(-1)^2 - 5(1)\right]$$

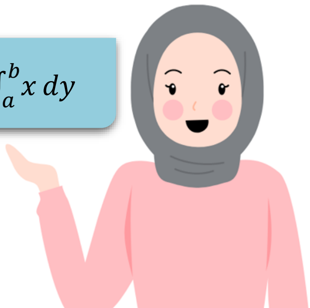
$$- \left[\frac{(-4)^3}{3} + 2(-4)^2 - 5(-4)\right]$$

Step 4 : Solve

$$= \left(-12 - \left(-\frac{35}{2}\right)\right) + \left(-\frac{10}{3} - \frac{92}{3}\right)$$

$$= -\frac{57}{2}$$

Formula Area: $A = \int_a^b x \, dy$

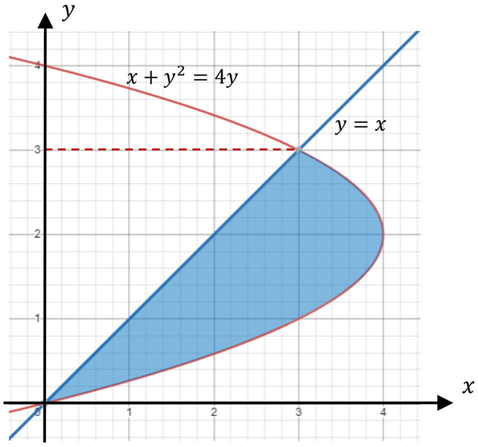


Step 5 : Write the answer

$\therefore Area\ is\ \frac{57}{2} unit^2$

Example 8

Find the followings area of the region bounded by the curve along the y-axis and the straight line:



Solution :

Step 1 : Identify the shaded region (near or far from x-axis)

$$A = \int_0^3 4y - y^2\, dy - \int_0^3 y\, dy$$

$$= \int_0^3 4y - y^2 - y\, dy$$

$$= \int_0^3 3y - y^2\, dy$$

DON'T FORGET

MINUS ➔ shaded region far from x-axis

Step 2 : Integrate

$$= \left[\frac{3y^2}{2} - \frac{y^3}{3}\right]_0^3$$

Step 3 : Substitute value of the limit into the equation

$$= \left[\frac{3(3)^2}{2} - \frac{(3)^3}{3}\right] - \left[\frac{3(0)^2}{2} - \frac{(0)^3}{3}\right]$$

UPPER limit – LOWER limit

Step 4 : Solve

$$= \frac{9}{2} - 0$$

$$= \frac{9}{2}$$

Formula Area: $A = \int_a^b x\, dy$

Step 5 : Write the answer

$\therefore Area\ is\ \frac{9}{2} unit^2$

Exercise

Find the followings shaded area bounded by the curve and the line along the axis mentioned in the bracket.

a) A curve $y = 4x - x^2$ and a line $2x + y = 8$ for $x = 0$ & $x = 4$

b) A curve $y - 2 = x^2 - 2x$ and a line $y = x + 6$ for $y = 2$ & $y = 6$

c) A curve $x^2 + y = 5x$ and a line $y = 2x$ for $x = 0$ & $x = 3$

d) A curve $y^2 - 6 = 2x$ and a line $y - 3 = x$ for $y = 0$ & $y = 2$

e) A curve $x^2 = 6x - y$ and a line $y = 6 - x$ for $x = 0$ & $x = 6$

f) A curve $x + y^2 = 3y + 4$ and a line $y + x = 4$ for $y = 0$ & $y = 3$

g) A curve $y + x^2 = 6x$ and line $y = 6 - x$ for $x = 1$ & $x = 6$

h) A curve $y + 2 = -6x - x^2$ and a line $y - x = 8$ for $x = -5$ & $x = -2$

i) A curve $2y^2 - 6y = 8 - x$ and a line $y = \frac{x}{2} - 1$ for $x = -1$ & $x = 4$

j) A curve $x = -y^2 - 7y + 10$ and a line $y + x = 2$ for $y = -4$ & $y = 2$

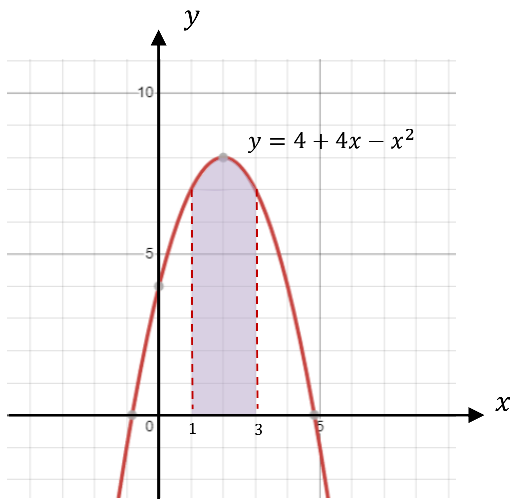
Answer

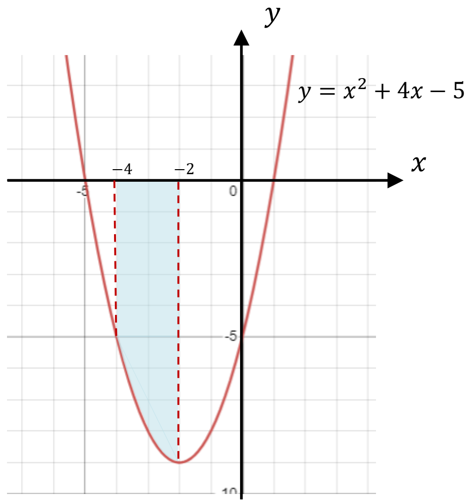
a)	$\frac{4}{3}$	b)	$\frac{51}{2}$	c)	9	d)	3.2839	e)	$\frac{91}{6}$
f)	$\frac{25}{3}$	g)	$\frac{125}{6}$	h)	$\frac{9}{2}$	i)	3.159	j)	36

VOLUME

Volume Under A Curve

x-axis





above x-axis

$$V = \pi \int_a^b y^2 \; dx$$

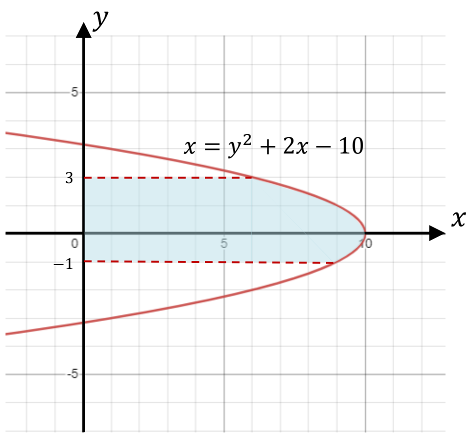
upper limit value

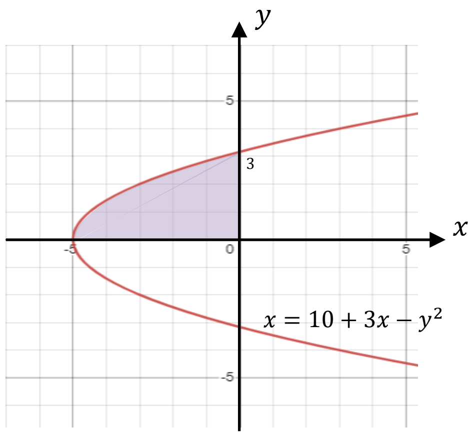
lower limit value

below x-axis

$$V = \pi \left| \int_a^b y^2 \; dx \right|$$

y-axis





right y-axis

$$V = \pi \int_a^b x^2 \; dy$$

upper limit value

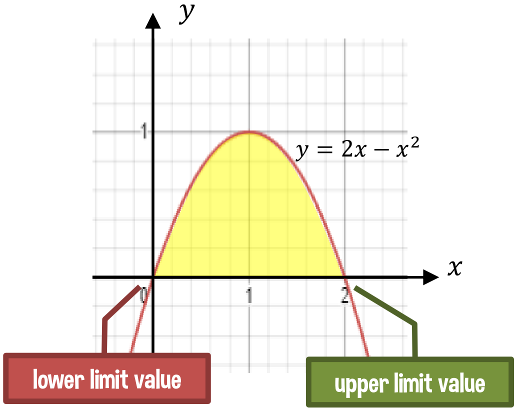
lower limit value

left y-axis

$$V = \pi \left| \int_a^b x^2 \; dy \right|$$

Example 1

Find the volume generated when the shaded region rotated 360° about the x-axis:



Solution :

Step 1 : Identify the shaded region (near or far from x-axis)

$$V = \pi \int_0^2 (2x - x^2)^2 \; dx$$

$$= \pi \int_0^2 4x^2 - 4x^3 + x^4 \, dx$$

Step 2 : Integrate

$$= \pi \left[\frac{4x^3}{3} - \frac{4x^4}{4} + \frac{x^5}{5} \right]_0^2$$

Step 3 : Substitute value of the limit into the equation

$$= \pi \left[\frac{4(2)^3}{3} - (2)^4 + \frac{(2)^5}{5} \right] - \left[\frac{4(0)^3}{3} - (0)^4 + \frac{(0)^5}{5} \right]$$

UPPER limit – LOWER limit

Step 4 : Solve

$$= \pi \left[\frac{16}{15} - 0 \right]$$

$$= \frac{16}{15}\pi$$

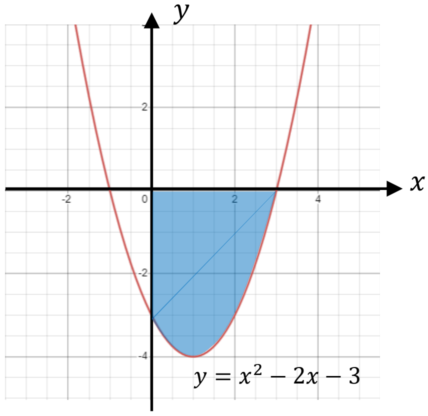
Step 5 : Write the answer

$$\therefore Volume \; is \frac{16}{15}\pi \; unit^3$$

Formula Volume: $V = \pi \int_a^b y^2 \, dx$

Example 2

Find the volume generated when the shaded region rotated 360° about the x-axis:



Solution :

Step 1 : Identify the shaded region (near or far from x-axis)

$$V = \pi \left| \int_0^3 (x^2 - 2x - 3)^2 \ dx \right|$$

$$= \pi \left| \int_0^3 x^4 - 4x^3 - 2x^2 + 12x + 9 \, dx \right|$$

Step 2 : Integrate

$$= \pi \left| \left[\frac{x^5}{5} - \frac{4x^4}{4} - \frac{2x^3}{3} + \frac{12x^2}{2} + 9x \right]_0^3 \right|$$

$$= \pi \left| \left[\frac{x^5}{5} - x^4 - \frac{2x^3}{3} + 6x^2 + 9x \right]_0^3 \right|$$

Step 3 : Substitute value of the limit into the equation

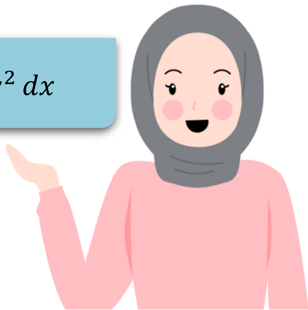
$$= \pi \left| \left[\frac{(3)^5}{5} - (3)^4 - \frac{2(3)^3}{3} + 6(3)^2 + 9(3) \right] - \left[\frac{(0)^5}{5} - (0)^4 - \frac{2(0)^3}{3} + 6(0)^2 + 9(0) \right] \right|$$

Step 4 : Solve

$$= \pi \left| \frac{153}{5} - 0 \right|$$

$$= \frac{153}{5} \pi$$

Formula Volume: $V = \pi \int_a^b y^2 \, dx$

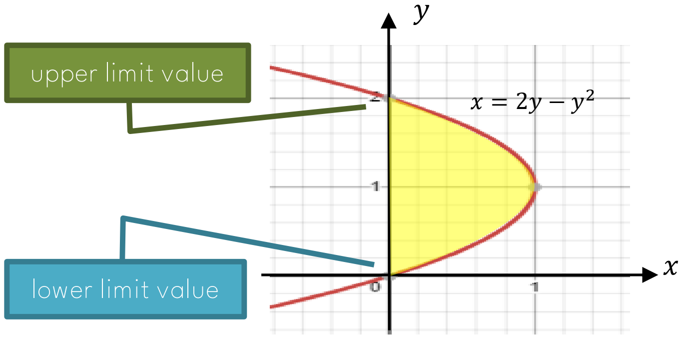


Step 5 : Write the answer

$$\therefore \ Volume \ is \ \frac{153}{5} \pi \ unit^3$$

Example 3

Find the volume generated when the shaded region rotated 360° about the y-axis:



Solution :

Step 1 : Identify the shaded region (near or far from x-axis)

$$V = \pi \int_0^2 (2y - y^2)^2 \; dy$$

$$= \pi \int_0^2 4y^2 - 4y^3 + y^4 \, dy$$

Step 2 : Integrate

$$= \pi \left[\frac{4y^3}{3} - \frac{4y^4}{4} + \frac{y^5}{5} \right]_0^2$$

$$= \pi \left[\frac{4y^3}{3} - y^4 + \frac{y^5}{5} \right]_0^2$$

Step 3 : Substitute value of the limit into the equation

$$= \pi \left[\frac{4(2)^3}{3} - (2)^4 + \frac{(2)^5}{5} \right] - \left[\frac{4(0)^3}{3} - (0)^4 + \frac{(0)^5}{5} \right]$$

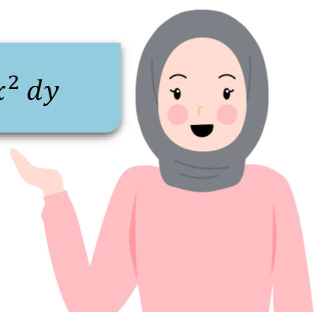
UPPER limit – LOWER limit

Step 4 : Solve

$$= \pi \left[\frac{16}{15} - 0 \right]$$

$$= \frac{16}{15} \pi$$

Formula Volume: $V = \pi \int_a^b x^2 \, dy$

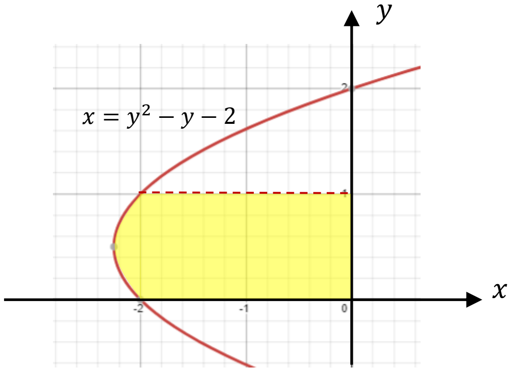


Step 5 : Write the answer

$\therefore \; Volume \; is \; \frac{16}{15} \pi \; unit^3$

Example 4

Find the volume generated when the shaded region rotated 360° about the y-axis:



Solution :

Step 1 : Identify the shaded region (near or far from x-axis)

$$V = \pi \left| \int_0^1 (y^2 - y - 2)^2 \, dy \right|$$

$$= \pi \left| \int_0^1 y^4 - 2y^3 - 3y^2 + 4y + 4 \, dy \right|$$

Step 2 : Integrate

$$= \pi \left| \left[\frac{y^5}{5} - \frac{2y^4}{4} - \frac{3y^3}{3} + \frac{4y^2}{2} + 4y \right]_0^1 \right|$$

$$= \pi \left| \left[\frac{y^5}{5} - \frac{y^4}{2} - y^3 + 2y^2 + 4y \right]_0^1 \right|$$

Step 3 : Substitute value of the limit into the equation

$$= \pi \left| \left[\frac{(1)^5}{5} - \frac{(1)^4}{2} - (1)^3 + 2(1)^2 + 4(1) \right] - \left[\frac{(0)^5}{5} - \frac{(0)^4}{2} - (0)^3 + 2(0)^2 + 4(0) \right] \right|$$

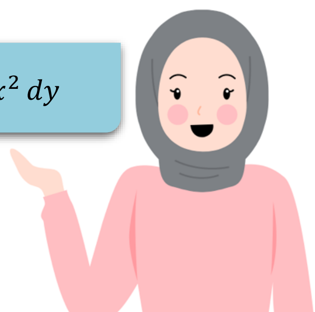
UPPER limit – LOWER limit

Step 4 : Solve

$$= \pi \left| \frac{47}{10} - 0 \right|$$

$$= \frac{47}{10} \pi$$

Formula Volume: $V = \pi \int_a^b x^2 \, dy$



Step 5 : Write the answer

$$\therefore Volume\ is\ \frac{47}{10} \pi\ unit^3$$

Exercise

Find the volume generated when the shaded region rotated 360° about the x-axis:

a) $y = x^2 + x - 6$ for $x = -2$ & $x = 2$

b) $y = 8 + 2x - x^2$ for $x = -1$ & $x = 3$

c) $y = (1 + x)(3 - x)$ for $x = -1$ & $x = 1$

d) $5x + y = x^2$ for $x = 0$ & $x = 4$

e) $y = \mp x^2 + 4$ for $x = -2$ & $x = 2$

f) $y = 2x^2 - 10x + 3$ for $x = 1$ & $x = 3$

Find the volume generated when the shaded region rotated 360° about the y-axis:

a) $x = y^2 + y - 12$ for $y = -3$ & $y = 2$

b) $x - 2y = 8 - y^2$ for $y = -2$ & $y = 3$

c) $x + 4y = y^2$ for $y = 0$ & $y = 4$

d) $y^2 = -2x - 16$ for $y = -3$ & $y = 3$

e) $x + y^2 = 4$ for $y = -2$ & $y = 2$

f) $x = (y - 3)(y + 2)$ for $y = 0$ & $y = 3$

ANSWER : $x - axis$

a) $\frac{625}{6}\pi$ b) $\frac{1024}{5}\pi$ c) $\frac{512}{15}\pi$

d) $\frac{921}{10}\pi$ e) $\frac{1792}{15}\pi$ f) $\frac{2134}{15}\pi$

ANSWER : $y - axis$

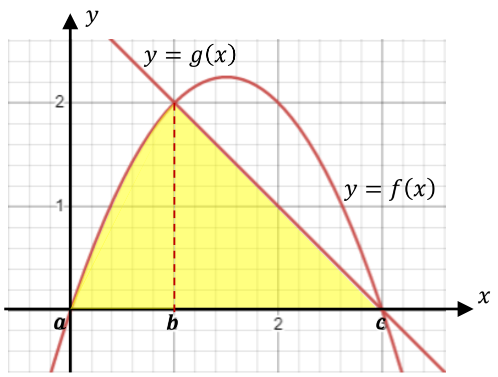
a) $\frac{16807}{30}\pi$ b) 54π c) $\frac{1024}{125}\pi$

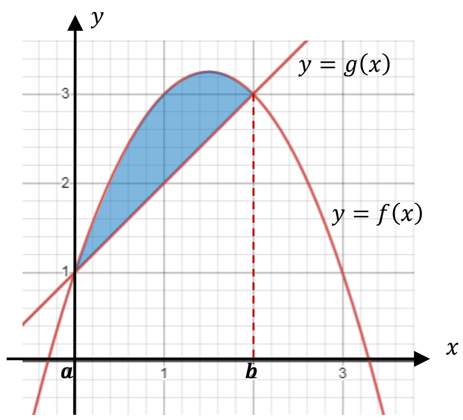
d) 128π e) $\frac{512}{15}\pi$ f) $\frac{4048}{105}\pi$

VOLUME

Volume Under A Curve & A Straight Line

x-axis





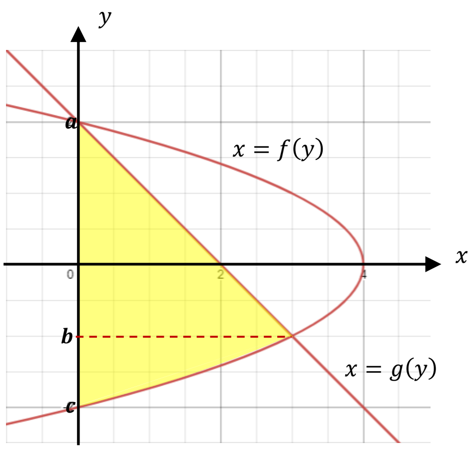
$$V = \pi \int_a^b [f(x)]^2 \, dx + \pi \int_b^c [g(x)]^2 \, dx$$

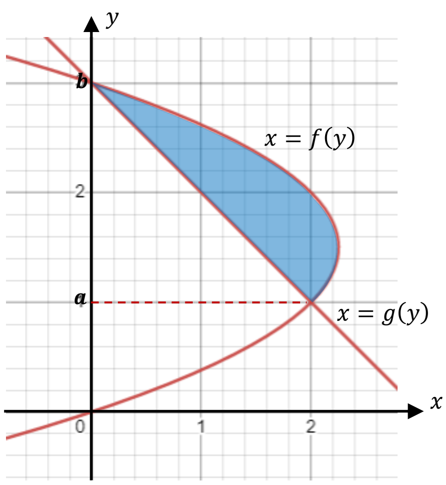
PLUS ➔ shaded region near the x-axis

$$V = \pi \int_a^b [f(x)]^2 \, dx - \pi \int_a^b [g(x)]^2 \, dx$$

MINUS ➔ shaded region far from x-axis
UPPER line – LOWER line

y-axis





$$V = \pi \int_a^b [g(y)]^2 \, dy + \pi \int_b^c [f(y)]^2 \, dy$$

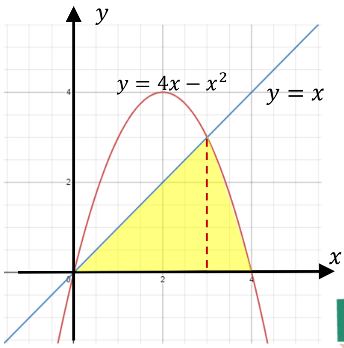
PLUS ➔ shaded region near the y-axis

$$V = \pi \int_a^b [g(y)]^2 \, dy - \pi \int_a^b [f(y)]^2 \, dy$$

MINUS ➔ shaded region far from y-axis
UPPER line – LOWER line

Example 1

Find the volume generated when the shaded region rotated 360° about the x-axis and the line :



Solution :

DON'T FORGET

PLUS ➔ shaded region near the x-axis

Step 1 : Identify the shaded region (near or far from x-axis)

$$V = \pi \int_0^3 x^2 \, dx + \pi \int_3^4 (4x - x^2)^2 \, dx$$

$$= \pi \int_0^3 x^2 \, dx + \pi \int_3^4 16x^2 - 8x^3 + x^4 \, dx$$

NOTES

You also CAN USE the Triangle formula instead of $A = \int_0^3 x^2 \, dx$

$\therefore \pi \left[\frac{1}{2} \times 3 \times 3\right] + \pi \int_3^4 (4x - x^2)^2 \, dx$

Step 2 : Integrate

$$= \pi \left[\frac{x^3}{3}\right]_0^3 + \pi \left[\frac{16x^3}{3} - \frac{8x^4}{4} + \frac{x^5}{5}\right]_3^4$$

$$= \pi \left[\frac{x^3}{3}\right]_0^3 + \pi \left[\frac{16x^3}{3} - 2x^4 + \frac{x^5}{5}\right]_3^4$$

Step 3 : Substitute value of the limit into the equation

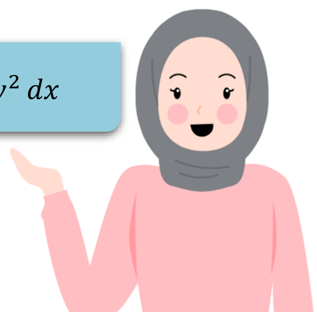
$$= \pi \left[\frac{3^3}{3} - \frac{0^3}{3}\right] + \pi \left[\left(\frac{16(4)^3}{3} - 2(4)^4 + \frac{(4)^5}{5}\right) - \left(\frac{16(3)^3}{3} - 2(3)^4 + \frac{(3)^5}{5}\right)\right]$$

Step 4 : Solve

$$= \pi \left[9 + \left(\frac{512}{15} - \frac{153}{5}\right)\right]$$

$$= \frac{188}{15}\pi$$

Formula Volume: $V = \pi \int_a^b y^2 \, dx$



Step 5 : Write the answer

$\therefore Volume\ is\ \frac{188}{15}\pi\ unit^3$

Example 2

Find the volume generated when the shaded region rotated 360° about the x-axis and the line :

Solution :

Step 1 : Identify the shaded region (near or far from x-axis)

$$V = \pi\int_{-3}^{-1}(-3x-x^2)^2\,dx - \pi\int_{-3}^{-1}(3+x)^2\,dx$$

$$= \pi\int_{-3}^{-1}9x^2+6x^3+x^4\ dx - \pi\int_{-3}^{-1}9+6x+x^2\,dx$$

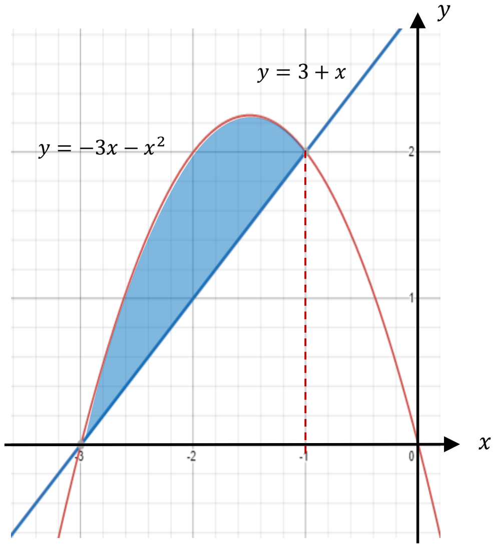
$$= \pi\int_{-3}^{-1}9x^2+6x^3+x^4-(9+6x+x^2)\,dx$$

$$= \pi\int_{-3}^{-1}9x^2+6x^3+x^4-9-6x-x^2\,dx$$

$$= \pi\int_{-3}^{-1}x^4+6x^3+8x^2-6x-9\,dx$$

DON'T FORGET

MINUS ➔ shaded region far from x-axis



Step 2 : Integrate

$$= \pi\left[\frac{x^5}{5}+\frac{6x^4}{4}+\frac{8x^3}{3}-\frac{6x^2}{2}-9x\right]_{-3}^{-1}$$

$$= \pi\left[\frac{x^5}{5}+\frac{3x^4}{2}+\frac{8x^3}{3}-3x^2-9x\right]_{-3}^{-1}$$

Step 3 : Substitute value of the limit into the equation

$$= \pi\left[\frac{(-1)^5}{5}+\frac{3(-1)^4}{2}+\frac{8(-1)^3}{3}-3(-1)^2-9(-1)\right]-\left[\frac{(-3)^5}{5}+\frac{3(-3)^4}{2}+\frac{8(-3)^3}{3}-3(-3)^2-9(-3)\right]$$

Step 4 : Solve

$$= \pi\left[\frac{139}{30}-\left(\frac{9}{10}\right)\right]$$

$$= \frac{56}{15}\pi$$

Formula Volume: $V = \pi\int_a^b y^2\,dx$

Step 5 : Write the answer

$$\therefore Volume\ is\ \frac{56}{15}\pi\ unit^3$$

Example 3

Find the volume generated when the shaded region rotated 360° about the x-axis and the line :

DON'T FORGET

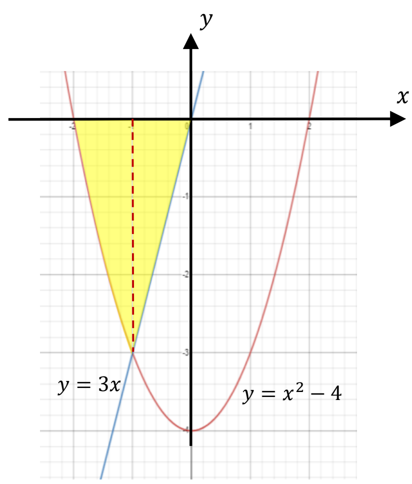
PLUS ➔ shaded region near the x-axis

Solution :

Step 1 : Identify the shaded region (near or far from x-axis)

$$V = \pi \int_{-2}^{-1} (x^2 - 4)^2 \, dx + \pi \int_{-1}^{0} (3x)^2 \, dx$$

$$= \pi \int_{-2}^{-1} x^4 - 8x^2 + 16 \, dx + \pi \int_{-1}^{0} 9x^2 \, dx$$



Step 2 : Integrate

$$= \pi \left[\frac{x^5}{5} - \frac{8x^3}{3} + 16x\right]_{-2}^{-1} + \pi \left[\frac{9x^3}{3}\right]_{-1}^{0}$$

$$= \pi \left[\frac{x^5}{5} - \frac{8x^3}{3} + 16x\right]_{-2}^{-1} + \pi [3x^3]_{-1}^{0}$$

Step 3 : Substitute value of the limit into the equation

$$= \pi \left[\left(\frac{(-1)^5}{5} - \frac{8(-1)^3}{3} + 16(-1)\right) - \left(\frac{(-2)^5}{5} - \frac{8(-2)^3}{3} + 16(-2)\right)\right] + \pi[3(0)^3 - 3(-1)^3]$$

Step 4 : Solve

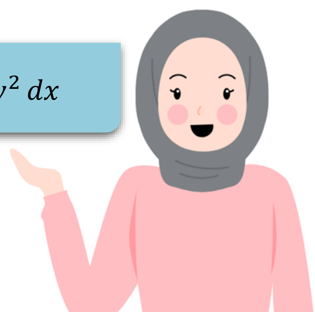
$$= \pi \left[\frac{53}{15} + 3\right]$$

$$= \frac{98}{15}\pi$$

Step 5 : Write the answer

$$\therefore Volume\ is\ \frac{98}{15}\pi\ unit^3$$

Formula Volume: $V = \pi \int_a^b y^2 \, dx$



Example 4

Find the volume generated when the shaded region rotated 360° about the x-axis and the line :

DON'T FORGET

MINUS → shaded region far from x-axis

Solution :

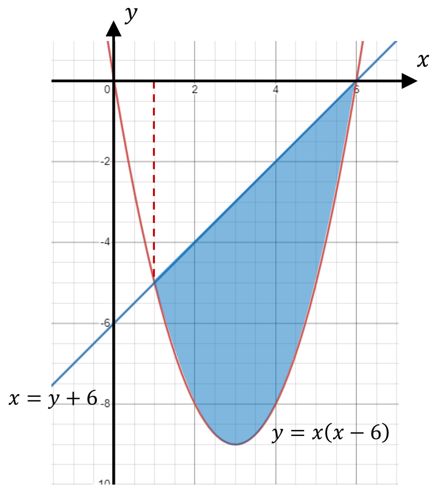
Step 1 : Identify the shaded region (near or far from x-axis)

$$V = \pi \int_1^6 (x^2 - 6x)^2 \, dx - \pi \int_1^6 (x-6)^2 \, dx$$

$$= \pi \int_1^6 x^4 - 12x^3 + 36x^2 - (x^2 - 12x + 36) dx$$

$$= \pi \int_1^6 x^4 - 12x^3 + 36x^2 - x^2 + 12x - 36 \, dx$$

$$= \pi \int_1^6 x^4 - 12x^3 + 35x^2 + 12x - 36 \, dx$$



Step 2 : Integrate

$$= \pi \left[\frac{x^5}{5} - \frac{12x^4}{4} + \frac{35x^3}{3} + \frac{12x^2}{2} - 36x \right]_1^6$$

$$= \pi \left[\frac{x^5}{5} - 3x^4 + \frac{35x^3}{3} + 6x^2 - 36x \right]_1^6$$

Step 3 : Substitute value of the limit into the equation

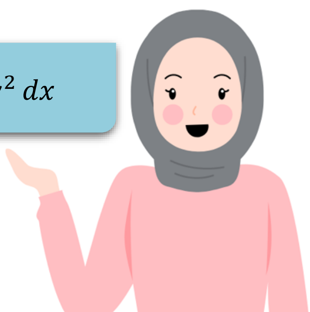
$$= \pi \left[\left(\frac{(6)^5}{5} - 3(6)^4 + \frac{35(6)^3}{3} + 6(6)^2 - 36(6) \right) - \left(\frac{(1)^5}{5} - 3(1)^4 + \frac{35(1)^3}{3} + 6(1)^2 - 36(1) \right) \right]$$

Step 4 : Solve

$$= \pi \left[\frac{936}{5} - \left(-\frac{317}{15} \right) \right]$$

$$= -\frac{625}{3}\pi$$

Formula Volume: $V = \pi \int_a^b y^2 \, dx$



Step 5 : Write the answer

$$\therefore Volume \ is \ \frac{625}{3}\pi \ unit^3$$

Example 5

Find the volume generated when the shaded region rotated 360° about the y-axis and the straight line :

Solution :

Step 1 : Identify the shaded region (near or far from x-axis)

$$V = \pi\int_{-1}^{2}(y^2-4)^2\,dy - \pi\int_{-1}^{2}(y-2)^2\,dy$$

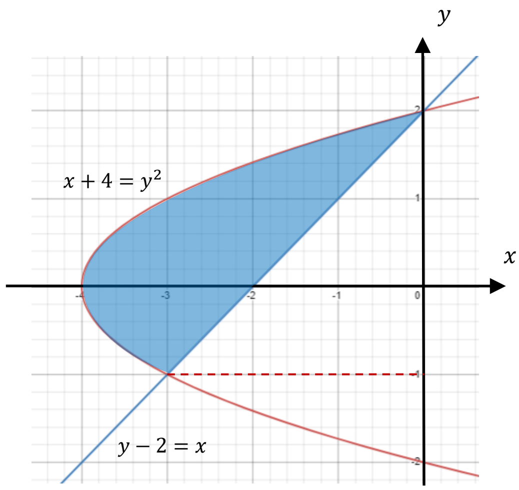
$$= \pi\int_{-1}^{2} y^4 - 8y^2 + 16 - (y^2 - 4y + 4)dy$$

$$= \pi\int_{-1}^{2} y^4 - 8y^2 + 16 - y^2 + 4y - 4\,dy$$

$$= \pi\int_{-1}^{2} y^4 - 9y^2 + 4y + 12\ dy$$

DON'T FORGET

MINUS ➔ shaded region far from x-axis



Step 2 : Integrate

$$= \pi\left[\frac{y^5}{5} - \frac{9y^3}{3} + \frac{4y^2}{2} + 12y\right]_{-1}^{2}$$

$$= \pi\left[\frac{y^5}{5} - 3y^3 + 2y^2 + 12y\right]_{-1}^{2}$$

Step 3 : Substitute value of the limit into the equation

$$= \pi\left[\left(\frac{(2)^5}{5} - 3(2)^3 + 2(2)^2 + 12(2)\right) - \left(\frac{(-1)^5}{5} - 3(-1)^3 + 2(-1)^2 + 12(-1)\right)\right]$$

Step 4 : Solve

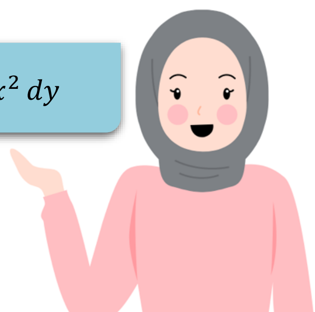
$$= \pi\left[-\frac{168}{5} - \left(-\frac{36}{5}\right)\right]$$

$$= -\frac{132}{5}\pi$$

Step 5 : Write the answer

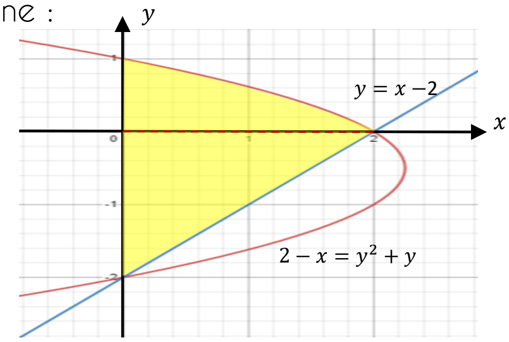
$$\therefore Volume\ is\ \frac{132}{5}\pi\ unit^3$$

Formula Volume: $V = \pi\int_a^b x^2\,dy$



Example 6

Find the volume generated when the shaded region rotated 360° about the y-axis and the straight line :



Solution :

Step 1 : Identify the shaded region (near or far from x-axis)

$$V = \pi\int_{-2}^{0}(y+2)^2\,dy + \pi\int_{0}^{1}(4-y-y^2)^2\,dy$$

$$= \pi\int_{-2}^{0} y^2+2y+4\,dy + \pi\int_{0}^{1} 16-8y-7y^2+2y^3+y^4\,dy$$

DON'T FORGET

PLUS ➔ shaded region near the x-axis

Step 2 : Integrate

$$= \pi\left[\frac{y^3}{3}+\frac{2y^2}{2}+4y\right]_{-2}^{0} + \pi\left[16y-\frac{8y^2}{2}-\frac{7y^3}{3}-\frac{2y^4}{4}+\frac{y^5}{5}\right]_{0}^{1}$$

$$= \pi\left[\frac{y^3}{3}+y^2+4y\right]_{-2}^{0} + \pi\left[16y-4y^2-\frac{7y^3}{3}-\frac{y^4}{2}+\frac{y^5}{5}\right]_{0}^{1}$$

Step 3 : Substitute value of the limit into the equation

$$= \pi\left[\left(\frac{(0)^3}{3}+(0)^2+4(0)\right)-\left(\frac{(-2)^3}{3}+(-2)^2+4(-2)\right)\right]$$

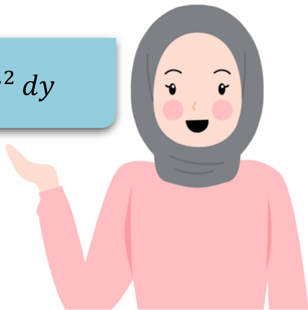
$$+\pi\left[\left(16(1)-4(1)^2-\frac{7(1)^3}{3}-\frac{(1)^4}{2}+\frac{(1)^5}{5}\right)-\left(16(0)-4(0)^2-\frac{7(0)^3}{3}-\frac{(0)^4}{2}+\frac{(0)^5}{5}\right)\right]$$

Step 4 : Solve

$$= \pi\left[\frac{20}{3}+\left(\frac{281}{30}\right)\right]$$

$$= \frac{481}{30}\pi$$

Formula Volume: $V = \pi\int_a^b x^2\,dy$



Step 5 : Write the answer

$\therefore$ *Volume is* $\frac{481}{30}\pi$ *unit*^{3}

Example 7

Find the volume generated when the shaded region rotated 360° about the y-axis and the straight line :

Solution :

Step 1 : Identify the shaded region (near or far from x-axis)

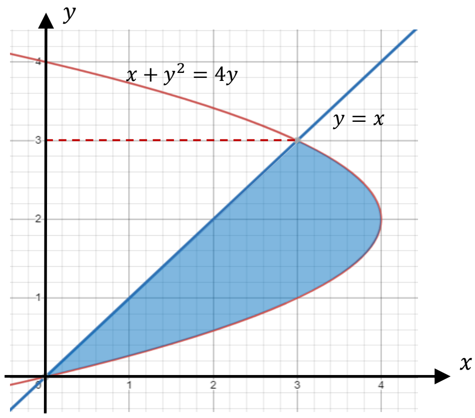
$$V = \pi \int_0^3 (4y - y^2)^2 \, dy - \pi \int_0^3 y^2 \, dy$$

$$= \pi \int_0^3 16y^2 - 8y^3 + y^4 - y^2 \, dy$$

$$= \pi \int_0^3 15y^2 - 8y^3 + y^4 \, dy$$

DON'T FORGET

MINUS ➔ shaded region far from x-axis



Step 2 : Integrate

$$= \pi \left[\frac{15y^3}{3} - \frac{8y^4}{4} + \frac{y^5}{5} \right]_0^3$$

$$= \pi \left[5y^3 + 2y^4 + \frac{y^5}{5} \right]_0^3$$

Step 3 : Substitute value of the limit into the equation

$$= \pi \left[\left(5(3)^3 + 2(3)^4 + \frac{(3)^5}{5} \right) - \left(5(0)^3 + 2(0)^4 + \frac{(0)^5}{5} \right) \right]$$

Step 4 : Solve

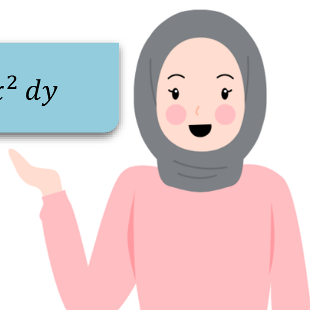
$$= \pi \left[-\frac{1728}{5} - (0) \right]$$

$$= -\frac{1728}{5}\pi$$

Step 5 : Write the answer

$$\therefore Volume\ is \frac{1728}{5}\pi\ unit^3$$

Formula Volume: $V = \pi \int_a^b x^2 \, dy$



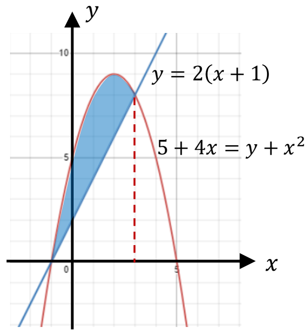
Exercise

1. Find the volume generated when the shaded region rotated 360° about the x-axis and the straight line :

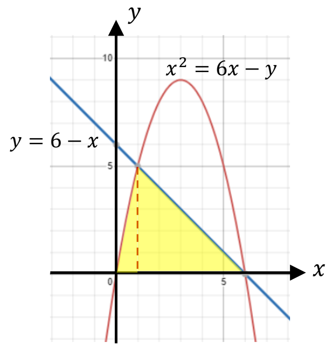
a) A curve $y = 4x - x^2$ and a line $x + y = 4$ for $x = 0$ and $x = 4$

b) A curve $x^2 + 6x = y$ and a line $y = x$ for $x = -6$ and $x = 0$

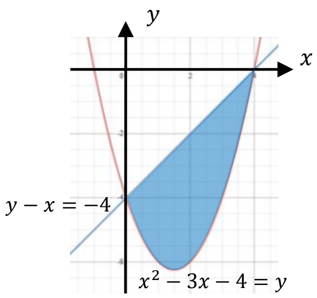
c)



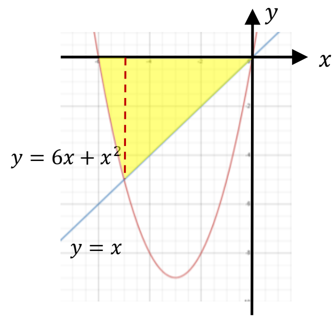
d)



e)



f)

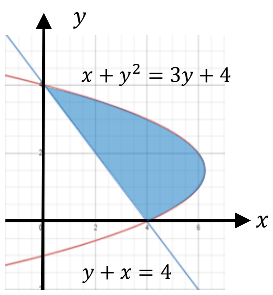


2. Find the volume generated when the shaded region rotated 360° about the y-axis and the straight line :

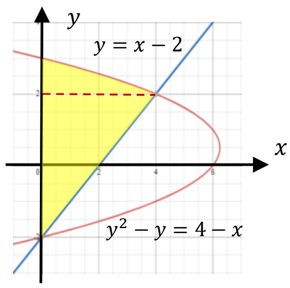
a) A curve $9 - y^2 = x$ and a line $y + 2 = \frac{4}{5}x$ for $y = -2$ and $y = 2$

b) A curve $-16 = 2x - y^2$ and a line $y = -x$ for $y = 0$ and $y = 4$

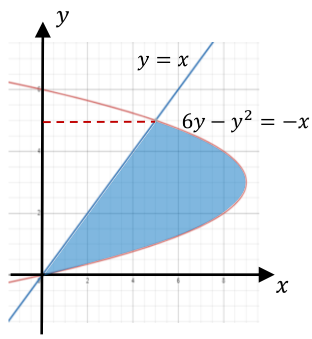
c)



d)



e)



ANSWER : $x - axis$

a) 12π b) $\frac{96}{5}\pi$ c) $\frac{944}{5}\pi$

d) $\frac{217}{10}\pi$ e) $\frac{1352}{15}\pi$ f) $\frac{40477}{10}\pi$

ANSWER : $y - axis$

a) 42π b) $\frac{31}{2}\pi$ c) $\frac{1352}{15}\pi$

d) $\frac{421}{30}\pi$ e) $\frac{512}{15}\pi$

REFERENCE

Abd Wahid Md Raji, Hamisan Rahmat, Ismail Kamis, Mohd Nor Mohamad & Ong Chee Tiong (2013). The First Course of Calculus for Science & Engineering Students. Universiti Teknologi Malaysia

Calculus 1 (article). (n.d.). Khan Academy. Retrieved 2021, from https://www.khanacademy.org/math/calculus-1

Irma Binti Maaman, Mohd Azim Bin Janis & Wan Hamiza Binti Wan Hasan (2018). Engineering Mathematics Vol 2. Politeknik Ibrahim Sultan

Suhana Ramli, Khairul Faizal Abdul Hamid & Intan Noor Dzalika Azis (2017). Engineering Mathematics 2 For Polytechnics (Volume 1). Politeknik Mukah

Zuraini Ibrahim, Myzatul Mansor, Suria Masnin, Baharudin Azit & Fatin Hamimah Mohamed Salleh (2018). Polytechnic Series Engineering Mathematics 2. Oxford Fajar.

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