CALCULUS

DIFFERENTIATION

for Science & Engineering Students

Junaliza Ishak
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Syafarizan Nasroddin

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Preface

Grateful to Allah because with His permission, the eBook Calculus - Differentiation For Science & Engineering Students was published. This eBook is written by lecturers who have been teaching in Engineering Mathematics for more than 15 years . This eBook can be used by all institutions of higher learning such as Polytechnics and Colleges as well as private and public universities. The purpose for this eBook was written is to make it easier for students to gain knowledge and review the topic of calculus in a simpler and more concise way.

Many examples in various forms of questions are included in this eBook with detailed steps of solution to make it easier for students to quickly understand the method of its solution. In addition, students will also able to improve and strengthen their understanding through the included practice questions. The authors hope that this eBook can benefit all students as well as educators around the world in the field of Engineering Mathematics.

Thank You So Much.

Suhana binti Ramli
Editor

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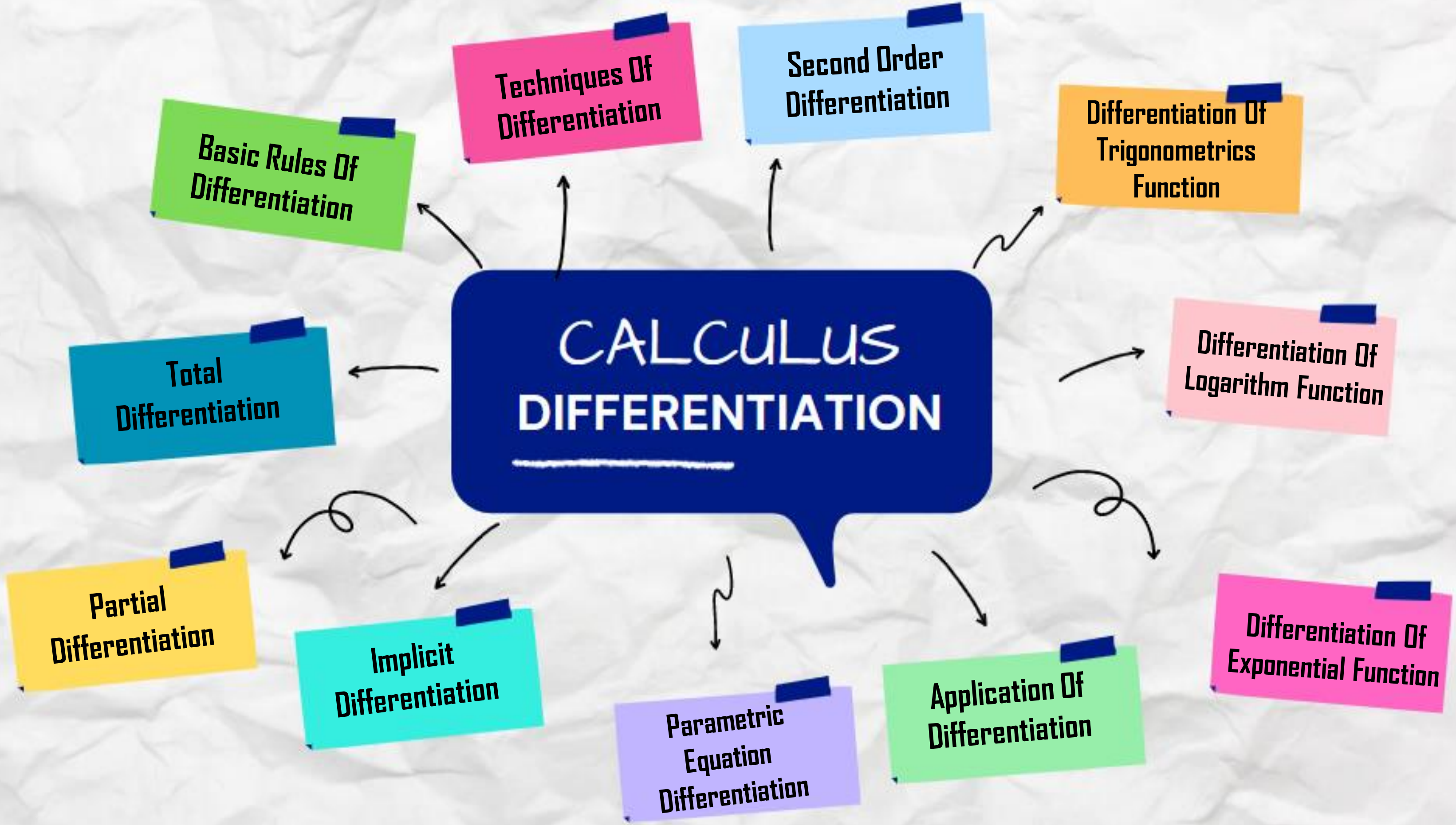
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Introduction Of Differentiation

Algebra vs Calculus

Algebra

- ❑ Algebra focuses on solving equations
- ❑ Algebra deals with operations on variables and numbers

Calculus

- ❑ Calculus is primarily focused on differentiation and integration problems.
- ❑ Calculus deals with operations on functions and their derivatives

Differentiation

- ➢ **The process of finding a derivative** is called differentiation.
- ➢ If given $y = f(x)$, the differentiation is written as $\frac{dy}{dx}$ **or** $f'(x)$
- ➢ **Application :**
  - i) to calculate rates of change
  - ii) to determine where the maximum and minimum values occur.

** *Among the uses of calculus in engineering are the study of gravity and planetary motion, fluid flow and ship design, and geometric curves and bridge engineering.*

Basic Rules Of Differentiation

- Determining the derivative of a function from first principles requires a long calculation and it is easy to make mistakes.
- However, we can use this method of finding the derivative from first principles to obtain rules which make finding the derivative of a function much simpler.

Basic Rules of Differentiation

a) Constant Rule

$$y = c$$
$$\frac{dy}{dx} = 0$$

b) Power Rule

$$y = x^n$$
$$\frac{dy}{dx} = nx^{n-1}$$

$$y = ax^n$$
$$\frac{dy}{dx} = nax^{n-1}$$

c) Sum & Difference Rule

$$y = f(x) \pm g(x)$$
$$\frac{dy}{dx} = f'(x) \pm g'(x)$$

d) Constant Multiple Rule

$$y = (ax^n \pm b)^k$$
$$\frac{dy}{dx} = k(ax^n \pm b)^{k-1} \frac{d}{dx}(ax^n \pm b)$$

IMPORTANT

a) Must have POWER $\quad y = \sqrt[4]{x^5} \longrightarrow y = x^{\frac{5}{4}}$

b) POWER must be numerator $\quad y = \frac{1}{3x^5} \longrightarrow y = \frac{1}{3}x^{-5}$

c) $x^0 = 1$

d) Expand $\quad y = (2x+1)(3-x) \longrightarrow y = 3 + 5x - 2x^2$

$y = (x-3)^2 \longrightarrow y = x^2 - 6x + 9$

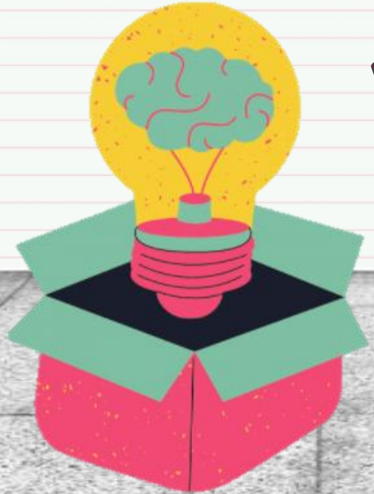
e) Separate $\quad y = \frac{x^6 - 4x^3 + 5}{x^3} \longrightarrow y = \frac{x^6}{x^3} - \frac{4x^3}{x^3} + \frac{5}{x^3}$

$y = x^3 - 4 + 5x^{-3}$

a) Constant Rule

$$y = c$$

$$\frac{dy}{dx} = 0$$



Differentiate the followings:

Example 1

$$y = \frac{4}{5}$$

Let's differentiate

Solution

$$\frac{dy}{dx} = 0$$

Example 2

$$y = 5\pi$$

Now, differentiate

Solution

$$\frac{dy}{dx} = 0$$

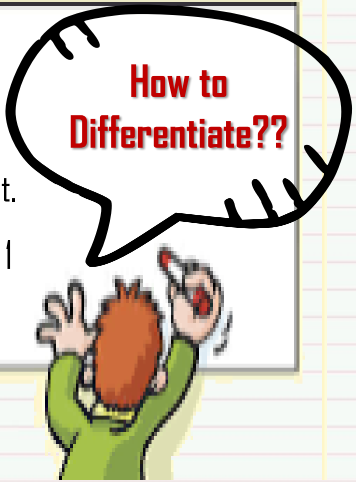
b) Power Function Rule

$$y = x^n$$
$$\frac{dy}{dx} = nx^{n-1}$$

$$y = ax^n$$
$$\frac{dy}{dx} = nax^{n-1}$$

STEPS

1. Bring the **POWER** up front.
2. **POWER** is subtracted by 1



Differentiate the followings:

Example 1

1. $y = x^5$

Let's differentiate

Solution

$$\frac{dy}{dx} = 5x^{5-1}$$
$$= 5x^4$$

Example 2

2. $y = 3x^4$

Now, differentiate

Solution

$$\frac{dy}{dx} = 4 \cdot 3x^{4-1}$$
$$= 12x^3$$

Calculus: Differentiation

Differentiate the followings:

Example 1

ATTENTION !
$\frac{x^2}{3} = \frac{1}{3}x^2$

$$y = 4x^4 - \frac{x^2}{3} - 9$$

Let's differentiate

Solution

$$\frac{dy}{dx} = (4)4x^{4-1} - (2)\frac{1}{3}x^{2-1} - 0$$
$$= 16x^3 - \frac{2}{3}x^1$$
$$= 16x^3 - \frac{2}{3}x$$

c) Sum & Difference Rule

$$y = f(x) \pm g(x)$$
$$\frac{dy}{dx} = f'(x) \pm g'(x)$$

Example 2

$$f(x) = x^5 - \frac{2}{3}x^6 - 9x$$

Now we can differentiate

Solution

$$f'(x) = (5)x^{5-1} - (6)\frac{2}{3}x^{6-1} - 9x^{1-1}$$
$$= 5x^4 - 4x^5 - 9x^0$$
$$= 5x^4 - 4x^5 - 9$$

REMEMBER !
$x^0 = 1$

Example 3

ATTENTION ! $\frac{3}{q^3} = 3q^{-3}$

$$p = 2q^3 - \frac{3}{q^3} + q^4$$

$$\therefore p = 2q^3 - 3q^{-3} + q^4$$

Solution

Then differentiate

$$\frac{dp}{dq} = (3)2q^{3-1} - (-3)3q^{-3-1} + 4q^{4-1}$$

$$= 6q^2 + 9q^{-4} + 4q^3$$

$$= 6q^2 + \frac{9}{q^4} + 4q^3$$

Example 4

ATTENTION ! $\sqrt{x^3} = x^{\frac{3}{2}}$

$$h(x) = \sqrt{x^3} - x^4 + 5x$$

$$\therefore h(x) = x^{\frac{3}{2}} - x^4 + 5x$$

Solution

Let's differentiate

$$h'(x) = \left(\frac{3}{2}\right)x^{\frac{3}{2}-1} - (4)x^{4-1} + 5x^{1-1}$$

$$= \frac{3}{2}x^{\frac{1}{2}} - 4x^3 + 5x^0$$

$$= \frac{3}{2}\sqrt{x} - 4x^3 + 5$$

REMEMBER ! $x^0 = 1$

Calculus: Differentiation

Example 5

Solution

$$y = (3x+1)(x-2)$$

$$\therefore y = 3x^2 - 5x - 2$$

REMEMBER !

$(3x+1)(x-2)$ (expand)

$= 3x^2 - 6x + x - 2$

$= 3x^2 - 5x - 2$

Now, let us differentiate

$$\frac{dy}{dx} = (2)3x^{2-1} - (1)5x^{1-1} - 0$$

$$= 6x^1 - 5x^0$$

$$= 6x - 5$$

Example 6

Solution

$$m = (2n-3)^2$$

$$\therefore m = 4n^2 - 12n + 9$$

REMEMBER !

$(2n-3)^2$ (expand)

$= (2n-3)(2n-3)$

$= 4n^2 - 6n - 6n + 9$

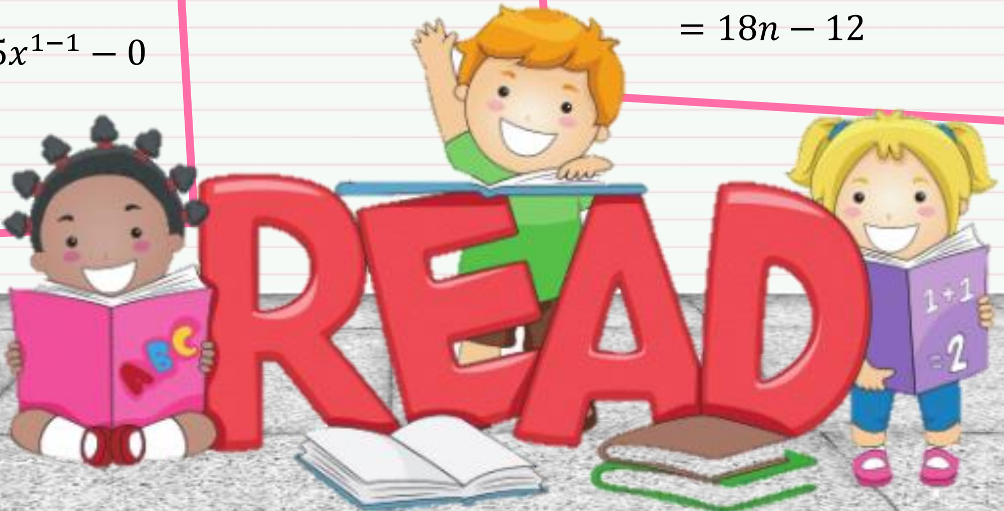
$= 4n^2 - 12n + 9$

Then we differentiate

$$\frac{dm}{dn} = (2)4n^{2-1} - (1)12n^{1-1} + 0$$

$$= 8n^1 - 12n^0$$

$$= 18n - 12$$



Example 7

$$y = \frac{x^4 + 5x^3 - \sqrt{5x}}{x}$$

$$y = x^3 + 5x^2 - \sqrt{5}x^{-\frac{1}{2}}$$

ATTENTION !

$$\sqrt{5x} = \sqrt{5}\sqrt{x}$$
$$= \sqrt{5}x^{\frac{1}{2}}$$

REMEMBER !

$$y = \frac{x^4 + 5x^3 - \sqrt{5x}}{x} \text{ (separate)}$$
$$= \frac{x^4}{x} + \frac{5x^3}{x} - \frac{\sqrt{5x}}{x}$$
$$= x^3 + 5x^2 - \sqrt{5}x^{-\frac{1}{2}}$$

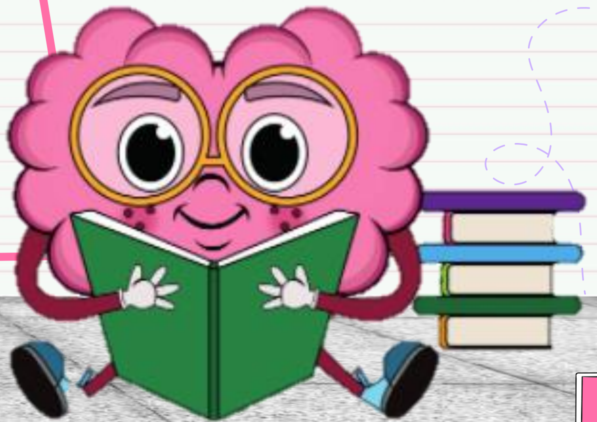
Solution

We can now differentiate

$$\frac{dy}{dx} = (3)x^{3-1} - (2)5x^{2-1} - \left(-\frac{1}{2}\right)\sqrt{5}x^{-\frac{1}{2}-1}$$

$$= 3x^2 - 10x^1 + \frac{\sqrt{5}}{2}x^{-\frac{3}{2}}$$

$$= 3x^2 - 10x + \frac{1}{2}\sqrt{\frac{5}{x^3}}$$



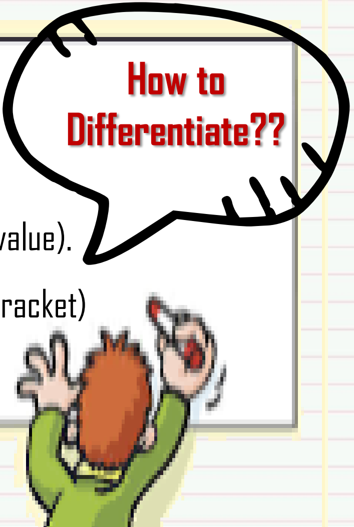
d) Constant Multiple Rule

$$y = (ax^n \pm b)^k$$

$$\frac{dy}{dx} = k(ax^n \pm b)^{k-1} \frac{d}{dx}(ax^n \pm b)$$

STEPS

1. Differentiate **OUTSIDE** (power value).
2. Differentiate **INSIDE** (value in bracket)



Differentiate the followings:

Example 1

$$y = (4x^2 + 3)^5$$

Solution

Let's differentiate

$$\frac{dy}{dx} = \underset{1}{5(4x^2+3)^{5-1}} \cdot \underset{2}{(8x)}$$

$$= 40x(4x^2+3)^4$$

Example 2

$$y = (x - 3x^2)^8$$

Solution

Differentiate the equation

$$\frac{dy}{dx} = \underset{1}{8(x-3x^2)^{8-1}} \cdot \underset{2}{(1-6x)}$$

$$= 8(1-6x)(x-3x^2)^7$$

Example 3

$$y = (x^3 - 8)^5$$

Solution

Let's differentiate

$$\frac{dy}{dx} = (5)(x^3 - 8)^{5-1} \cdot (3x^{3-1})$$

$$= 15x^2(x^3 - 8)^4$$

Example 4

$$f(x) = \frac{1}{6}(x^4 + 3x)^3$$

Solution

Now let's differentiate

$$f'(x) = (3)\frac{1}{6}(x^4 + 3x)^{3-1} \cdot (4x^{4-1} + 3x^{1-1})$$

$$= \frac{1}{2}(4x^3 + 3)(x^4 + 3x)^2$$

Example 5

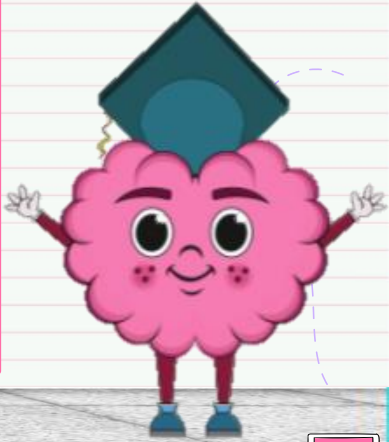
$$f(x) = (2x^3 + 5x - 3)^4$$

Solution

We can now differentiate

$$f'(x) = (4)(2x^3 + 5x - 3)^{4-1} \cdot (6x^{3-1} + 5x^{1-1})$$

$$= 4(6x^2 + 5)(2x^3 + 5x - 3)^3$$



Calculus: Differentiation

Example 6

Solution

$$y = \frac{2}{3(5-2x)^6}$$

change the form

$$y = \frac{2}{3}(5-2x)^{-6}$$

We can differentiate

$$\frac{dy}{dx} = (-6)\left(\frac{2}{3}\right)(5-2x)^{-6-1} \cdot (-2)$$

$$= 8(5-2x)^{-7}$$

$$= \frac{8}{(5-2x)^7}$$

Example 7

Solution

$$y = \sqrt[3]{x^4 - 5}$$

change the form

$$y = (x^4 - 5)^{\frac{1}{3}}$$

Then differentiate

$$\frac{dy}{dx} = \left(\frac{1}{3}\right)(x^4-5)^{\frac{1}{3}-1} \cdot (4x^{4-1})$$

$$= \frac{1}{3}(4x^3)\,(x^4-5)^{-\frac{2}{3}}$$

$$= \frac{4x^3}{3\sqrt[3]{(4x^3-5)^2}}$$

Techniques of Differentiation

CHAIN RULE

The chain rule has been known since Isaac Newton and Leibniz first discovered the calculus at the end of the 17th century. The chain rule states that **to compute the derivative of composite function,** f(g(x)) is f'(g(x))·g'(x).

a) Chain Rule

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$$

Calculus: Differentiation

a) Chain Rule

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$$

STEPS

1. Find u & y
   - u in the bracket & y the remain
2. Differentiate u & y
3. Substitute into the formula
4. Solve / simplify
5. Replace u

Differentiate the followings:

Example 1

$$y = (4x^2 + 3)^5$$

Solution

Step 1: $u = 4x^2 + 3$ $\qquad y = u^5$

Step 2: $\frac{du}{dx} = 8x$ $\qquad \frac{dy}{du} = 5u^4$

Step 3: $\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$

$\frac{dy}{dx} = 5u^4 \cdot (8x)$

Step 4: $= 40x(u)^4$

Step 5: $= 40x(4x^2 + 3)^4$

Example 2

$$y = (2x^3 + 5x - 3)^4$$

Solution

Step 1 $u = 2x^3 + 5x - 3 \qquad y = u^4$

Step 2 $\dfrac{du}{dx} = 6x + 5 \qquad \dfrac{dy}{du} = 4u^3$

Step 3 $\dfrac{dy}{dx} = \dfrac{dy}{du} \times \dfrac{du}{dx}$

$$\frac{dy}{dx} = 4u^3 \cdot (6x + 5)$$

Step 4 $= 4(6x + 5)(u)^3$

Step 5 $= 4(6x^2 + 5)(2x^3 + 5x - 3)^3$

Example 3

$$y = \frac{1}{6}(x^4 + 3x)^3$$

Solution

Step 1 $u = x^4 + 3x \qquad y = \dfrac{1}{6}u^3$

Step 2 $\dfrac{du}{dx} = 4x^3 + 3 \qquad \dfrac{dy}{du} = \dfrac{1}{2}u^2$

Step 3 $\dfrac{dy}{dx} = \dfrac{dy}{du} \times \dfrac{du}{dx}$

$$\frac{dy}{dx} = \frac{1}{2}u^2 \cdot (4x^3 + 3)$$

Step 4 $= \dfrac{1}{2}(4x^3 + 3)(u)^2$

Step 5 $= \dfrac{1}{2}(4x^3 + 3)(x^4 + 3x)^2$

Calculus: Differentiation

Example 4

$$y = \frac{2}{3(5-2x)^6}$$

change the form

Solution

$$y = \frac{2}{3}(5-2x)^{-6}$$

Step 1 $u = 5 - 2x$ $\qquad y = \frac{2}{3}u^{-6}$

Step 2 $\frac{du}{dx} = -2$ $\qquad \frac{dy}{du} = -4u^{-7}$

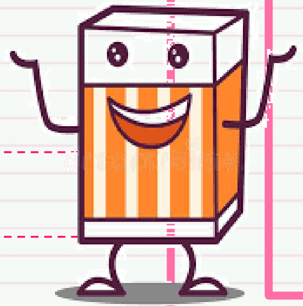
Step 3

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$$

$$\frac{dy}{dx} = -4u^{-7} \cdot (-2)$$

Step 4 $= -8(u)^{-7}$

Step 5 $= -\frac{8}{(5-2x)^7}$



Example 5

$$y = \sqrt[3]{x^4 - 5}$$

change the form

Solution

$$y = (x^4 - 5)^{\frac{1}{3}}$$

Step 1 $u = x^4 - 5$ $\qquad y = u^{\frac{1}{3}}$

Step 2 $\frac{du}{dx} = 4x^3$ $\qquad \frac{dy}{du} = \frac{1}{3}u^{-\frac{2}{3}}$

Step 3

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$$

$$\frac{dy}{dx} = \frac{1}{3}u^{-\frac{2}{3}} \cdot (4x^3)$$

Step 4 $= \frac{1}{3}(4x^3)\,(u)^{-\frac{2}{3}}$

Step 5 $= \frac{4x^3}{3\sqrt[3]{(4x^3-5)^2}}$

CONCLUSION:

If given question $y = (7 - 6x^2)^8$ therefore it can be solved either by using **Composite Function** or **Chain Rule**.

*** Refer to what method the question want !!!*

Composite Function

Solution

$$y = (7 - 6x^2)^8$$

$$\frac{dy}{dx} = 8(7 - 6x^2)^7 \cdot (-12x)$$

$$= -96x(7 - 6x^2)^7$$

Chain Rule

Solution

$$y = (7 - 6x^2)^8$$

$$u = 7 - 6x^2 \qquad y = u^8$$

$$\frac{du}{dx} = -6x \qquad \frac{dy}{du} = 8u^7$$

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$$

$$\frac{dy}{dx} = 8u^7 \cdot (-6x)$$

$$= -48x(u)^7$$

$$= -48x(7 - 6x^2)^7$$

Techniques Of Differentiation

PRODUCT RULE

If $y = u.v$ is the product of a function, then to obtain the derivative of the product of that function, **the product rule** must be used.

b) Product Rule

$$\frac{dy}{dx} = v\frac{du}{dx} + u\frac{dv}{dx}$$

ATTENTION:

If given questions as followings:

① $y = x^3(7 - 6x^2)$ Expand

$= 7x^3 - 6x^5$

② $y = (x + 2)(7 - 6x^2)$ Expand

$= 7x - 6x^3 + 14 - 12x^2$

bracket has NO POWER!

③ $y = (7 - 6x^2)^2$

$= (7 - 6x^2)(7 - 6x^2)$ Expand

$= 47 - 12x^2 + 36x^4$

bracket has the POWER of 2!

Bracket has a POWER

① $y = x^3(7 - 6x^2)^3$

② $y = (2x + 3)^4(7 - 6x^2)^3$

Differentiate using PRODUCT RULE!

Calculus: Differentiation

b) Product Rule

$$\frac{dy}{dx} = v\frac{du}{dx} + u\frac{dv}{dx}$$

STEPS

1. Find u & v
   - u in front & v at the back
2. Differentiate u & v
3. Substitute into the formula
4. Factorize
5. Solve / simplify

Example 1

Differentiate $y = 2x^3(x^2+3)^5$

Solution

Step 1 $u = 2x^3$ $\qquad v = (x^2+3)^5$

Step 2 $\frac{du}{dx} = 6x^2$ $\qquad \frac{dv}{dx} = 10x(x^2+3)^4$

Step 3

$$\frac{dy}{dx} = v\frac{du}{dx} + u\frac{dv}{dx}$$

$$\frac{dy}{dx} = (x^2+3)^5 \cdot (6x^2) + (2x^3) \cdot 10x\,(x^2+3)^4$$

$$= 6x^2(x^2+3)^5 + 20x^4(x^2+3)^4$$

Step 4

$$= 2x^2(x^2+3)^4[3(x^2+3) + 10x^2]$$

Step 5

$$= 2x^2(x^2+3)^4(3x^2+9+10x^2)$$

$$= 2x^2(x^2+3)^4(13x^2+9)$$

Example 2

Differentiate $y = (2x^3 + 1)(1 - 4x)^3$

Solution

Step 1 $u = 2x^3 + 1$ $\qquad v = (1 - 4x)^3$

Step 2 $\frac{du}{dx} = 6x^2$ $\qquad \frac{dv}{dx} = 3(1 - 4x)^2(-4)$

$= -12(1 - 4x)^2$

Step 3

$$\frac{dy}{dx} = v\frac{du}{dx} + u\frac{dv}{dx}$$

$$\frac{dy}{dx} = (1 - 4x)^3 \cdot 6x^2 + (2x^3 + 1\ ) \cdot -12(1 - 4x)^2$$

$$= 6x^2(1 - 4x)^3 - 12(2x^3 + 1\ )(1 - 4x)^2$$

Step 4

$$= 6(1 - 4x)^2[x^2(1 - 4x) - 2(2x^3 + 1)]$$

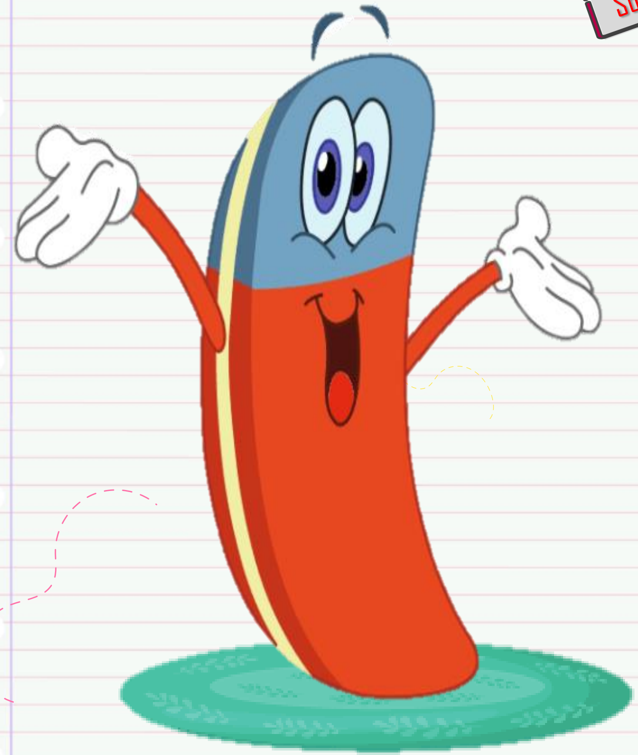
Step 5

$$= 6(1 - 4x)^2(x^2 - 4x^3 - 4x^3 - 2)$$

$$= 6(1 - 4x)^2(x^2 - 8x^3 - 2)$$

Example 3

Differentiate $y = (x-1)^3(3x+1)^5$



Solution

Step 1

$u = (x-1)^3$ $\qquad$ $v = (3x+1)^5$

Step 2

$$\frac{du}{dx} = 3(x-1)^2(1) = 3(x-1)^2$$

$$\frac{dv}{dx} = 5(3x+1)^4(3) = 15(3x+1)^4$$

Step 3

$$\frac{dy}{dx} = v\frac{du}{dx} + u\frac{dv}{dx}$$

$$\frac{dy}{dx} = (3x+1)^5 \cdot 3(x-1)^2 + (x-1)^3 \cdot 15(3x+1)^4$$

$$= 3(3x+1)^5(x-1)^2 + 15(x-1)^3(3x+1)^4$$

Step 4

$$= 3(3x+1)^4(x-1)^2[(3x+1) + 5(x-1)]$$

Step 5

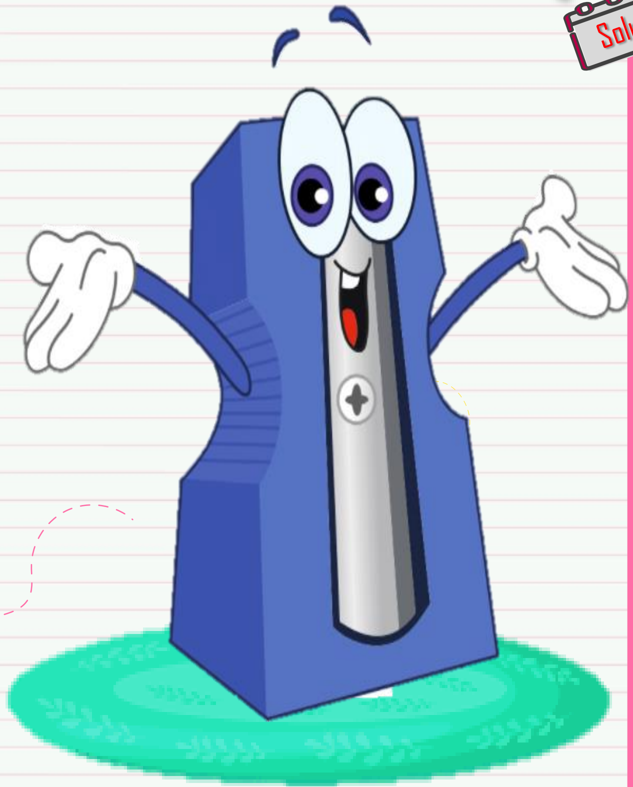
$$= 3(3x+1)^4(x-1)^2(3x+1+5x-5)$$

$$= 3(3x+1)^4(x-1)^2(8x-4)$$

$$= 12(3x+1)^4(x-1)^2(2x-1)$$

Example 4

Find the derivative for $y = \sqrt{2x+3}(2-3x^2)^5$



Solution

Step 1

$$u = \sqrt{2x+3} = (2x+3)^{\frac{1}{2}}$$

$$v = (2-3x^2)^5$$

Step 2

$$\frac{du}{dx} = \frac{1}{2}(2x+3)^{-\frac{1}{2}}(2) = (2x+3)^{-\frac{1}{2}}$$

$$\frac{dv}{dx} = 5(2-3x^2)^4(-6x) = -30x(2-3x^2)^4$$

Step 3

$$\frac{dy}{dx} = v\frac{du}{dx} + u\frac{dv}{dx}$$

$$\frac{dy}{dx} = (2-3x^2)^5 \cdot (2x+3)^{-\frac{1}{2}} + (2x+3)^{\frac{1}{2}}[-30x(2-3x^2)^4]$$

$$= (2-3x^2)^5 \cdot (2x+3)^{-\frac{1}{2}} - 30x(2x+3)^{\frac{1}{2}}(2-3x^2)^4$$

Step 4

$$= (2-3x^2)^4(2x+3)^{-\frac{1}{2}}[(2-3x^2) - 30x(2x+3)]$$

Step 5

$$= (2-3x^2)^4(2x+3)^{-\frac{1}{2}}[2-3x^2-60x^2-90x]$$

$$= (2-3x^2)^4(2x+3)^{-\frac{1}{2}}[2-63x^2-90x]$$

Techniques Of Differentiation

QUOTIENT RULE

The quotient rule is a method of finding the derivative of a function that is the ratio of two differentiable functions.

Its function is in the form $\frac{u}{v}$

c) Quotient Rule

$$\frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$$

Calculus: Differentiation

Quotient Rule

$$\frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$$

STEPS

1. Find u & v
   - u is nominator & v is denominator
2. Differentiate u & v
3. Substitute into the formula
4. Factorize
5. Solve / simplify

Example 1

Differentiate $y = \frac{2x^3}{(x^2+3)^5}$

Solution

Step 1 $u = 2x^3$ $\qquad v = (x^2+3)^5$

Step 2 $\frac{du}{dx} = 6x^2$ $\qquad \frac{dv}{dx} = 5(x^2+3)^4(2x)$

$$= 10x(x^2+3)^4$$

Step 3

$$\frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$$

$$\frac{dy}{dx} = \frac{(x^2+3)^5 \cdot (6x^2) - (2x^3) \cdot 10x(x^2+3)^4}{\left((x^2+3)^5\right)^2}$$

$$= \frac{6x^2(x^2+3)^5 - 20x^4(x^2+3)^4}{(x^2+3)^{10}}$$

Step 4

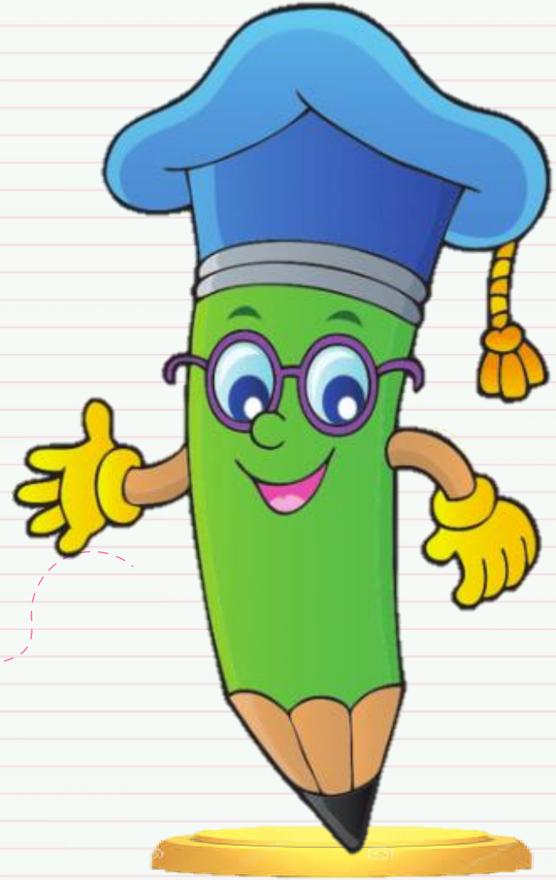
$$= \frac{2x^2(x^2+3)^4[3(x^2+3) - 10x^2]}{(x^2+3)^{10}}$$

Step 5

$$= \frac{2x^2(3x^2+9-10x^2)}{(x^2+3)^6} = \frac{2x^2(9-7x^2)}{(x^2+3)^6}$$

Example 2

Find the derivative for $y = \frac{(2x^3+1)}{(1-4x)^3}$



Solution

Step 1

$$u = 2x^3 + 1 \qquad v = (1 - 4x)^3$$

Step 2

$$\frac{du}{dx} = 6x^2 \qquad \frac{dv}{dx} = 3(1 - 4x)^2(-4)$$

$$= -12(1 - 4x)^2$$

Step 3

$$\frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$$

$$\frac{dy}{dx} = \frac{(1 - 4x)^3 \cdot 6x^2 - (2x^3 + 1\,) \cdot -12(1 - 4x)^2}{\left((1 - 4x)^3\right)^2}$$

$$= \frac{6x^2(1 - 4x)^3 + 12(2x^3 + 1\,)(1 - 4x)^2}{(1 - 4x)^6}$$

Step 4

$$= \frac{6(1 - 4x)^2[x^2(1 - 4x) + 2(2x^3 + 1\,)]}{(1 - 4x)^6}$$

Step 5

$$= \frac{6(x^2 - 4x^3 + 4x^3 + 2)}{(1 - 4x)^4} = \frac{6(x^2 + 2)}{(1 - 4x)^4}$$

Example 3

Find the derivative for $y = \frac{(x-1)^3}{(3x+1)^5}$

Solution

Step 1

$u = (x-1)^3$ $\qquad$ $v = (3x+1)^5$

Step 2

$$\frac{du}{dx} = 3(x-1)^2(1) = 3(x-1)^2$$

$$\frac{dv}{dx} = 5(3x+1)^4(3) = 15(3x+1)^4$$

Step 3

$$\frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$$

$$\frac{dy}{dx} = \frac{(3x+1)^5 \cdot 3(x-1)^2 - (x-1)^3 \cdot 15(3x+1)^4}{\left((3x+1)^5\right)^2}$$

$$= \frac{3(3x+1)^5(x-1)^2 - 15(x-1)^3(3x+1)^4}{(3x+1)^{10}}$$

Step 4

$$= \frac{3(3x+1)^4(x-1)^2[(3x+1) - 5(x-1)]}{(3x+1)^{10}}$$

Step 5

$$= \frac{3(x-1)^2(3x+1-5x+5)}{(3x+1)^6} = \frac{3(x-1)^2(6-2x)}{(3x+1)^6} = \frac{6(x-1)^2(3-x)}{(3x+1)^6}$$

Calculus: Differentiation

Example 4

Find the derivative for $s = \frac{(7t+2)^2}{4(t^3+1)^3}$

Solution

Step 1

$u = (7t+2)^2 \qquad v = 4(t^3+1)^3$

Step 2

$$\frac{du}{dt} = 2(7t+2)(7) = 14(7t+2)$$

$$\frac{dv}{dt} = 12(t^3+1)^2(3t^2) = 36t^2(t^3+1)^2$$

Step 3

$$\frac{ds}{dt} = \frac{v\frac{du}{dt} - u\frac{dv}{dt}}{v^2}$$

$$\frac{ds}{dt} = \frac{4(t^3+1)^3 \cdot 14(7t+2) - (7t+2)^2 \cdot 36t^2(t^3+1)^2}{(4(t^3+1)^3)^2}$$

$$= \frac{56(t^3+1)^3 \cdot (7t+2) - 36t^2(t^3+1)^2(7t+2)^2}{16(t^3+1)^6}$$

Step 4

$$= \frac{4(t^3+1)^2(7t+2)[14(t^3+1) - 9t^2(7t+2)]}{16(t^3+1)^6}$$

Step 5

$$= \frac{4(t^3+1)^2(7t+2)(14t^3+14-63t^3-18t^2)}{16(t^3+1)^6}$$

$$= \frac{(7t+2)(14-18t^2-49t^3)}{4(t^3+1)^4}$$

Second Order Of Differentiation

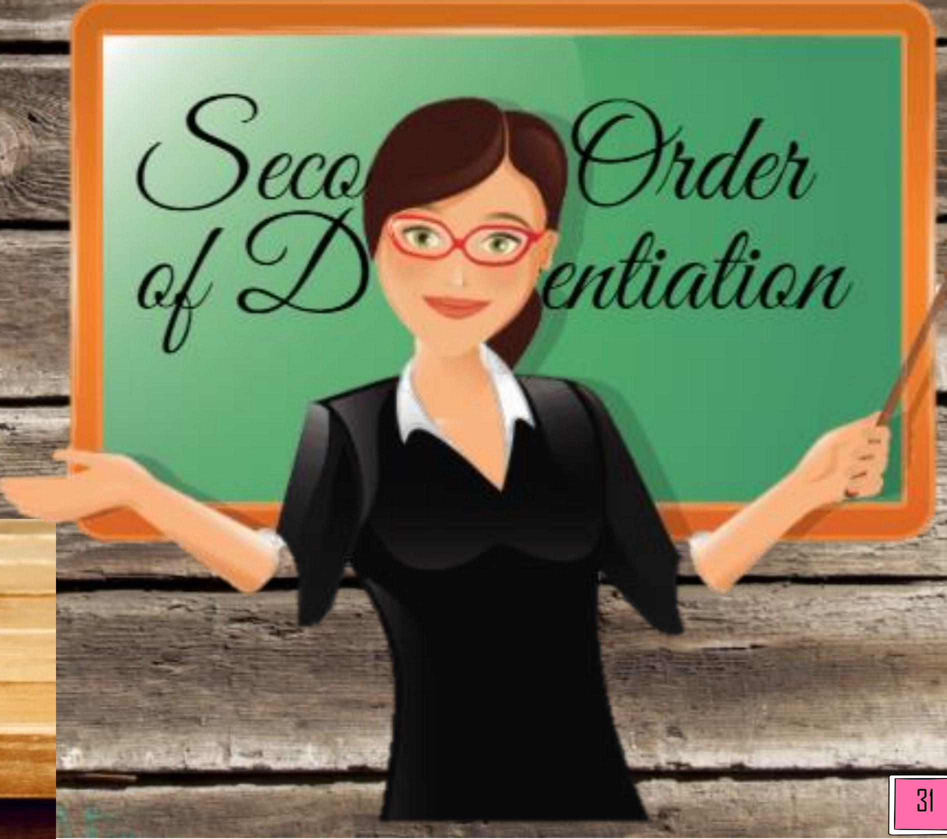
- **The second order of differentiation** is when the differentiation made twice
- The second derivatives of a function $f(x)$ is usually denoted as $f''(x)$ and for $y = f(x)$, then the second order written as $\frac{d^2y}{dx^2}$
- Among the uses of the second order of differentiation is to determine the optimum value.

differentiate first time → differentiate second time

$$f(x) \rightarrow f'(x) \rightarrow f''(x)$$

differentiate first time → differentiate second time

$$y \rightarrow \frac{dy}{dx} \rightarrow \frac{d^2y}{dx^2}$$



Example 1

Find the derivative for the followings:

1. $y = 4x^3 + 2x$

Solution

$$\frac{dy}{dx} = 12x^2 + 2$$

$$\frac{d^2y}{dx^2} = 24x$$

2. $y = (2 + 3x)(x - 1)$

Solution

change the form

$$y = 3x^2 - x - 2$$

$$\frac{dy}{dx} = 6x - 1$$

$$\frac{d^2y}{dx^2} = 6$$

3. $y = (5 + 3x^4)^5$

Solution

$$\frac{dy}{dx} = 5(5 + 3x^4)^4 \cdot 12x^3$$

$$= 60x^3(5 + 3x^4)^4$$

For 2nd order derivative, must use product rule : $\frac{dy}{dx} = v\frac{du}{dx} + u\frac{dv}{dx}$

$$\frac{d^2y}{dx^2} = 60x^3 \cdot 4(5 + 3x^4)^3 12x^3 + (5 + 3x^4)^4 180x^2$$

$$= 2880x^6(5 + 3x^4)^3 + 180x^2(5 + 3x^4)^4$$

$$= 20x^2(5 + 3x^4)^3[144x^4 + 9\,(5 + 3x^4)]$$

$$= 20x^2(5 + 3x^4)^3(144x^4 + 45 + 27x^4)$$

Example 2

Find the value of $\frac{d^2y}{dx^2}$ when $x = 3$ for each of the following

1 $y = 5x^3 - \frac{6}{x^2} + 9x + 3$

Solution

change the form

$y = 5x^3 - 6x^{-2} + 9x + 3$

$\frac{dy}{dx} = 15x^2 + 12x^{-3} + 9$

$\frac{d^2y}{dx^2} = 30x - 36x^{-4}$

$= 30x - \frac{36}{x^4}$

Substitute $x = 3$ into $\frac{d^2y}{dx^2}$

$\frac{d^2y}{dx^2} = 30(3) - \frac{36}{3^4} = \frac{806}{9}$

2 $y = x(2x - 1)^4$

Must use product rule for 1st order and 2nd order derivative :

$\frac{dy}{dx} = v\frac{du}{dx} + u\frac{dv}{dx}$

Solution

$\frac{dy}{dx} = x \cdot 4(2x-1)^3 \cdot 2 + (1)(2x-1)^4 \cdot 2$

$= 8x(2x-1)^3 + 2(2x-1)^4$

$\frac{d^2y}{dx^2} = [8x \cdot 3(2x-1)^2 \cdot 2 + (2x-1)^3 \cdot 8] + [4 \cdot 2(2x-1)^3 \cdot 2\ ]$

$= 48x(2x-1)^2 + 8(2x-1)^3 + 16(2x-1)^3$

$= 48x(2x-1)^2 + 24(2x-1)^3$

$= 6(2x-1)^2[8x + 4(2x-1)]$

$= 6(2x-1)^2(16x-4)$

Substitute $x = 3$ into $\frac{d^2y}{dx^2}$

$\frac{d^2y}{dx^2} = 6(2(3)-1)^2[16(3)-4] = 6600$

Differentiation of Trigonometric Function

- **Trigonometric differentiation** needs to use a fixed formula to solve it.
- It can be solved either using the trigonometric differentiation formula or a chain rule.
- A combination of either a product rule or a quotient rule should be used if it involves two functions.

Trigonometry Function

Basic Derivatives

$$\frac{d}{dx}(sinx) = cosx$$

$$\frac{d}{dx}(cosx) = -sinx$$

$$\frac{d}{dx}(tanx) = sec^2x$$

$$\frac{d}{dx}(cosecx) = -cosec\ x\ cotx$$

$$\frac{d}{dx}(secx) = secx\ tanx$$

$$\frac{d}{dx}(cotx) = -cosec^2x$$

Reciprocal Identities

$$tanx = \frac{sinx}{cosx}$$

$$cosecx = \frac{1}{sinx}$$

$$secx = \frac{1}{cosx}$$

$$cotx = \frac{1}{tanx} = \frac{cosx}{sinx}$$

Trigonometric Identities

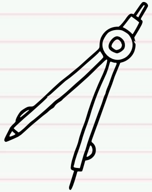
$$cos^2x + sin^2x = 1$$

$$1 + tan^2x = sec^2x$$

$$cot^2x + 1 = cosec^2x$$

How To Derive Trigonometry Function

Method 1 (trigonometry with power 1)



STEPS

1. Differentiate trigonometry function
2. Differentiate the angle (in bracket)

Examples :

$y = sin5x,$

$y = \sin(5x + 4)$

Method 2 (trigonometry with power > 1)

STEPS

1. Differentiate Power
2. Differentiate trigonometry function
3. Differentiate the angle (in bracket)

Examples :

$y = sin^2 5x,$

$y = sin^4 (5x + 4)$

Trigonometry Function

1 STEPS

1) Differentiate trigonometry function
2) Differentiate the angle (in bracket)

$y = sin2x$

Solution

Let's differentiate

$$\frac{dy}{dx} = \overset{1}{\boxed{cos2x}} \cdot \overset{2}{\boxed{(2)}}$$

$$= 2\ cos2x$$

$y = sin(1 - 2x)$

Solution

Now, differentiate

$$\frac{dy}{dx} = \overset{1}{\boxed{cos(1 - 2x)}} \cdot \overset{2}{\boxed{(-2)}}$$

$$= -2\ cos(1 - 2x)$$

2 STEPS

1) Differentiate Power
2) Differentiate trigonometry function
3) Differentiate the angle (in bracket)

$y = sin^3 2x$

Solution

Then differentiate

$$\frac{dy}{dx} = \overset{1}{\boxed{3sin^2 2x}} \cdot \overset{2}{\boxed{cos2x}} \cdot \overset{3}{\boxed{(2)}}$$

$$= 6\ sin^2 2x\ cos2x$$

Calculus: Differentiation

Example 1

Find the derivative for the followings:

1

$y = cos3x$

Solution

$$\frac{dy}{dx} = -sin3x \cdot (3)$$

$$= -3\ sin3x$$

2

$y = \frac{1}{2}tan4x$

Solution

$$\frac{dy}{dx} = \frac{1}{2}sec^2 4x \cdot (4)$$

$$= 2\ sec^2(1-4x)$$

3

$y = sin\frac{2}{3}x$

Solution

$$\frac{dy}{dx} = cos\frac{2}{3}x \cdot \left(\frac{2}{3}\right)$$

$$= \frac{2}{3}\ cos\frac{2}{3}x$$

4

$y = cot\sqrt{x}$

$= cotx^{\frac{1}{2}}$

change the form

Solution

$$\frac{dy}{dx} = -cosec\sqrt{x} \cdot \left(\frac{1}{2}x^{-\frac{1}{2}}\right)$$

$$= -\frac{cosec^2\sqrt{x}}{2\sqrt{x}}$$

5

$y = \frac{3}{8}\cos\frac{2}{x^4}$

$= \frac{3}{8}\cos(2x^{-4})$

change the form

Solution

$$\frac{dy}{dx} = \frac{3}{8}(-sin2x^{-4}) \cdot (-8x^{-5})$$

$$= \frac{3}{x^5}\ sin\frac{2}{x^4}$$

Example 2

Find the derivative for the followings:

1 Solution

$$y = cos(3x + 2)$$

$$\frac{dy}{dx} = -sin(3x+2) \cdot (3)$$

$$= -3\ sin(3x+2)$$

2 Solution

$$y = \frac{3}{8} sin(1-4x)^3$$

composite function

$$\frac{dy}{dx} = \frac{3}{8} cos(1-4x)^3 \cdot [(3(1-4x)^2)(-4)]$$

$$= -\frac{9}{2}(1-4x)^2\ cos(1-4x)^3$$

3 Solution

$$y = cos(5x + x^3)$$

$$\frac{dy}{dx} = -sin(5x+x^3) \cdot (5+3x^2)$$

$$= -(5+3x^2)\ sin(5x+x^3)$$

4 Solution

$$y = tan\left(7 - \frac{2}{x^4}\right)$$

$$= tan(7 - 2x^{-4})$$

change the form

$$\frac{dy}{dx} = sec^2(7-2x^{-4}) \cdot (-8x^{-5})$$

$$= -\frac{8}{x^5} sec^2\left(7 - \frac{2}{x^4}\right)$$

5 Solution

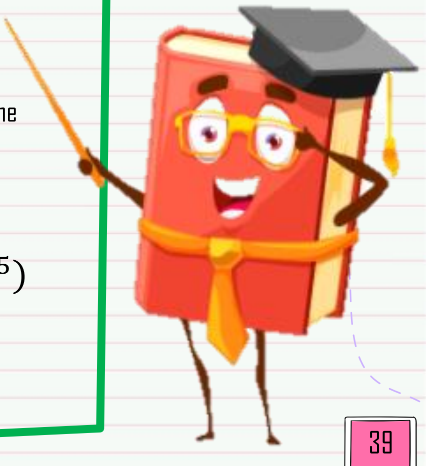
$$y = 4cot(1-\sqrt{x})$$

$$= 4cot\left(1 - x^{\frac{1}{2}}\right)$$

change the form

$$\frac{dy}{dx} = 4\left(-cosec^2(1-\sqrt{x})\right) \cdot \left(-\frac{1}{2}x^{-\frac{1}{2}}\right)$$

$$= \frac{2cosec^2(1-\sqrt{x})}{\sqrt{x}}$$



Calculus: Differentiation

Example 3

Find the derivative for the followings:

1

Solution

$$y = cos^3 4x$$

$$\frac{dy}{dx} = 3 \cdot cos^2 4x \cdot -\sin 4x\,(4)$$

$$= -12\, cos^2 4x\, sin 4x$$

3

Solution

$$y = 2cos^2(5x + x^3)$$

$$\frac{dy}{dx} = 2 \cdot 2cos(5x + x^3) \cdot -sin(5x + x^3) \cdot (5 + 3x^2)$$

$$= -4(5 + 3x^2)cos(5x + x^3)sin(5x + x^3)$$

2

Solution

$$y = \frac{3}{5} sin^5(1 - 4x)$$

$$\frac{dy}{dx} = 5 \cdot \frac{3}{5} sin^4(1 - 4x) \cdot cos(1 - 4x) \cdot (-4)$$

$$= -12 sin^4(1 - 4x) \cdot cos(1 - 4x)$$

4

Solution

$$y = tan^4\left(7 - \frac{1}{x^4}\right)$$

change the form

$$= tan^4(7 - x^{-4})$$

$$\frac{dy}{dx} = 4 \cdot tan^3(7 - x^{-4}) \cdot sec^2(7 - x^{-4}) \cdot (4x^{-5})$$

$$= \frac{16}{x^5} tan^3(7 - x^{-4}) sec^2\left(7 - \frac{1}{x^4}\right)$$

Example 4

Find the derivative for the followings:

A — Chain Rule

$$y = sin^3(tanx^3)$$

Solution

$u = tanx^3$ $\quad\quad$ $y = sin^3u$

$\frac{du}{dx} = sec^2x^3(3x^2)$ $\quad\quad$ $\frac{dy}{du} = 3sin^2u\ cosu$

$= 3x^2sec^2x^3$

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$$

$$\frac{dy}{dx} = 3sin^2u\ cosu \cdot (3x^2sec^2x^3)$$

$$= 9x^2sec^2x^3sin^2u\ cosu$$

$$= 9x^2\ sec^2x^3\ sin^2(tanx^3)\ cos(tanx^3)$$

B — Product Rule

$$y = x^4\ sin(5 - 4x^2)$$

Solution

$u = x^4$ $\quad\quad$ $v = sin(5 - 4x^2)$

$\frac{du}{dx} = 4x^3$ $\quad\quad$ $\frac{dv}{dx} = cos(5 - 4x^2)(-8x)$

$= -8x\ cos(5 - 4x^2)$

$$\frac{dy}{dx} = v\frac{du}{dx} + u\frac{dv}{dx}$$

$$\frac{dy}{dx} = sin(5 - 4x^2) \cdot 4x^3 + x^4 \cdot -8x\ cos(5 - 4x^2)$$

$$= 4x^3sin(5 - 4x^2) - 8x^5cos(5 - 4x^2)$$

$$= 4x^3[sin(5 - 4x^2) - 2x^2cos(5 - 4x^2)]$$

Quotient Rule

$$y = \frac{cos(x^2+5)}{(x^3+2x^2)}$$

Solution

$u = cos(x^2+5)$

$\frac{du}{dx} = -sin(x^2+5) \cdot 2x$

$= -2x\, sin(x^2+5)$

$v = (x^3+2x^2)$

$\frac{dv}{dx} = 3x^2+4x$

$$\frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$$

$$\frac{dy}{dx} = \frac{(x^3+2x^2) \cdot -2x\, sin(x^2+5) - cos(x^2+5) \cdot (3x^2+4x)}{(x^3+2x^2)^2}$$

$$= \frac{-2x(x^3+2x^2)sin(x^2+5) - (3x^2+4x)cos(x^2+5)}{(x^3+2x^2)^2}$$

Differentiation Of Logarithm Function

- The expression for the derivative of the natural logarithm function is $y = ln|x| = (ln|x|)' = \frac{1}{x}$.
- It can be solved either using the logarithm differentiation formula or a chain rule.
- The law of logarithms should be applied first if necessary before differentiation.
- Logarithm differentiation can also involve the **product rule** or **quotient rule.**

Logarithmic Function

Formula Differentiation of Logarithm Function

a) $\frac{d}{dx} ln|x| = \frac{1}{x}$

b) $\frac{d}{dx} ln|ax+b| = \frac{1}{ax+b}\frac{d}{dx}(ax+b)$

c) $\frac{d}{dx}\ln u = \frac{1}{u}\cdot\frac{du}{dx}$

where u is any value in bracket

d) $\frac{d}{dx}\ln f(x) = \frac{f'(x)}{f(x)}$

REMEMBER

Law of Logarithm

$log_a M + log_a N = log_a MN$

$log_a M - log_a N = log_a \frac{M}{N}$

$log_a M^b = b\, log_a M$

$log_a a = 1$

$log_N M = \frac{log_a M}{log_a N}$

How To Derive Logarithmic Function

STEPS

1) Write $\frac{1}{ax+b}$
2) Differentiate $(ax+b)$

OR

1) Write $\frac{1}{u}$
2) Differentiate u

(*u is any value in bracket*)

1 $y = \ln 2x$

Solution

Now, differentiate

$$\frac{dy}{dx} = \frac{1}{2x} \cdot (2)$$

$$= \frac{1}{x}$$

u = 2x

STEPS

1) Write $\frac{1}{u}$
2) Differentiate u

2 $y = \ln(1-2x)$

Solution

Let's differentiate

$$\frac{dy}{dx} = \frac{1}{(1-2x)} \cdot (-2)$$

u = 1 - 2x

STEPS

1) Write $\frac{1}{u}$
2) Differentiate u

OR $\frac{d}{dx}\ln f(x) = \frac{f'(x)}{f(x)}$

Example 1

Find the derivative for the followings:

1

$y = \ln 6$

Solution

$\frac{dy}{dx} = 0$

differentiate constant

2

$y = 2 \ln 5x$

Solution

$\frac{dy}{dx} = 2 \cdot \frac{1}{5x} \cdot 5$

$= \frac{2}{x}$

3

$y = \ln 2x^5$

$= 5 \ln 2x$

Simplify using law of log
$log_a M^b = b\ log_a M$

Solution

$\frac{dy}{dx} = 5 \cdot \frac{1}{2x} \cdot 2$

$= \frac{5}{x}$

4

$y = \ln|5x - 3|$

Solution

$\frac{dy}{dx} = \frac{1}{(5x-3)} \cdot 5$

$= \frac{5}{(5x-3)}$

Calculus: Differentiation

5

$$y = \ln(3x-4)^5$$

$$= 5\ln(3x-4)$$

Simplify using law of log
$log_a M^b = b\ log_a M$

Solution

$$\frac{dy}{dx} = 5\cdot\frac{1}{(3x-4)}\cdot 3$$

$$= \frac{15}{(3x-4)}$$

6

$$y = \sqrt{3}\ln(7-3x-4x^2)$$

Solution

$$\frac{dy}{dx} = \sqrt{3}\cdot\frac{1}{(7-3x-4x^2)}\cdot -3-8x$$

$$= \frac{\sqrt{3}(-3-8x)}{(7-3x-4x^2)}$$

7

$$y = \ln\sqrt[3]{5x-2}$$

$$= \ln(5x-2)^{\frac{1}{3}}$$

$$= \frac{1}{3}\ln(5x-2)$$

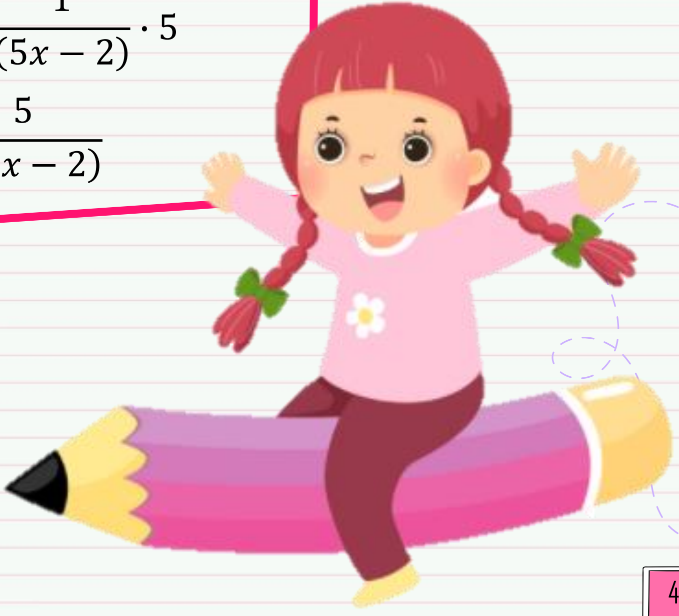
change the form

Simplify using law of log
$log_a M^b = b\ log_a M$

Solution

$$\frac{dy}{dx} = \frac{1}{3}\cdot\frac{1}{(5x-2)}\cdot 5$$

$$= \frac{5}{3(5x-2)}$$



8

$$f(x) = \ln(5x-3)^2(3x^2+4)$$

$$= \ln(5x-3)^2 + \ln(3x^2+4)$$

Simplify using law of log
$log_2 mn = log_2 m + log_2 m$

$$= 2\ln(5x-3) + \ln(3x^2+4)$$

Simplify using law of log
$log_a M^b = b\, log_a\, M$

Solution

$$f'(x) = 2 \cdot \frac{1}{(5x-3)} \cdot 5 + \frac{1}{(3x^2+4)} \cdot 6x$$

$$= \frac{10}{(5x-3)} + \frac{6x}{(3x^2+4)}$$

9

$$y = \ln\frac{7x}{1-3x^2}$$

$$= \ln 7x - \ln(1-3x^2)$$

Simplify using law of log
$log_2 \frac{m}{n} = log_2 m - log_2 m$

Solution

$$\frac{dy}{dx} = \frac{1}{7x} \cdot 7 - \frac{1}{(1-3x^2)} \cdot -6x$$

$$= \frac{1}{x} + \frac{6x}{(1-3x^2)}$$

Calculus: Differentiation

Example 2

Find the derivative for the followings:

Product Rule

1 $g(x) = 2x^4 \ln(5-2x)$

Solution

$u = 2x^4$ $\qquad$ $v = \ln(5-2x)$

$\frac{du}{dx} = 8x^3$ $\qquad$ $\frac{dv}{dx} = \frac{1}{(5-2x)} \cdot -2$

$$= \frac{-2}{(5-2x)}$$

$$\frac{dy}{dx} = v\frac{du}{dx} + u\frac{dv}{dx}$$

$$g'(x) = \ln(5-2x) \cdot 8x^3 + 2x^4 \cdot \frac{-2}{(5-2x)}$$

$$= 8x^3 \ln(5-2x) - \frac{4x^4}{(5-2x)}$$

Factorize

$$= 4x^3\left(2\ln(5-2x) - \frac{x}{(5-2x)}\right)$$

Quotient Rule

2 $f(x) = \frac{\ln(5-2x)}{2x^4}$

Remember!
$\frac{ln|5-2x|}{2x^4} \neq ln\frac{|5-2x|}{|2x^4|}$

Solution

$u = \ln(5-2x)$ $\qquad$ $v = 2x^4$

$\frac{du}{dx} = \frac{-2}{(5-2x)}$ $\qquad$ $\frac{dv}{dx} = 8x^3$

$$\frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$$

$$f'(x) = \frac{2x^4 \cdot \frac{-2}{(5-2x)} - \ln(5-2x) \cdot 8x^3}{(2x^4)^2}$$

$$= \frac{-4x^3\left[\frac{x}{(5-2x)} + 2\ln(5-2x)\right]}{4x^8}$$

Factorize

$$= -\frac{1}{x^5}\left(\frac{x}{(5-2x)} + 2\ln(5-2x)\right)$$

Example 3

Find the derivative for the followings:

Chain Rule

1 $y = tan(\ln 3x)$

Solution

$u = \ln 3x$

$\frac{du}{dx} = \frac{1}{3x} \cdot 3 = \frac{1}{x}$

$y = tanu$

$\frac{dy}{du} = sec^2 u$

$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$

$\frac{dy}{dx} = sec^2 u \cdot \frac{1}{x}$

$= \frac{sec^2(\ln 3x)}{x}$

Chain Rule

2 $y = ln(\cos 2x^3)$

Solution

$u = \cos 2x^3$

$\frac{du}{dx} = -sin\, 2x^3 \cdot 6x^2$

$= -6x^2 \, sin\, 2x^3$

$y = lnu$

$\frac{dy}{du} = \frac{1}{u}$

$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$

$\frac{dy}{dx} = \frac{1}{u} \cdot (-6x^2 \, sin\, 2x^3)$

$= \frac{-6x^2 \, sin\, 2x^3}{u}$

$= \frac{-6x^2 \, sin\, 2x^3}{\cos 2x^3}$

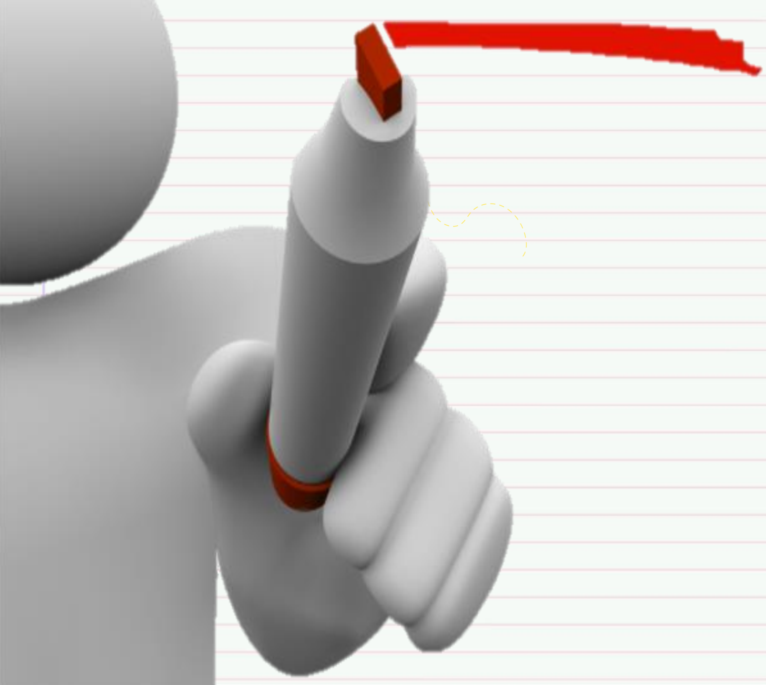
$= -6x^2 tan2x^3$

Simplify using reciprocal identities

$tanx = \frac{sinx}{cosx}$

#TakeNote

IMPORTANT!



$g(x) = ln(2x-3)(2-x^3)$	Simplify using law of log $log_2mn = log_2m + log_2m$
$g(x) = ln(2x-3)ln(2-x^3)$	Use Product Rule $\frac{dy}{dx} = v\frac{du}{dx} + u\frac{dv}{dx}$
$y = ln\frac{7x}{1-3x^2}$	Simplify using law of log $log_2\frac{m}{n} = log_2m - log_2m$
$y = \frac{ln7x}{ln(2-3x^2)}$	Use Quotient Rule $\frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$
$f(x) = ln(5x-3)^2$	Simplify using law of log $log_aM^b = b\ log_a M$
$f(x) = \left(ln(5x-3)\right)^2$	Use Chain Rule $\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$

Differentiation Of Exponential Function

- **Exponential functions** have the form $f(x) = e^x$, then the derivative is given by $\frac{d}{dx}(e^x) = (e^x)' = e^x$
- Its can be solved either using the exponent differentiation formula or a chain rule.
- A combination of either a product rule or a quotient rule should be used if it involves two functions.

Exponential Function

Formula Differentiation of Exponential Function

a) $\frac{d}{dx}(e^x) = e^x$

b) $\frac{d}{dx}(e^{ax+b}) = e^{ax+b} \cdot \frac{d}{dx}(ax+b)$

c) $\frac{d}{dx}e^u = e^u \cdot \frac{du}{dx}$

where u is any value in bracket

d) $\frac{d}{dx}e^{f(x)} = e^{f(x)} \cdot f'(x)$

REMEMBER

Law of Indices / Exponent

$a^m \times a^n = a^{m+n}$

$a^m \div a^n = a^{m-n}$

$(a^m)^n = a^{mn}$

$(ab)^n = a^n b^n$

$a^{-n} = \frac{1}{a^n}$

$\sqrt[n]{a^m} = a^{\frac{m}{n}}$

$a^0 = 1$

How To Derive Exponential Function

STEPS

1) Write the exponent function
2) Differentiate the POWER

OR

1) Write e^u
2) Differentiate u
   (*u is power of exponent*)

① $y = e^{2x}$

Solution

Let's differentiate

$$\frac{dy}{dx} = \overset{1}{e^{2x}} \cdot \overset{2}{(2)}$$

$$= 2e^{2x}$$

u = 2x

STEPS

1) Write e^u
2) Differentiate u

② $y = 2e^{7x-3}$

Solution

Let's differentiate

$$\frac{dy}{dx} = \overset{1}{2e^{7x-3}} \cdot \overset{2}{(7)}$$

$$= 14e^{7x-3}$$

u = 7x - 3

STEPS

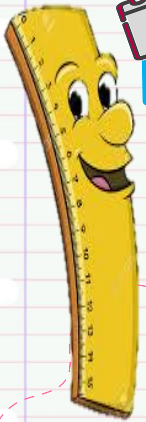
1) Write e^u
2) Differentiate u

OR $$\frac{d}{dx}e^{f(x)} = e^{f(x)}f'(x)$$

Calculus: Differentiation

Example 1

Find the derivative for the followings:



1 $y = e^2$

Solution

$$\frac{dy}{dx} = 0$$

differentiate constant

2 $y = e^{\sqrt{3}x}$

Solution

$$\frac{dy}{dx} = e^{\sqrt{3}x} \cdot \sqrt{3}$$

$$= \sqrt{3}e^{\sqrt{3}x}$$

3 $y = 6e^{4x}$

Solution

$$\frac{dy}{dx} = 6 \cdot e^{4x} \cdot 4$$

$$= 24e^{4x}$$

4 $y = e^{2x-3}$

Solution

$$\frac{dy}{dx} = e^{2x-3} \cdot 2$$

$$= 2e^{2x-3}$$

5 $y = (2e^{4x})^2$

$$= 4e^{8x}$$

Solution

$$\frac{dy}{dx} = 4e^{8x} \cdot 8$$

$$= 32e^{8x}$$

6 $y = \frac{5}{3e^{3x}}$

$$= \frac{5}{3}e^{-3x}$$

Solution

$$\frac{dy}{dx} = \frac{5}{3}e^{-3x} \cdot -3$$

$$= -\frac{15}{3e^{3x}}$$

Example 2

Find the derivative for the followings:

1

$y = e^{2-8x}(6e^{3x})$

Simplify using law of index $a^m \times a^n = a^{m+n}$

Solution

$= 6e^{2-8x+3x}$

$= 6e^{2-5x}$

$\frac{dy}{dx} = 6e^{2-5x} \cdot -5$

$= -30e^{2-5x}$

2

$y = \frac{3e^{5-2x}}{8e^{4x}}$

Simplify using law of index $a^m \div a^n = a^{m-n}$

Solution

$= \frac{3}{8}e^{5-2x-4x}$

$= \frac{3}{8}e^{5-6x}$

$\frac{dy}{dx} = \frac{3}{8}e^{5-6x} \cdot -6$

$= -\frac{9}{4}e^{5-6x}$

3

$y = \sqrt[4]{81e^{8x}}$

Simplify using law of index $\sqrt[n]{a^m} = a^{\frac{m}{n}}$, $(ab)^n = a^n b^n$

Solution

$= (81e^{8x})^{\frac{1}{4}}$

$= 81^{\frac{1}{4}} \cdot e^{8x(\frac{1}{4})}$

$= 3e^{2x}$

$\frac{dy}{dx} = 3e^{2x} \cdot 2$

$= 6e^{2x}$

Calculus: Differentiation

Example 3

Find the derivative for the followings:

1

$y = e^{3x}(2e^{3x} - 6)$ — expand

$= 2e^{6x} - 6e^{3x}$

Solution

$\frac{dy}{dx} = 2e^{6x} \cdot 6 - 6e^{3x} \cdot 3$

$= 12e^{6x} - 18e^{3x}$

$= 6e^{3x}(2e^{3x} - 3)$

2

$y = \frac{e^{8x} + 2e^{x}}{e^{3x}}$ — separate

Solution

$= \frac{e^{8x}}{e^{3x}} + \frac{2e^{x}}{e^{3x}}$

$= e^{5x} + 2e^{-2x}$

$\frac{dy}{dx} = e^{5x} \cdot 5 + 2e^{-2x} \cdot -2$

$= 5e^{5x} - \frac{4}{e^{2x}}$

$= \frac{1}{e^{2x}}(5e^{7x} - 4)$

Factorize

3

$y = 2e^{6x} - \frac{5}{e^{3x}}$ — change the form

$= 2e^{6x} - 5e^{-3x}$

Solution

$\frac{dy}{dx} = 2e^{6x} \cdot 6 - 5e^{-3x} \cdot -3$

$= 12e^{6x} + \frac{15}{e^{3x}}$

$= \frac{3}{e^{3x}}(4e^{9x} + 5)$

Factorize

Example 4

Find the derivative for the followings:

Chain Rule

$$y = e^{x^4}$$

Solution

$u = x^4$ $\qquad$ $y = e^u$

$\frac{du}{dx} = 4x^3$ $\qquad$ $\frac{dy}{du} = e^u$

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$$

$$\frac{dy}{dx} = e^u \cdot 4x^3$$

$$= 4x^3 e^{x^4}$$

Chain Rule

$$y = ln(e^{2x})$$

Solution

$u = e^{2x}$ $\qquad$ $y = lnu$

$\frac{du}{dx} = 2e^{2x}$ $\qquad$ $\frac{dy}{du} = \frac{1}{u}$

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$$

$$\frac{dy}{dx} = \frac{1}{u} \cdot 2e^{2x}$$

$$= \frac{2e^{2x}}{e^{2x}}$$

$$= 2$$

Chain Rule

$$y = cos(e^{4x+1})$$

Solution

$u = e^{4x+1}$ $\qquad$ $y = cosu$

$\frac{du}{dx} = 4e^{4x+1}$ $\qquad$ $\frac{dy}{du} = -sinu$

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$$

$$\frac{dy}{dx} = -sinu \cdot 4e^{4x+1}$$

$$= -4e^{4x+1} sin(e^{4x+1})$$

Example 5

Find the derivative for the followings:

Product Rule

$$g(x) = 2e^{3x}(x^3+2)^4$$

Solution

$u = 2e^{3x}$ $\qquad$ $v = (x^3+2)^4$

$\dfrac{du}{dx} = 6e^{3x}$ $\qquad$ $\dfrac{dv}{dx} = 4(x^3+2)^3 \cdot 3x^2 = 12x^2(x^3+2)^3$

$$\frac{dy}{dx} = v\frac{du}{dx} + u\frac{dv}{dx}$$

$$g'(x) = (x^3+2)^4 \cdot 6e^{3x} + 2e^{3x} \cdot 12x^2(x^3+2)^3$$

$$= 6e^{3x}(x^3+2)^4 + 24x^2e^{3x}(x^3+2)^3$$

$$= 6e^{3x}(x^3+2)^3(x^3+2+4x^2)$$

Factorize

Quotient Rule

$$y = \frac{e^{6x+1}}{2x^4}$$

Solution

$u = e^{6x+1}$ $\qquad$ $v = 2x^4$

$\dfrac{du}{dx} = 6e^{6x+1}$ $\qquad$ $\dfrac{dv}{dx} = 8x^3$

$$\frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$$

$$\frac{dy}{dx} = \frac{2x^4 \cdot 6e^{6x+1} - e^{6x+1} \cdot 8x^3}{(2x^4)^2}$$

$$= \frac{12x^4e^{6x+1} - 8x^3e^{6x+1}}{4x^8}$$

$$= \frac{4x^3e^{6x+1}(3x-2)}{4x^8}$$

Factorize

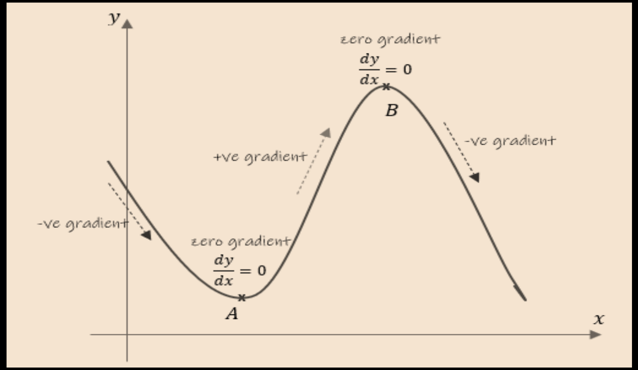
$$= \frac{e^{6x+1}(3x-2)}{x^5}$$

Application Of Differentiation

- Calculus (differentiation & integration) was developed to improve the understanding of application of differentiation.
- Differentiation & integration can help us solve various types of real-world problems.
- We use the derivative **to determine the maximum and minimum** values of particular functions (e.g. cost, strength, amount of material used in a building, profit, loss, etc.).
- Its also used **to solve rate of change** (e.g. area, volume, length, velocity etc.)

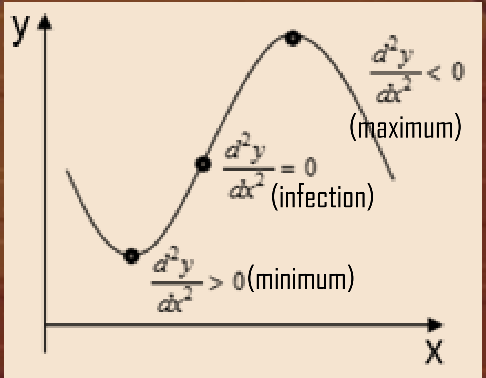
a) Maximum, Minimum And Infection Point

- We use differentiation to determine whether a function is increasing or decreasing.
- A point where a function changes from an increasing to a decreasing function or visa-versa is known as a **turning point**.
- Turning point is also known as stationary point or critical point.
- A function is increasing if the derivative is positive, and if the derivative is negative then the function is decreasing.



Maximum, Minimum And Infection Point

- **Stationary points** are points on a curve where the gradient is zero.
- This is where the curve reaches a minimum or maximum.
- A maximum is a high point and a minimum is a low point on the curve.
- There are three types of stationary points: maximum, minimum and inflection/ inflexion points.



How to find stationary point and determine the nature of the point?

REMEMBER

1. At all stationary point, the gradient is zero where $\frac{dy}{dx} = 0$
2. Second derivative will tell the nature of the curve whether it is a maximum point, a minimum point or a point of inflection.
   - $\frac{d^2y}{dx^2} > 0$ *minimum point*
   - $\frac{d^2y}{dx^2} < 0$ *maximum point*
   - $\frac{d^2y}{dx^2} = 0$ *inflection point*

STEPS

1. Differentiate 1st time $\left(\frac{dy}{dx}\right)$
2. Let $\frac{dy}{dx} = 0$, then find the value of x
3. Find value of y (*substitute x into the function given from question*)
4. State coordinate of the turning point.
5. Differentiate 2nd time $\left(\frac{d^2y}{dx^2}\right)$ and find the value of $\frac{d^2y}{dx^2}$ (*when necessary*)
6. State the nature of the point (*max or min or inflection point*)

Example 1

Find the stationary points on the graph of $y = x^2 + 4x + 3$ and state their nature.

Solution

$$y = x^2 + 4x + 3$$

Step 1

$$\frac{dy}{dx} = 2x + 4$$

Step 2

$$\frac{dy}{dx} = 0 \qquad \therefore 2x + 4 = 0$$

$$2x = -4$$

$$x = -2$$

Step 3

$when\ x = -2$

$$y = (-2)^2 + 4(-2) + 3$$

$$y = -1$$

Step 4

$turning\ point\ is\ (-2, -1)$

Step 5

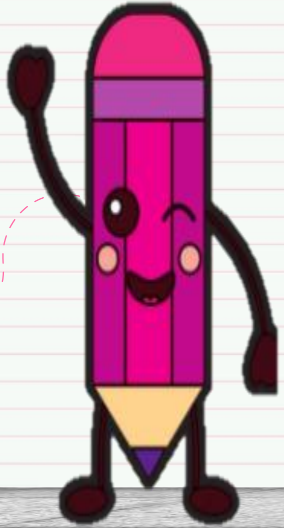
$$\frac{d^2y}{dx^2} = 2 > 0$$

REMEMBER

$\frac{d^2y}{dx^2} > 0$ *minimum point*

Step 6

$Point\ (-2, -1)\ is\ a\ minimum\ point$



Example 2

Find the stationary points on the graph of $y = x^3 - 2x^2 - 4x + 5$ and state their nature.

2 stationary points

Solution

$$y = x^3 - 2x^2 - 4x + 5$$

Step 1

$$\frac{dy}{dx} = 3x^2 - 4x - 4$$

Step 2

$$\frac{dy}{dx} = 0 \qquad \therefore 3x^2 - 4x - 4 = 0$$

$$x = 2 \text{ and } x = -\frac{2}{3}$$

2 values of x

Step 3

when $x = 2$ $\quad y = (2)^3 - 2(2)^2 - 4(2) + 5$

$$\therefore y = -3$$

2 values of y

when $x = -\frac{2}{3}$ $\quad y = \left(-\frac{2}{3}\right)^3 - 2\left(-\frac{2}{3}\right)^2 - 4\left(-\frac{2}{3}\right) + 5$

$$\therefore y = \frac{175}{27}$$

Step 4

turning points are $(2, -3)$ & $\left(-\frac{2}{3}, \frac{175}{27}\right)$

Step 5

$$\frac{d^2y}{dx^2} = 6x - 4$$

substitute values of x

when $x = 2$ $\quad \frac{d^2y}{dx^2} = 6(2) - 4 = 8 \quad > 0$

when $x = -\frac{2}{3}$ $\quad \frac{d^2y}{dx^2} = 6\left(-\frac{2}{3}\right) - 4 = -8 \quad < 0$

Step 6

Point $(2, -3)$ *is a minimum point &*

point $\left(-\frac{2}{3}, \frac{175}{27}\right)$ *is a maximum point*

REMEMBER

$\frac{d^2y}{dx^2} > 0$ *minimum point*

$\frac{d^2y}{dx^2} < 0$ *maximum point*

Example 3

Find the stationary points on the graph of $y = 2x^2 - 8x + 8$ and state their nature. Then sketch the graph.

Solution

$y = 2x^2 - 8x + 8$

Step 1 $\frac{dy}{dx} = 4x - 8$

Step 2 $\frac{dy}{dx} = 0$ $\quad \therefore 4x - 8 = 0$

$4x = 8$

$x = 2$

Step 3 $when\ x = 2$ $\quad y = 2(2)^2 - 8(2) + 8$

$y = 0$

Step 4 $turning\ point\ is\ (2, 0)$

Step 5 $\frac{d^2y}{dx^2} = 4 > 0$

Step 6 $Point\ (2, 0)\ is\ a\ minimum\ point$

REMEMBER

$\frac{d^2y}{dx^2} > 0$ *minimum point*

How to plot the graph

$y = 2x^2 - 8x + 8$

intercept x − axis, $y = 0$

$\therefore 2x^2 - 8x + 8 = 0$

$x = 2$

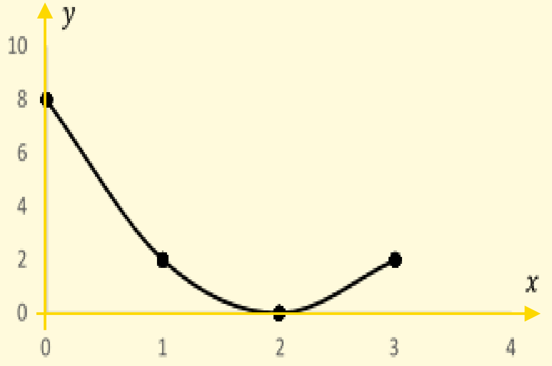
intercept y − axis, $x = 0$

$\therefore y = 2(0)^2 - 8(0) + 8$

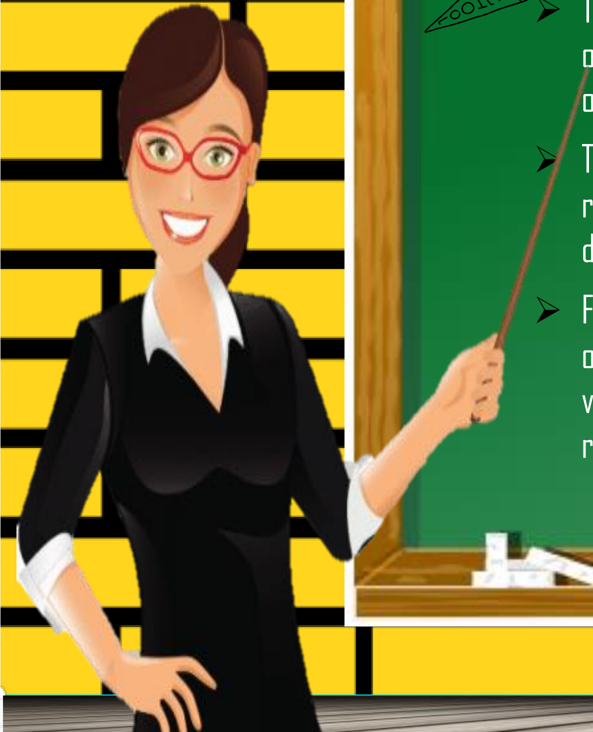
$= 8$

points on graph

1. (2, 0)
2. (0, 8)
3. (2, 0)



b) Rates of Change



- The **rate of change** is the rate of increasing or decreasing of a quantity with the respect of time, (t).
- The rate of change of a function $f(x)$ with respect to x can be found by finding the derived function $f'(x)$.
- For example in mechanics, the rate of change of displacement (with respect to time) is the velocity. The rate of change of velocity (with respect to time) is the acceleration.

If given $y = f(t)$,

Then the related chain rule will be

$$\frac{dy}{dt} = \frac{dy}{dx} \times \frac{dx}{dt}$$

where $\frac{dy}{dt}$ is the rate of change of y with the respect of time, t and $\frac{dy}{dx}$ is the rate of change of x with the respect of time, t.

Rate increase = +ve values

Rate decrease = -ve values

How to solve rate of change ?

Related Chain Rule

a) Rates which contain $y = f(x)$

$$\frac{dy}{dt} = \frac{dy}{dx} \times \frac{dx}{dt}$$

b) Rates which contain radius :

volume, $\frac{dV}{dt} = \frac{dV}{dr} \times \frac{dr}{dt}$

area, $\frac{dA}{dt} = \frac{dA}{dr} \times \frac{dr}{dt}$

c) Rates which contain length :

volume, $\frac{dV}{dt} = \frac{dV}{dx} \times \frac{dx}{dt}$

area, $\frac{dA}{dt} = \frac{dA}{dx} \times \frac{dx}{dt}$

STEPS

1. Extract **ALL** the given information from question. Including **WHAT** to find.
2. Solve related formula to find value of related variable (*if the value is given or when necessary*).
3. Write the related equation or formula (*usually WHAT rate to find*).
4. Then **differentiate** the related formula.
5. State related **chain rule**.
6. Substitute **ALL** the value.
7. Solve/simplify

Example 1

If the expression of y given by $y = x^3 + x^2 - 5$ Find the rate of change of y if x increase at 0.5 $^{unit}/_{s}$ when $x = 5$.

Solution

Step 1 $\frac{dy}{dt} = ?$ $\frac{dx}{dt} = 0.5$ $x = 5$

Step 3 $y = x^3 + x^2 - 5$

Step 4 $\frac{dy}{dx} = 3x^2 + 2x$

Step 5 $\frac{dy}{dt} = \frac{dy}{dx} \times \frac{dx}{dt}$

Rates which contain $y = f(x)$
$\frac{dy}{dt} = \frac{dy}{dx} \times \frac{dx}{dt}$

Step 6 $\frac{dy}{dt} = (3x^2 + 2x) \times (0.5)$

Given that $x = 5$

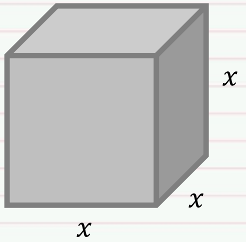
Step 7 $= (3(5)^2 + 2(5)\ ) \times (0.5)$

$= 42.5\ ^{unit}/_{sec}$

∴ Rate of y increase at 42.5 $^{unit}/_{second}$

Example 2

The volume of a cube is decreasing at the rate of 4 cms^{-1}.
Find the rate of decrease of the length when the volume is $216\ \text{cm}^3$.



Solution

Step 1 $\frac{dx}{dt} = ?$ $\quad \frac{dV}{dt} = -4$ (decreased) $\quad V = 216$

Step 3 $V = x^3$ (formula of volume)

$216 = x^3$

$x = 6$

Step 4 $\frac{dV}{dx} = 3x^2$

Step 5 $\frac{dV}{dt} = \frac{dV}{dx} \times \frac{dx}{dt}$

Rates which contain length
$\frac{dV}{dt} = \frac{dV}{dx} \times \frac{dx}{dt}$

Step 6 $-4 = 3x^2 \times \frac{dx}{dt}$

Step 7 $\frac{dx}{dt} = \frac{-4}{3x^2}$

$= \frac{1.2}{3(6)^2}$

From Step 3: $x = 6$

$= 0.037\ {cm^2}/{sec}$

∴ Rate of the volume decrease at $0.037\ {cm^3}/{sec}$

Example 3

The area A of a circle is increasing at a constant rate of $1.2\ ^{cm^2}/_{s}$. Calculate the rate at which the radius, r of the circle is increasing when the perimeter of the circle is $5\ cm$.

Solution

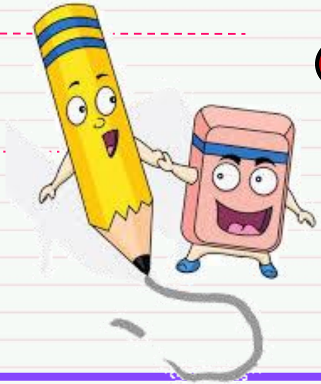
Step 1 $\frac{dr}{dt} = ?$ $\frac{dA}{dt} = 1.2$ $P = 5$

Step 2 $Perimeter, P = 2\pi r$ (formula of parameter)

$\therefore 2\pi r = 5$

Step 3 $A = \pi r^2$

Step 4 $\frac{dA}{dr} = 2\pi r$



Step 5 $\frac{dA}{dt} = \frac{dA}{dr} \times \frac{dr}{dt}$

Rates which contain radius
$\frac{dA}{dt} = \frac{dA}{dr} \times \frac{dr}{dt}$

Step 6 $1.2 = 2\pi r \times \frac{dr}{dt}$

Step 7 $\frac{dr}{dt} = \frac{1.2}{2\pi r}$

From Step 2: $2\pi r = 5$

$= \frac{1.2}{5}$

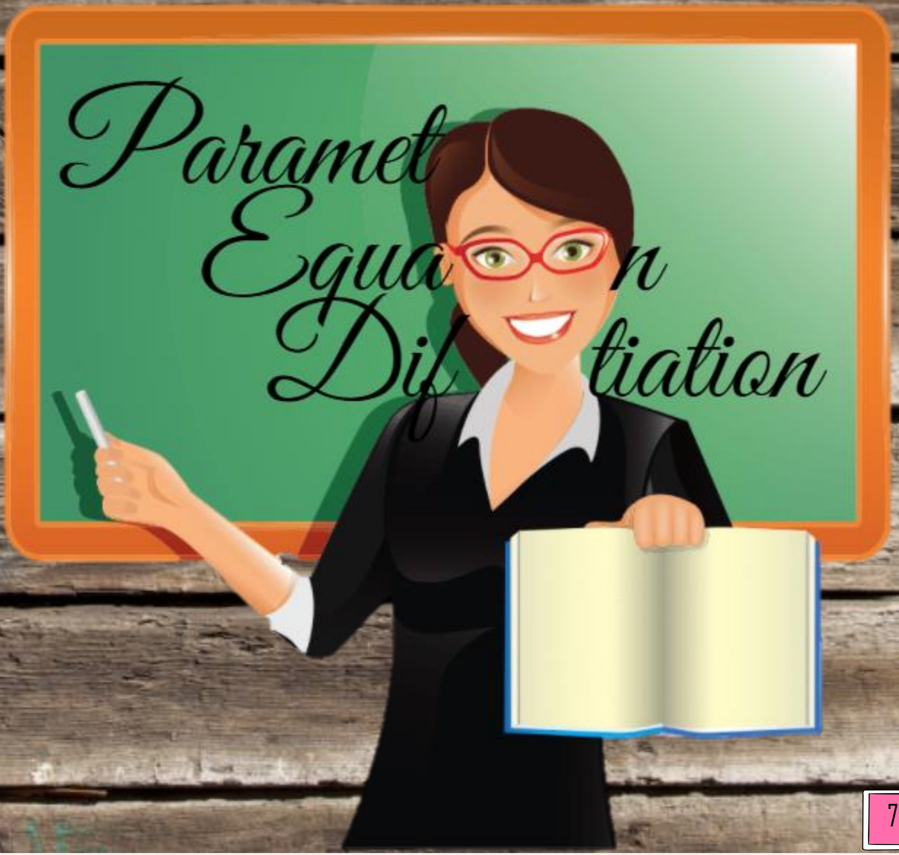
$= 0.24\ ^{cm^2}/_{sec}$

$\therefore$ Rate of the area increase at $0.24\ ^{cm^2}/_{sec}$

Parametric Equation Differentiation

- A **parametric equation** is where the x and y coordinates are both written in terms of another letter.
- The third variable is called a **parameter** and is usually given the letter t or θ.
- The differentiation of functions given in parametric form is carried out using the **Chain Rule**.

$$\frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx} \quad \textbf{where} \quad \frac{dt}{dx} = \frac{1}{\frac{dx}{dt}}$$



How to solve parametric equation differentiation ??

STEPS

1. Differentiate each of the parametric equations for the parameter.
2. State the related Chain Rule.
3. Substitute the resulting expression for the parameter into the Chain Rule.
4. Simplify the equation.

Calculus: Differentiation

Example 1

Find $\frac{dy}{dx}$ when $x = 2t^2 + 3t$ and $y = 5t - 3$.

Solution

$y = 5t - 3$ $\qquad$ $x = 2t^2 + 3t$

Step 1

$\frac{dy}{dt} = 5$ $\qquad$ $\frac{dx}{dt} = 4t + 3$

change the position

$\frac{dt}{dx} = \frac{1}{4t + 3}$

Step 2

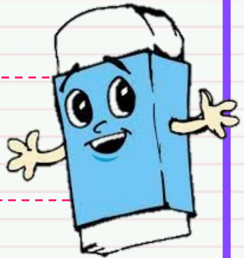
$\frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx}$

Step 3

$\frac{dy}{dx} = (5) \times \left(\frac{1}{4t + 3}\right)$

Step 4

$= \frac{5}{4t + 3}$



Example 2

Given $x = 3\ sin2\theta$ and $y = 6\ cos2\theta$ then find $\frac{dy}{dx}$.

Solution

$y = 6\ cos2\theta$ $\qquad$ $x = 3\ sin2\theta$

Step 1

$\frac{dy}{d\theta} = -12\ sin2\theta$ $\qquad$ $\frac{dx}{d\theta} = 6\ cos2\theta$

change the position

$\frac{d\theta}{dx} = \frac{1}{6\ cos2\theta}$

Step 2

$\frac{dy}{dx} = \frac{dy}{d\theta} \times \frac{d\theta}{dx}$

Step 3

$\frac{dy}{dx} = (-12\ sin2\theta) \times \left(\frac{1}{6\ cos2\theta}\right)$

Step 4

$= \frac{-12\ sin2\theta}{6\ cos2\theta}$

$= -2\ tan2\theta$

Simplify using reciprocal identities
$tanx = \frac{sinx}{cosx}$

Calculus: Differentiation

Example 3

Given $x = 4\,ln3t$ and $y = 5t^3 - 3t$, find $\frac{dy}{dx}$

Solution

$y = 5t^3 - 3t \qquad x = 4\,ln3t$

Step 1

$\frac{dy}{dt} = 15t^2 - 3 \qquad \frac{dx}{dt} = \frac{4}{3t} \cdot 3$

$= \frac{4}{t}$

change the position

$\frac{dt}{dx} = \frac{t}{4}$

Step 2

$\frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx}$

Step 3

$\frac{dy}{dx} = (15t^2 - 3) \times \left(\frac{t}{4}\right)$

Step 4

$= \frac{t(15t^2 - 3)}{4}$

Example 4

Find $\frac{dy}{dx}$ when $x = \frac{e^{4t}}{2}$ and $y = \frac{2}{1-7t}$.

Solution

$y = \frac{2}{1-7t} \qquad x = \frac{e^{4t}}{2}$

$y = 2(1-7t)^{-1}$

Step 1

$\frac{dy}{dt} = -2(1-7t)^{-2}. -7 \qquad \frac{dx}{dt} = 4 \cdot \frac{e^{4t}}{2}$

$= \frac{14}{(1-7t)^2} \qquad = 2e^{4t}$

change the position

$\frac{dt}{dx} = \frac{1}{2e^{4t}}$

Step 2

$\frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx}$

Step 3

$\frac{dy}{dx} = \left(\frac{14}{(1-7t)^2}\right) \times \left(\frac{1}{2e^{4t}}\right)$

Step 4

$= \frac{7}{e^{4t}(1-7t)^2}$

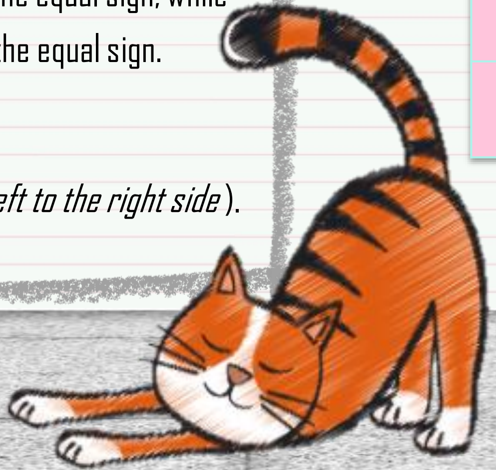
Implicit Differentiation

- **Implicit differentiation** is the process of finding the derivative of a dependent variable in an implicit function by differentiating each term separately.
- The technique of implicit differentiation allows you to find the derivative of y with respect to x without having to solve the given equation for y.
- The **chain rule** must be used whenever the function y is being differentiated because of we assume that y may be expressed as a function of x.

How to solve implicit differentiation ??

STEPS

1. Differentiate x and y with respect to x (*differentiate y followed by* $\frac{dy}{dx}$).
1. Put all $\frac{dy}{dx}$ expression on the left side of the equal sign, while the other expression on the right side of the equal sign.
2. Factorize $\frac{dy}{dx}$ expression on the left side.
3. Find $\frac{dy}{dx}$ (*move all the expression on the left to the right side*).



Basic implicit differentiation

Function	Differentiation
x	1
y	$1\frac{dy}{dx}$
xy	Differentiate by using product rule
$\frac{y}{x}$	Differentiate by using quotient rule

Calculus: Differentiation

Example 1

Find $\frac{dy}{dx}$ for $3x^3 + 2y = 8 - 2y^4 + 5x$.

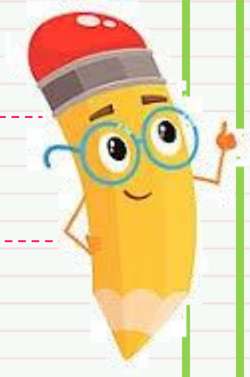
$$3x^3 + 2y = 8 - 2y^4 + 5x$$

Step 1 $$9x^2 + 2\frac{dy}{dx} = 0 - 8y^3\frac{dy}{dx} + 5$$

Step 2 $$2\frac{dy}{dx} + 8y^3\frac{dy}{dx} = 5 - 9x^2$$

Step 3 $$\frac{dy}{dx}(2 + 8y^3) = 5 - 9x^2$$

Step 4 $$\frac{dy}{dx} = \frac{5 - 9x^2}{2 + 8y^3}$$



Example 2

Differentiate using **Product Rule**

Find $\frac{dy}{dx}$ for $2x^4 + 8 = 2x^2y^3 - 5y^3$.

$$2x^4 + 8 = 2x^2y^3 - 5y^3$$

Step 1 $$8x^3 + 0 = \left[4xy^3 + 6x^2y^2\frac{dy}{dx}\right] - 15y^2\frac{dy}{dx}$$

Step 2 $$15y^2\frac{dy}{dx} - 6x^2y^2\frac{dy}{dx} = 4xy^3 - 8x^3$$

Step 3 $$\frac{dy}{dx}(15y^2 - 6x^2y^2) = 4xy^3 - 8x^3$$

Step 4 $$\frac{dy}{dx} = \frac{4xy^3 - 8x^3}{15y^2 - 6x^2y^2}$$

$$= \frac{4x(y^3 - 2x^2)}{3y^2(5 - 2x^2)}$$

Factorize

Calculus: Differentiation

Example 3

Given $2x^4 - 2 = y^2 - 3e^{2x+2y}$, find $\frac{dy}{dx}$

$$2x^4 - 2 = y^2 - 4e^{3x+2y}$$

Step 1

$$8x^3 - 0 = 2y\frac{dy}{dx} - 4e^{3x+2y}\left[3 + 2\frac{dy}{dx}\right]$$

expand

$$8x^3 = 2y\frac{dy}{dx} - 12e^{3x+2y} - 8e^{3x+2y}\frac{dy}{dx}$$

Step 2

$$2y\frac{dy}{dx} - 8e^{3x+2y}\frac{dy}{dx} = 8x^3 + 12e^{3x+2y}$$

Step 3

$$\frac{dy}{dx}(2y - 8e^{3x+2y}) = 8x^3 + 12e^{3x+2y}$$

Step 4

$$\frac{dy}{dx} = \frac{8x^3 + 12e^{3x+2y}}{2y - 8e^{3x+2y}}$$

Factorize

$$= \frac{4(2x^3 - 3e^{3x+2y})}{2(y + 4e^{3x+2y})} = \frac{2(2x^3 - 3e^{3x+2y})}{y + 4e^{3x+2y}}$$

Example 4

Find $\frac{dy}{dx}$ for $x^5 - 2sin3y = ln(2x + 3y)$.

$$x^5 - 2sin3y = ln(2x + 3y)$$

Step 1

$$5x^4 - 6\cos3y\frac{dy}{dx} = \frac{1}{2x + 3y}\left[2 + 3\frac{dy}{dx}\right]$$

expand

$$5x^4 - 6\cos3y\frac{dy}{dx} = \frac{2}{2x + 3y} + \frac{3}{2x + 3y}\frac{dy}{dx}$$

Step 2

$$\frac{3}{2x + 3y}\frac{dy}{dx} + 6\cos3y\frac{dy}{dx} = 5x^4 - \frac{2}{2x + 3y}$$

Step 3

$$\frac{dy}{dx}\left(\frac{3}{2x + 3y} + 6\cos3y\right) = 5x^4 - \frac{2}{2x + 3y}$$

Step 4

$$\frac{dy}{dx} = \frac{5x^4 - \frac{2}{2x + 3y}}{\frac{3}{2x + 3y} + 6\cos3y}$$

Calculus: Differentiation

Example 5

Differentiate using **Quotient Rule**

Find $\frac{dy}{dx}$ for $2x^4 - 15 = \frac{2x}{3y} + y^3$.

$$2x^4 - 15 = \frac{2x}{3y} + y^3$$

Step 1

$$8x^3 - 0 = \left(\frac{3y(2) - 2x(3\frac{dy}{dx})}{(3y)^2}\right) + 3y^2\frac{dy}{dx}$$

$$8x^3 = \frac{6y - 6x\frac{dy}{dx}}{9y^2} + 3y^2\frac{dy}{dx}$$

$$= \frac{6y - 6x\frac{dy}{dx} + 27y^4\frac{dy}{dx}}{9y^2}$$

$$72x^3y^2 = 6y - 6x\frac{dy}{dx} + 27y^4\frac{dy}{dx}$$

Step 2

$$27y^4\frac{dy}{dx} - 6x\frac{dy}{dx} = 72x^3y^2 - 6y$$

Step 3

$$\frac{dy}{dx}(27y^4 - 6x) = 72x^3y^2 - 6y$$

Step 4

$$\frac{dy}{dx} = \frac{72x^3y^2 - 6y}{27y^4 - 6x}$$

$$= \frac{6\,y(12x^3y - 1)}{3(9y^4 - 2x)}$$ Factorize

$$= \frac{2y(12x^3y - 1)}{(9y^4 - 2x)}$$

IMPORTANT

How do You Solve these Problem?

1

$\sin(x^2+y^2)$

$= \cos(x^2+y^2)\cdot\left(2x+2y\frac{dy}{dx}\right)$

$= 2x\cos(x^2+y^2) + 2y\cos(x^2+y^2)\frac{dy}{dx}$

2

$\sin(x^2y^2)$

$= \cos(x^2y^2)\cdot\left(2xy^2+2x^2y\frac{dy}{dx}\right)$

$= 2xy^2\cos(x^2+y^2) + 2x^2y\cos(x^2y^2)\frac{dy}{dx}$

3

$\ln(x^2+y^2)$

$= \frac{1}{x^2+y^2}\cdot\left(2x+2y\frac{dy}{dx}\right)$

$= \frac{2x}{x^2+y^2} + \frac{2y}{x^2+y^2}\frac{dy}{dx}$

4

$\ln(x^2y^2)$

$= \frac{1}{x^2y^2}\cdot\left(2xy^2+2x^2y\frac{dy}{dx}\right)$

$= \frac{2xy^2}{x^2y^2} + \frac{2x^2y}{x^2y^2}\frac{dy}{dx} = \frac{2}{x} + \frac{2}{y}\frac{dy}{dx}$

5

$e^{(x^2+y^2)}$

$= e^{(x^2+y^2)}\cdot\left(2x+2y\frac{dy}{dx}\right)$

$= 2xe^{(x^2+y^2)} + 2ye^{(x^2+y^2)}\frac{dy}{dx}$

6

$e^{(x^2y^2)}$

$= e^{(x^2y^2)}\cdot\left(2xy^2+2x^2y\frac{dy}{dx}\right)$

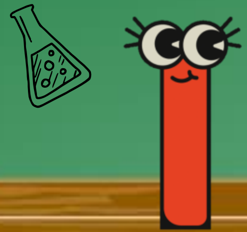
$= 2xy^2e^{(x^2y^2)} + 2x^2ye^{(x^2y^2)}\frac{dy}{dx}$

Partial Differentiation

- **Partial differentiation** are defined as derivatives of a function where the variable of differentiation is indicated while all the other variables are held constant.
- Suppose f is a function in x and y then it will be expressed by $f(x,y)$. So, the partial differentiation of f with respect to x will be $\frac{\partial f}{\partial x}$ **keeping y as constant** and partial differentiation of f with respect to y will be $\frac{\partial f}{\partial y}$ **keeping x as constant.**

a) First Order Of Partial Differentiation

- If $f(x,y)$ is a function of two variables, there are two first order partial differentiation of f that is partial derivative of f with respect to x and the partial derivative of f with respect to y
- The first partial derivatives of the equation $z=f(x,y)$ be a function with two variables
  - with respect to x: $f_x = \frac{\partial f}{\partial x}$ OR $\frac{\partial z}{\partial x}$
  - with respect to x: $f_y = \frac{\partial f}{\partial y}$ OR $\frac{\partial z}{\partial y}$



Calculus: Differentiation

How to solve 1st order partial differentiation ??

$z = x^2 + 2y$
Differentiate y ,and
x held as CONSTANT

$z = x^2 + 2y$
Differentiate x,and
y held as CONSTANT

When x and y stands alone

If given $f(x, y) =$
Then f_x and f_y

$z = 2x^2y$
Differentiate x,and
y COPY back

$z = 2x^2y$
Differentiate y,and
x COPY back

When x and y comes together

If given $z =$
Then $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$

Calculus: Differentiation

How to differentiate ??

Example 1

Given $z = x^3 + 5y^2 - 8$. Find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$

When x and y stands alone

Solution

respect to x

$$\frac{\partial z}{\partial x} = 3x^2 + 0 - 0$$
$$= 3x^2$$

Differentiate x, and y held as CONSTANT

respect to y

$$\frac{\partial z}{\partial y} = 0 + 10y - 0$$
$$= 10y$$

Differentiate y, and x held as CONSTANT

Example 2

Given $z = x^3 + 5y^2 - 8x^3y^2$. Find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$

When x and y comes together

Solution

respect to x

$$\frac{\partial z}{\partial x} = 3x^2 + 0 - 24x^2y^2$$
$$= 3x^2 - 24x^2y^2$$

Differentiate x, and y COPY back

respect to y

$$\frac{\partial z}{\partial y} = 0 + 10y - 16yx^3$$
$$= 10y - 16x^3y$$

Differentiate y, and x COPY back

Calculus: Differentiation

Example 3

Find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ for the followings:

1

$$z = x^4 + 5 - 3y^2$$

$$\frac{\partial z}{\partial x} = 4x^3 + 0 + 0$$
$$= 4x^3$$

$$\frac{\partial z}{\partial y} = 0 + 0 - 6y$$
$$= -6y$$

2

$$z = 4x + 2x^4y^2 - y^5$$

$$\frac{\partial z}{\partial x} = 4 + 8x^3y^2 - 0$$
$$= 4 + 8x^3y^2$$

$$\frac{\partial z}{\partial y} = 0 + 4yx^4 - 5y^4$$
$$= 4x^4y - 5y^4$$

3

$$z = (x^2 + 2)(1 - 3y)$$
$$= x^2 - 3x^2y + 2 - 6y$$

expand

$$\frac{\partial z}{\partial x} = 2x - 6xy + 0 - 0$$
$$= 2x - 6xy$$

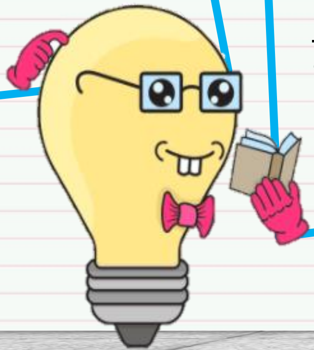
$$\frac{\partial z}{\partial y} = 0 - 3x^2(1) + 0 - 6$$
$$= 3x^2 - 6$$

4

$$z = 3x + x\,cos2y - 5y$$

$$\frac{\partial z}{\partial x} = 3 + (1)cos2y - 0$$
$$= 3 + cos2y$$

$$\frac{\partial z}{\partial y} = 0 - x\,sin2y(2) - 5$$
$$= -2x\,sin2y - 5$$



Example 4

Find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ for the followings:

1

$$z = 5e^x y + 3 - e^{2x+3y}$$

$$\frac{\partial z}{\partial x} = 5e^x y + 0 - e^{2x+3y}(2)$$

$$= 5e^x y - 2e^{2x+3y}$$

$$\frac{\partial z}{\partial y} = 5e^x(1) - e^{2x+3y}(3)$$

$$= 5e^x - 3e^{2x+3y}$$

2

$$z = ln(x^2 + y^2) + 4x - 3y$$

$$\frac{\partial z}{\partial x} = \left[\frac{1}{x^2 + y^2} \cdot 2x\right] + 4 - 0$$

$$= \frac{2x}{x^2 + y^2} + 4$$

$$\frac{\partial z}{\partial y} = \left[\frac{1}{x^2 + y^2} \cdot 2y\right] + 0 - 3$$

$$= \frac{2y}{x^2 + y^2} - 3$$

3

$$z = (x^3 + y^2)^5 + 4x - 3y$$

$$\frac{\partial z}{\partial x} = 5(x^3 + y^2)^4(3x^2 + 0) + 4 - 0$$

$$= 15x^2(x^3 + y^2)^4 + 4$$

$$\frac{\partial z}{\partial y} = 5(x^3 + y^2)^4(0 + 2y) + 0 - 3$$

$$= 10y(x^3 + y^2)^4 - 3$$

Example 5

Find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ for the followings:

Product Rule

Solution

$$z = 2x^4 - 4x^2y\left(\cos(x^3+3y)\right) + 5y$$

$4x^2y\left(\cos(x^3+3y)\right)$ must use **product rule** for derivative with respect to x and y

with respect to x

$u = 4x^2y$ $\quad v = \cos(x^3+3y)$

$\frac{\partial u}{\partial x} = 8xy$ $\quad \frac{\partial v}{\partial x} = -3x^2\sin(x^3+3y)$

$$\frac{\partial z}{\partial x} = 8x^3 - \left[\cos(x^3+3y)\cdot 8xy + 4x^2y\cdot\left(-3x^2\sin(x^3+3y)\right)\right] + 0$$

$$= 8x^3 - 8xy\cos(x^3+3y) + 12x^4y\sin(x^3+3y)$$

with respect to y

$u = 4x^2y$ $\quad v = \cos(x^3+3y)$

$\frac{\partial u}{\partial y} = 4x^2$ $\quad \frac{\partial v}{\partial y} = -3\sin(x^3+3y)$

$$\frac{\partial z}{\partial y} = 0 - \left[\cos(x^3+3y)\cdot 4x^2 + 4x^2y\cdot\left(-3\sin(x^3+3y)\right)\right] + 5$$

$$= -4x^2\cos(x^3+3y) + 12x^2y\sin(x^3+3y) + 5$$

Calculus: Differentiation

Example 6

Find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ for the followings:

Quotient Rule

$$z = y^2 - \frac{4x^2+3y}{5x-y^3} + x^5$$

Solution

$$\frac{\partial z}{\partial x} = 0 - \left[\frac{(5x-y^3)\cdot 8x - (4x^2+3y)(5)}{(5x-y^3)^2}\right] + 5x^4$$

$$= -\left[\frac{40x^2 - 8xy^3 - 20x^2 - 15y}{(5x-y^3)^2}\right] + 5x^4$$

$$= -\left[\frac{20x^2 - 8xy^3 - 15y}{(5x-y^3)^2}\right] + 5x^4$$

$$\frac{\partial z}{\partial y} = 2y - \left[\frac{(5x-y^3)\cdot(3) - (4x^2+3y)(-3y^2)}{(5x-y^3)^2}\right] + 0$$

$$= 2y - \left[\frac{15x - 3y^3 + 12x^2y^2 + 9y^3}{(5x-y^3)^2}\right]$$

$$= 2y - \left[\frac{15x + 12x^2y^2 + 6y^3}{(5x-y^3)^2}\right]$$

$\frac{4x^2+3y}{5x-y^3}$ must use **quotient rule** for derivative with respect to x and y

with respect to x

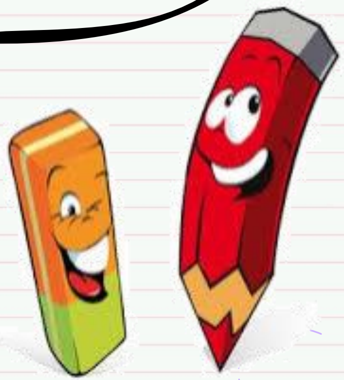
$u = 4x^2 + 3y \qquad v = 5x - y^3$

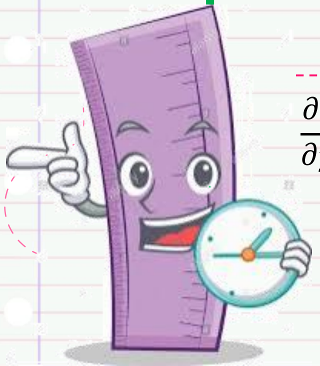
$\frac{\partial u}{\partial x} = 8x \qquad \frac{\partial v}{\partial x} = 5$

with respect to y

$u = 4x^2 + 3y \qquad v = 5x - y^3$

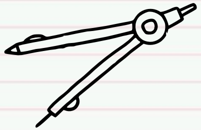
$\frac{\partial u}{\partial y} = 3 \qquad \frac{\partial v}{\partial y} = -3y^2$





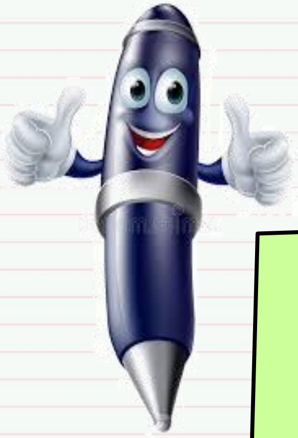
b) Second Order Of Partial Differentiation

- Given a function $f(x,y)$, the function is said to be differentiable if f_x and f_y exist.
- If we can differentiate z=$f(x,y)$, therefore its 1st order derivatives can be differentiated again and we can delineate the 2nd order partial derivatives of z=$f(x,y)$ as follows:



How to solve 2nd order partial differentiation ??

Partial Derivative	respect to x	respect to y
1st order	$\frac{\partial z}{\partial x}$	$\frac{\partial z}{\partial x}$
2nd order	$\frac{\partial^2 z}{\partial x^2}$	$\frac{\partial^2 z}{\partial y^2}$
	respect to y	respect to x
2nd order	$\frac{\partial^2 z}{\partial y \partial x}$	$\frac{\partial^2 z}{\partial y \partial x}$



Note !!!

The answer for $\frac{\partial^2 z}{\partial y \partial x}$ **should be the same as** $\frac{\partial^2 z}{\partial y \partial x}$

REMEMBER

If given $f(x, y) =$

- **First order**

  f_x and f_y

- **Second order**

  f_{xx} and f_{yx}

  f_{yy} and f_{xy}

If given $z =$

- **First order**

  $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$

- **Second order**

  $\frac{\partial^2 z}{\partial x^2}$ and $\frac{\partial^2 z}{\partial y \partial x}$

  $\frac{\partial^2 z}{\partial y^2}$ and $\frac{\partial^2 z}{\partial x \partial y}$

How to differentiation ??

Example 1

Given $z = 3x^3 + 2y^3 - 5x^3y^2$. Find $\frac{\partial^2 z}{\partial x^2}$, $\frac{\partial^2 z}{\partial y^2}$ and $\frac{\partial^2 z}{\partial x \partial y}$

First Order Partial Differentiation

Differentiate Z respect to $\boldsymbol{x}$

$$\frac{\partial z}{\partial x} = 9x^2 + 0 - 15x^2y^2$$
$$= 9x^2 - 15x^2y^2$$

Differentiate Z respect to $\boldsymbol{y}$

$$\frac{\partial z}{\partial y} = 0 + 6y^2 - 10yx^3$$
$$= 6y^2 - 10yx^3$$

Second Order Partial Differentiation

Differentiate Z respect to $\boldsymbol{x}$ **(2nd time)**

$$\frac{\partial^2 z}{\partial x^2} = 18x - 30xy^2$$

Differentiate Z respect to $\boldsymbol{y}$ **(2nd time)**

$$\frac{\partial^2 z}{\partial y^2} = 12x - 10x^3$$

Differentiate $\frac{\partial z}{\partial x}$ respect to $\boldsymbol{y}$

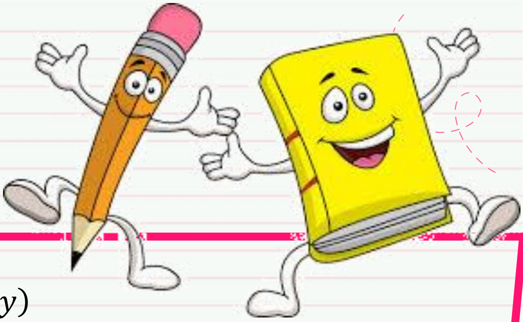
$$\frac{\partial^2 z}{\partial y \partial x} = 0 - 30yx^2$$
$$= -30yx^2$$

Differentiate $\frac{\partial z}{\partial y}$ respect to $\boldsymbol{x}$

$$\frac{\partial^2 z}{\partial x \partial y} = 0 - 30x^2y$$
$$= -30x^2y$$

The answer SHOULD be the same

Calculus: Differentiation



Example 2

Find $\frac{\partial^2 z}{\partial x^2}$, $\frac{\partial^2 z}{\partial y^2}$ and $\frac{\partial^2 z}{\partial x \partial y}$ for the followings:

Solution

1 $z = 5x + x^5y^6 + 4y + 25$

$$\frac{\partial z}{\partial x} = 5 + 5x^4y^6 + 0 + 0$$
$$= 5 + 5x^4y^6$$

$$\frac{\partial z}{\partial y} = 0 + 6y^5x^5 + 4 + 0$$
$$= 6x^5y^5 + 4$$

$$\frac{\partial^2 z}{\partial x^2} = 0 + 20x^3y^6$$
$$= 20x^3y^6$$

$$\frac{\partial^2 z}{\partial y^2} = 30y^4x^5 + 0$$
$$= 30x^5y^4$$

$$\frac{\partial^2 z}{\partial y \partial x} = 0 + 30y^5x^4$$
$$= 30x^4y^5$$

$$\frac{\partial^2 z}{\partial x \partial y} = 30x^4y^5 + 0$$
$$= 30x^4y^5$$

Solution

2 $z = e^{2x+3y} + ln(3x - 4y)$

$$\frac{\partial z}{\partial x} = e^{2x+3y}(2) + \frac{1}{3x-4y}(3)$$
$$= 2e^{2x+3y} + \frac{3}{3x-4y}$$

$$\frac{\partial z}{\partial y} = e^{2x+3y}(3) + \frac{1}{3x-4y}(-4)$$
$$= 3e^{2x+3y} - \frac{4}{3x-4y}$$

$$\frac{\partial^2 z}{\partial x^2} = 2e^{2x+3y}(2) - \frac{3}{(3x-4y)^2}(3)$$
$$= 4e^{2x+3y} - \frac{9}{(3x-4y)^2}$$

$$\frac{\partial^2 z}{\partial y^2} = 3e^{2x+3y}(3) + \frac{4}{(3x-4y)^2}(-4)$$
$$= 9e^{2x+3y} - \frac{16}{(3x-4y)^2}$$

$$\frac{\partial^2 z}{\partial x \partial y} = 3e^{2x+3y}(2) - \frac{4}{(3x-4y)^2}(3)$$
$$= 6e^{2x+3y} - \frac{12}{(3x-4y)^2}$$

$$\frac{\partial^2 z}{\partial y \partial x} = 2e^{2x+3y}(3) + \frac{3}{(3x-4y)^2}(-4)$$
$$= 6e^{2x+3y} - \frac{12}{(3x-4y)^2}$$

Calculus: Differentiation

Example 3

$$f(x,y) = 2xe^{4y} + cos(3x^2 + y^3)$$

Solution

$$f_x = 2e^{4y} - sin(3x^2 + y^3) \cdot 6x$$

$$= 2e^{4y} - 6x\ sin(3x^2 + y^3)$$

$6x\ sin(3x^2 + y^3)$ must use product rule for 2nd order derivative with respect to x

$$f_y = 2xe^{4y} \cdot 4 - sin(3x^2 + y^3) \cdot 3y^2$$

$$= 8xe^{4y} - 3y^2\ sin(3x^2 + y^3)$$

$3y^2\ sin(3x^2 + y^3)$ must use product rule for 2nd order derivative with respect to y

$$f_{xx} = 0 - [sin(3x^2 + y^3) \cdot 6 + 6x \cdot cos(3x^2 + y^3) \cdot 6x]$$

$$= -6sin(3x^2 + y^3) - 36x^2cos(3x^2 + y^3)$$

$$f_{yy} = 8xe^{4y} \cdot 4 - [sin(3x^2 + y^3) \cdot 6y + 3y^2cos(3x^2 + y^3) \cdot 3y^2]$$

$$= 32xe^{4y} - 6y\ sin(3x^2 + y^3) - 9y^4cos(3x^2 + y^3)$$

$$f_{xy} = 2e^{4y} \cdot 4 - 6x\ cos(3x^2 + y^3) \cdot 3y^2$$

$$= 8e^{4y} - 18xy^2cos(3x^2 + y^3)$$

$$f_{yx} = 8e^{4y} - 3y^2cos(3x^2 + y^3) \cdot 6x$$

$$= 8e^{4y} - 18xy^2cos(3x^2 + y^3)$$

Total Differentiation

- Total derivatives are often used in **related rates** problems; for example, finding the rate of change of volume when two parameters are changing with time.
- A function $w=f(x, y, z)$ and t with x, y, z being functions of t, Then the total derivatives of w with respect to t is given by.

$$\frac{dw}{dt}=\frac{\partial w}{\partial x}\cdot\frac{dx}{dt}+\frac{\partial w}{\partial y}\cdot\frac{dy}{dt}+\frac{\partial w}{\partial z}\cdot\frac{dz}{dt}$$

Calculus: Differentiation

How to solve total derivative ??

STEPS

1. Extract **ALL given** info from question. Including WHAT to find.
2. Differentiate function using partial differentiation method.
3. Derive related Chain Rule of total derivative.
4. Substitute the partial differentiation into the Chain Rule.
5. Then substitute all related values and solve the differentiation

Example 1

Find total derivative for $z = x^2y^3 - 2x + \frac{3}{5y}$

Step 2

$$\frac{\partial z}{\partial x} = 2xy^3 - 2$$

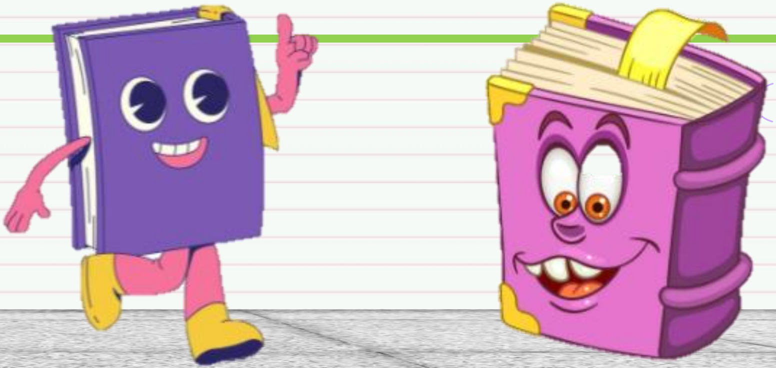
$$\frac{\partial z}{\partial y} = 3x^2y^2 - \frac{3}{5y^2}$$

Step 3

$$dz = \frac{\partial z}{\partial x} \cdot dx + \frac{\partial z}{\partial y} \cdot dy$$

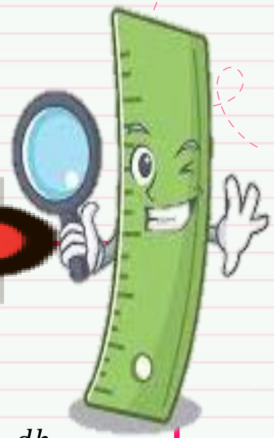
Step 4

$$= (2xy^3 - 2) \cdot dx + \left(3x^2y^2 - \frac{3}{5y^2}\right) \cdot dy$$

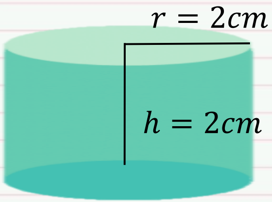


Example 2

The radius and height of a cylinder are both 2 cm. The radius is decreased at 1 $^{cm}/_{sec}$ and the height is increasing at 2 $^{cm}/_{sec}$. Calculate the change in volume with respect to time at this instant?



Solution



Step 1 $\frac{dr}{dt} = -1$ $\quad \frac{dh}{dt} = 2$ $\quad r = h = 2$

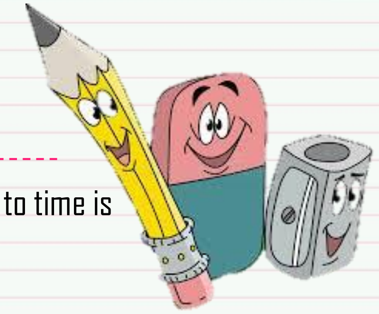
Step 2 The volume of cylinder is:

$V = \pi r^2 h$

$\frac{\partial V}{\partial r} = 2\pi r h$ $\quad \frac{\partial V}{\partial h} = \pi r^2$

Step 3 The total derivative of this with respect to time is

$$\frac{dV}{dt} = \frac{\partial V}{\partial r} \cdot \frac{dr}{dt} + \frac{\partial V}{\partial h} \cdot \frac{dh}{dt}$$



Step 4

$$\frac{dV}{dt} = (2\pi r h) \cdot \frac{dr}{dt} + (\pi r^2) \cdot \frac{dh}{dt}$$

Step 5 Substitute r $= 2, h = 2, \frac{dr}{dt} = -1$ and $\frac{dh}{dt} = 2$ into step 4

$= 2\pi(2)(2) \cdot (-1) + \pi(2)^2 \cdot (2)$

$= -8\pi + 8\pi$

$= 0$

$\therefore$ there is no changes in volume of the cylinder change in x & y

$(1, 2)$ → $(0.8, 2.3)$

change of x: $dx = 0.8 - 1 = -0.2$

change of y: $dy = 2.3 - 2 = 0.3$

Example 3

If $z = 1 - 2x^2 + 3xy + y^3$, Find the total derivative of z when (x, y) changes from $(1, 2)$ to $(0.8, 2.3)$.

Step 1

$dx = -0.2$ $\qquad dy = 0.3$

Step 2

$$\frac{\partial z}{\partial x} = -4x + 3y$$

$$\frac{\partial z}{\partial y} = -3x + 3y^2$$

Step 3

The total derivative is

$$dz = \frac{\partial z}{\partial x} \cdot dx + \frac{\partial z}{\partial y} \cdot dy$$

Step 4

$$dz = (-4x + 3y) \cdot dx + (-3x + 3y^2) \cdot dy$$

Step 5

Substitute $x = 1, y = 2$, $dx = -0.2$ and $dy = 0.3$ into step 4

$$= (-4(1) + 3(2)) \cdot (-0.2) + +(-3(1) + 3(2)^2) \cdot (0.3)$$

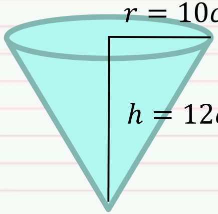
$$= 2.9 \; unit$$

$\therefore$ there is 2.9 $unit$ changes in z

Example 4

Find the total derivative of volume of a cone when the height decrease by $0.03\ ^{cm}/_{s}$ and the radius increase by $0.05\ ^{cm}/_{s}$. The height and radius of the cone is $12\ cm$ and $10\ cm$?

Solution



Step 1

$$\frac{dr}{dt} = 0.05 \qquad \frac{dh}{dt} = -0.03 \qquad h = 12 \qquad r = 10$$

Step 2

The volume of cone is:

$$V = \frac{1}{3}\pi r^2 h$$

$$\frac{\partial V}{\partial r} = \frac{2}{3}\pi r h \qquad \frac{\partial V}{\partial h} = \frac{1}{3}\pi r^2$$

Step 3

The total derivative of this with respect to time is

$$\frac{dV}{dt} = \frac{\partial V}{\partial r}\cdot\frac{dr}{dt} + \frac{\partial V}{\partial h}\cdot\frac{dh}{dt}$$

Step 4

$$\frac{dV}{dt} = \left(\frac{2}{3}\pi r h\right)\cdot\frac{dr}{dt} + \left(\frac{1}{3}\pi r^2\right)\cdot\frac{dh}{dt}$$

Step 5

Substitute $\mathrm{r} = 10, h = 12, \frac{dr}{dt} = 0.05$ and $\frac{dh}{dt} = -0.03$ into step 4

$$= \frac{2}{3}\pi(10)(12)\cdot(0.05) + \frac{1}{3}\pi(10)^2\cdot(-0.03)$$

$$= 4\pi - \pi$$

$$= 3\pi$$

$\therefore$ there is $3\pi\ ^{cm^3}/_{s}$ changes in volume of the cone

Example 5

Consider a cuboid where all the edges are increasing. Given that the height is $15\ cm$, length is $12\ cm$ and the width is $7\ cm$. Evaluate $\frac{dV}{dt}$ when $\frac{dl}{dt} = 0.3$, $\frac{dh}{dt} = 0.5$ and $\frac{dw}{dt} = 0.8$?

Solution

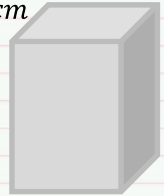
Step 1

$$\frac{dl}{dt} = 0.3 \qquad \frac{dh}{dt} = 0.5 \qquad \frac{dw}{dt} = 0.8$$

$w = 7$

$l = 12$

$h = 15$



Step 2

The volume of cuboid is:

$$V = h \cdot w \cdot l$$

$$\frac{\partial V}{\partial l} = hw \qquad \frac{\partial V}{\partial w} = lw \qquad \frac{\partial V}{\partial h} = wl$$

Step 3

The total derivative of this with respect to time is

$$\frac{dV}{dt} = \frac{\partial V}{\partial l} \cdot \frac{dl}{dt} + \frac{\partial V}{\partial w} \cdot \frac{dw}{dt} + \frac{\partial V}{\partial h} \cdot \frac{dh}{dt}$$

Step 4

$$\frac{dV}{dt} = hw \cdot \frac{dl}{dt} + lw \cdot \frac{dw}{dt} + wl \cdot \frac{dh}{dt}$$

Step 5

Substitute $l = 12, w = 7, h = 15, \frac{dl}{dt} = 0.3,$ $\frac{dh}{dt} = 0.5$ and $\frac{dw}{dt} = 0.8$ into step 4

$$= (15)(7) \cdot 0.3 + (12)(7) \cdot 0.8 + (7)(12) \cdot 0.5$$

$$= 140.7$$

$\therefore$ there is $140\ {}^{cm^3}/_{s}$ changes in volume of the cuboid

EXERCISE

SCAN ME

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