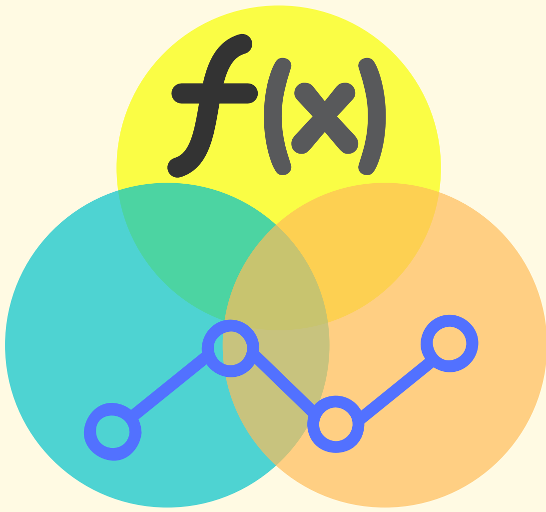


DISCRETE Mathematics

Exploring Sets, Relations & Functions

Nur Intan Syazwani Binti Jamaludin
Raja Noorliyana Binti Raja Almanan

Discrete Mathematics

Exploring Sets, Relations & Functions

Author

Nur Intan Syazwani Binti Jamaludin

Raja Noorliyana Binti Raja Almanan

Editor

Nur Intan Syazwani Binti Jamaludin

Department of Mathematics, Science and Computer

Politeknik Tuanku Syed Sirajuddin

Publisher
Department of Mathematics, Science and Computer,
Politeknik Tuanku Syed Sirajuddin,
Pauh Putra, 02600 Arau, Perlis
Tel: 04-988 6200
Fax: 04-988 6300
http://www.ptss.edu.my/

Discrete Mathematics
Exploring Sets, Relations & Functions

e ISBN 978-967-2258-83-4

Appreciation

It has been our good fortune to have advice and guidance of many expert whose knowledge and skills have enhanced this book in many ways. For the valuable help, we thank you very much.

Nur Intan Syazwani Binti Jamaludin
Raja Noorliyana Binti Raja Almanan

Abstract

Discrete Mathematics Exploring Sets, Relations and Functions is designed specially for Digital Technology students in polytechnics. It aims to boost their understanding of this related content area. The goal of this book is to be the student's special reference anywhere and anytime. It can also be practiced by individuals who can benefit from the content as basic concepts provided have been comprised of simple and understandable forms.

Each chapter in this book is filled with diagrams, tips, compact notes and sample questions with solutions for quick references. Exercises given are completed with answers as guidelines.

Hopefully, this book can stimulate students' interest in learning and become a catalyst for their success.

Table of Content

Appreciation ii

Abstract iii

Topic 1: Sets 1

Topic 2: Relations 26

Topic 3: Functions 39

References 56

SETS AND SET OPERATIONS

1.1 Basic Concept of Sets Terminology

Set theory is a foundation of contemporary arithmetic, and is being used in official explanations. The concept of set is reserved as basic; therefore, we do not attempt to describe what a set is, but we can give an easy description, define significant properties of sets, and provide illustrations. All supplementary ideas of mathematics can be constructed based on the concept of set. Comparable but in casual arguments, sets also mentioned as collection, group, and aggregate.

A set generally is a collection of objects known as the members or elements of that set. Let's say, we have a set and we denote that some items that fit into this set are inside the set while items that do not fit into this set are not inside the set. A set is an abstract thing where its members do not necessarily have to be physically assembled mutually for them to create a set. Furthermore, we also state that sets contain their elements.

For example, the set of trainees in an apartment; the list of alphabets in a word; or the set of real numbers. Thus, sets may be made up of elements of countless natures such as colour, people, physical objects, numbers, and signs. We will use the words object in an extensive method to comprise all these diverse types of items.

1.2 Definition of Sets

A set is a collection of objects, things and symbols which have a certain common property. Sets can be defined in 3 ways which are using words (description method), by listing elements that separated by commas (roster method), and by combination of words and listing elements (set-builder notation). The notation used for sets is braces, {}.

{prime numbers}

Description method

Read as 'set of prime number'.

$\{2, 3, 5, \ldots\}$

Roster method

Read as 'elements of the set are 2, 3, 5 and so on'.

Note!

3 dots are used to indicate that elements have been omitted when listing all the elements is impractical.

$\{x : x \geq 1, x \text{ is an odd positive integer}\}$

Set-builder notation

Read as 'set of all x such that x is an odd positive integer'.

a. Members or Elements

The individual objects in a set are called the members or elements of the set.

The same elements in a set need not be repeated.

elements

$A = \{a, b, c, d, e\}$

Note!

Sets are usually denoted by capital letters.

Read as 'members of set A are a, b, c, d and e'.

Z = {letters in the word *'HELLO'*}

$Z = \{H, E, L, O\}$

Epsilon, ∈ , is used to denote an object 'is an element of' or 'is a member of' a set. While ∉ is used to denote 'is not an element of' or 'is not a member of'.

If $A = \{5, 6, 7, 8\}$, then $6 \in A$ and $9 \notin A$

b. Number of Elements

The in a set A is written as n(A).

If $A = \{p, q, r, s\}$, then n(A) = 4

c. Empty Set

A null or empty set has no elements in it and is represented by the symbol ϕ or $\{\}$.

> $A = \{$month with 40 days$\}$
>
> There is no element in set A
>
> $\therefore A = \{\}$
>
> $B = \{x : x$ is an even number $< 10\}$
>
> $B = \{2, 4, 6, 8\}$
>
> $\therefore B \neq \{\}$

Note!

- $A = \{0\}$ or $\{\phi\}$ does not denote that A is an empty set.
- $A = \{0\}$ means that there is an element '0' in set A.
- $A = \{\phi\}$ means that there is an element 'ϕ' in set A.

d. Subset

If every element of set A is a member of set B, then set A is a subset of B. Using symbols, this is written as $A \subset B$. To calculate the number of subsets that can be obtained from a given set, we can use the formula 2^n where n is the number of elements in the set. Empty set is a subset of all sets and every set is a subset of itself.

> Number of subsets = 2^n,
>
> n = number of elements in the set

> $A = \{2, 4, 6\}$, $B = \{2, 4, 6, 8, 10\}$
>
> $\therefore A \subset B$

e. Universal set

A universal set is a set which includes all the sets in a discussion. It is usually denoted by ξ.

$$\xi = \{x : x \text{ is real number}\}$$

f. Equal set

Two sets are equal, denoted by $A = B$ if the elements in both sets are same. If set A is equal to set B, then each element in set A belongs to set B and vice versa. Nonetheless, if there exist one element that is not common to both, then the sets are said to be unequal and denoted by $A \neq B$

Equal sets: $A = B$
Unequal sets: $A \neq B$

Note!

A set does not change if the arrangement of its elements is reshuffled.

Note!

Equivalent sets have the same number of elements but do not necessarily equal. Hence, equivalent sets are not the same as equal sets.

Example 1

Given that $\xi = \{x : x \text{ is positive integer}\}$, set $A = \{x : x \text{ is odd number} < 6\}$.

a. List all the elements of set A.

b. State the number of elements in set A.

c. Determine whether set A is empty set or not.

d. Calculate the number of subsets for set A.

e. List all the subsets of set A.

f. If set $B = \{5, 3, y - 2\}$ and $A = B$, find the value of y.

Solution

a. Set $A = \{x : x \text{ is odd number} < 6\}$

$= \{1, 3, 5\}$

Only consider elements that belong to the universal set provided.

b. $n(A) = 3$

c. A is not an empty set

$\therefore A \neq \{\}$ or $A \neq \phi$

d. Number of subsets $= 2^3$

$= 8$

e. Subsets of set $A = \{\}, \{1\}, \{3\}, \{5\}, \{1, 3\}, \{1, 5\}, \{3, 5\}, \{1, 3, 5\}$

f. $A = B$

$\therefore y - 2 = 1$

$y = 3$

Compare each element in set A and set B give:
$\{1, 3, 5\} = \{5, 3, y - 2\}$

Exercise 1

1. Given that set P = {letters in the word '*MALAYSIA*'} and set Q = {factors of 8}.

   a. List all the elements of each of the following.

      i. set P

      ii. set Q

   b. State the value of each of the following and determine whether the sets are empty set or not.

      i. n(P)

      ii. n(Q)

2. Calculate the number of subsets for each of the following sets.

   a. $V = \{x, y\}$

   b. $W = \{0, 1, 2\}$

3. List the subsets of each of the following sets.

   a. $V = \{y, z\}$

   b. $W = \{1, 2, 3\}$

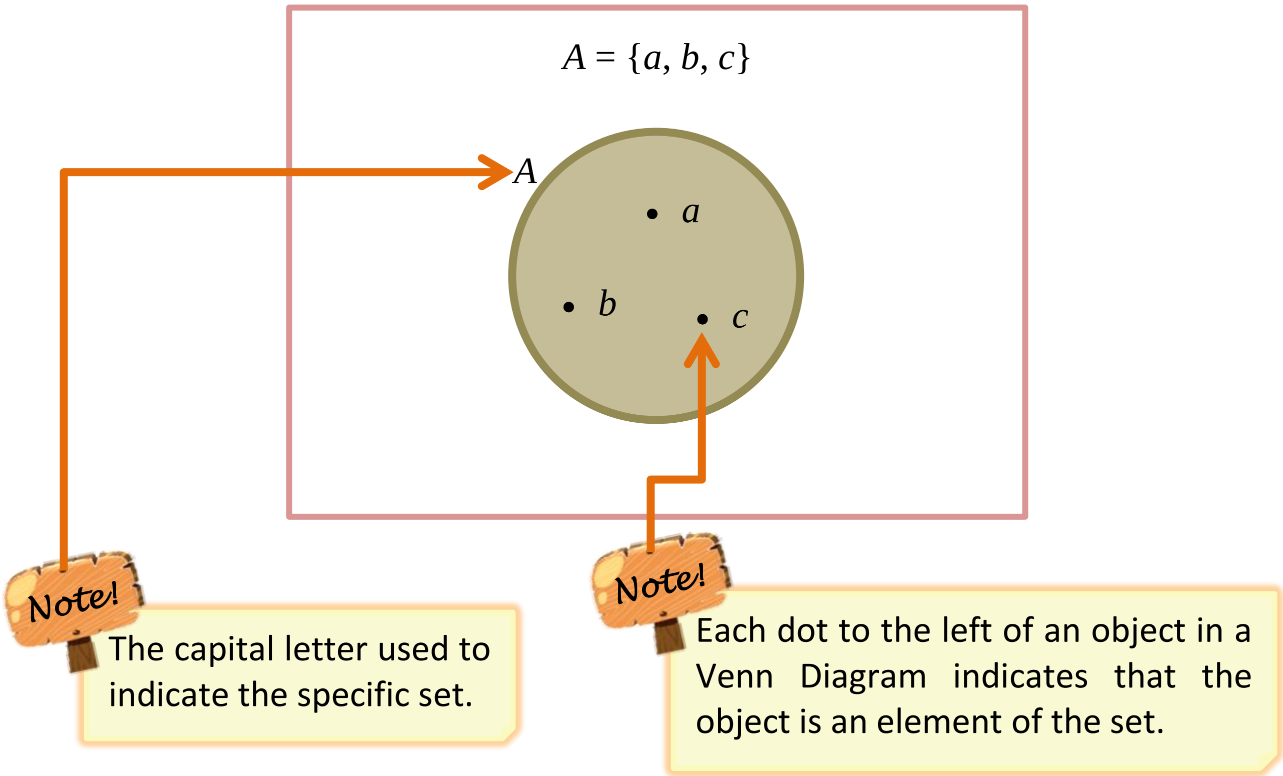
4. Determine whether each of the following pair of sets are equal or not.

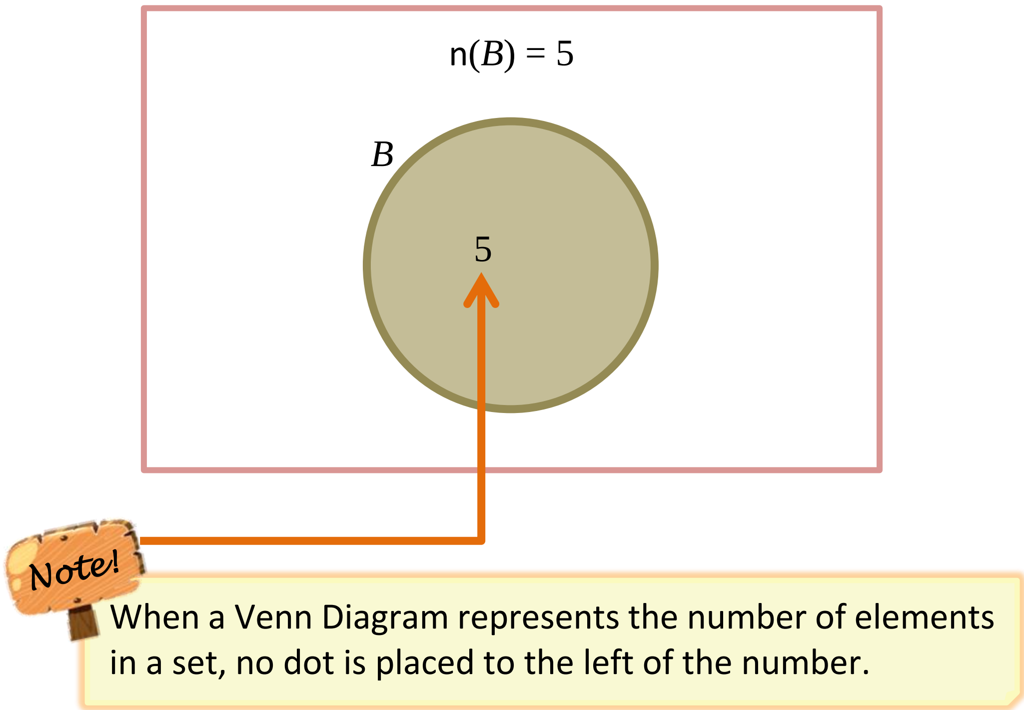
   a. $R = \{x, y, z\}$ and $S = \{y, z, x\}$

   b. P = {multiples of 3 less than 6} and Q = {factors of 3}

1.3 Venn Diagrams

Geometrical shapes such as circles, ovals, rectangles and triangles can be used to enclose elements of a set in representing the sets. This illustration is called as Venn Diagrams.





Example 2

Draw a Venn diagram to represent the following sets.

a. $E = \{4, 5, 6\}$

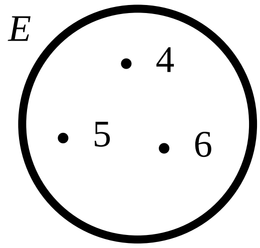
b. $F = \{w, x, y, z\}$

c. $n(G) = 12$

d. $n(H) = 0$

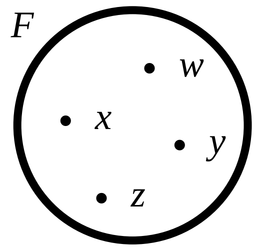
Solution

a.

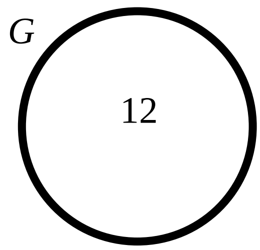


Each element must have dot to the left of the number.

b.

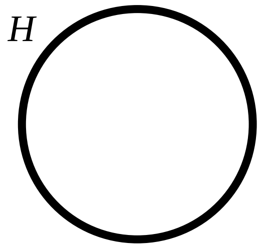


c.



No dot is placed to the left of the number.

d.



No object inside the circle since the number of elements for set H is 0.

Exercise 2

1. Draw a Venn diagram to illustrate each of the following sets.

   a. $J = \{3, 6, 9\}$

   b. K = {vowels in the word '*BEST*'}

   c. $L = \{x : -2 < x < 2$, x is an integer$\}$

   d. M = {two-digit numbers less than 14 whose units' digit is less than 3}

2. Draw a Venn diagram to represent the number of elements in each of the following sets.

   a. P = {factors of 9}

   b. Q = {letters in the word '*SCORE*'}

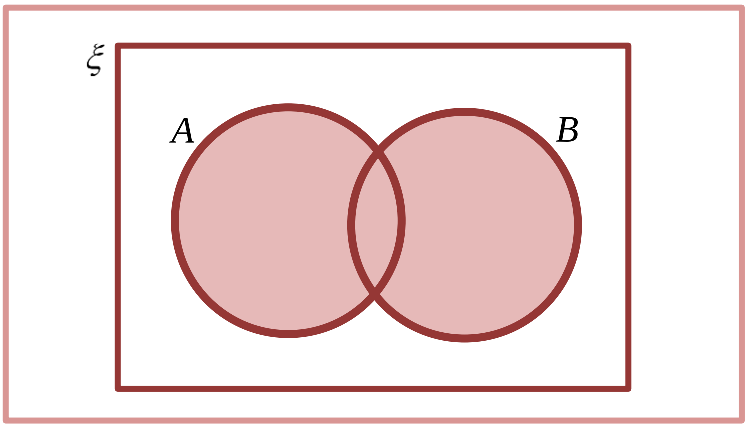
   c. $R = \{x : 20 < x < 29$, x is an even integer$\}$

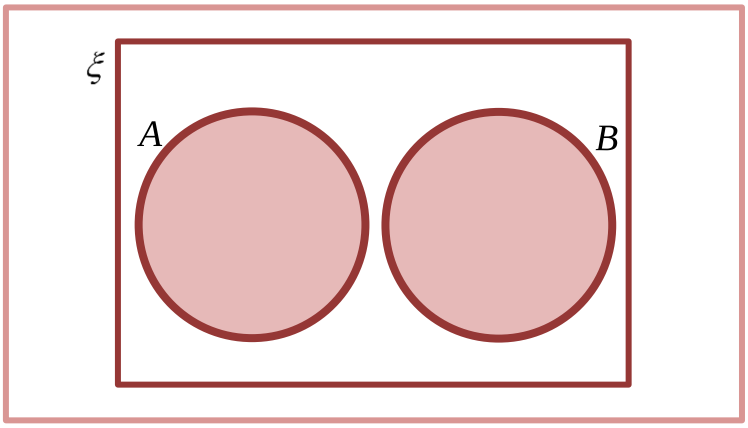
   d. S = {multiples of 13 less than 50}

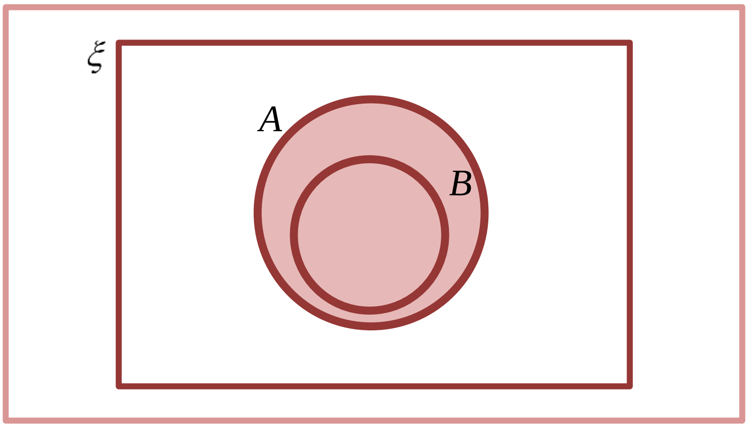
1.4 Set Notation and Operation

a. Union

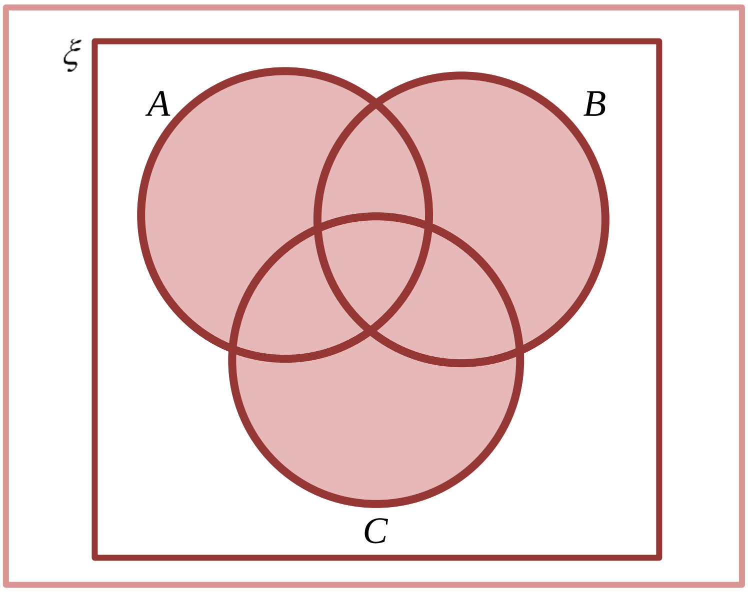
The union of sets A and B, denoted by $A \cup B$ is the set of all elements which belong to A or B or both A and B. The shaded region of each of the following Venn diagrams illustrates the relationship of $A \cup B$.







The union of three sets A, B and C denoted by $A \cup B \cup C$ being the set of elements which belong to either one, two or all three sets. This relationship can be illustrated by the Venn diagram below:



If $A = \{a, b, c\}$, $B = \{b, c, d\}$,

then $A \cup B = \{a, b, c, d\}$ and $n(A \cup B) = 4$

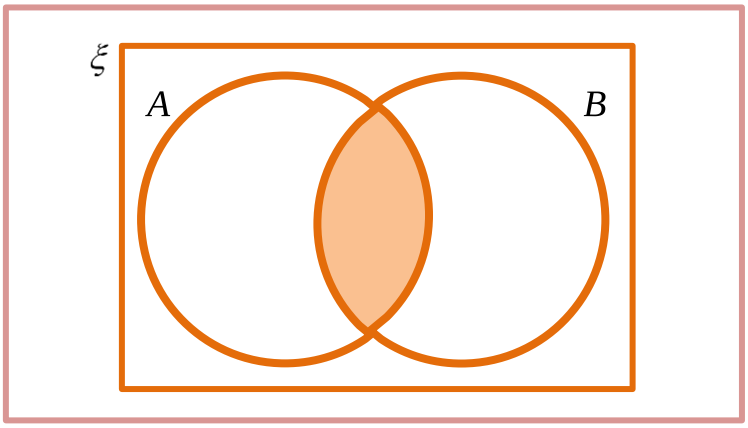
Answer written as
$A \cup B = \{a, b, c, b, c, d\}$ is incorrect.

Note!

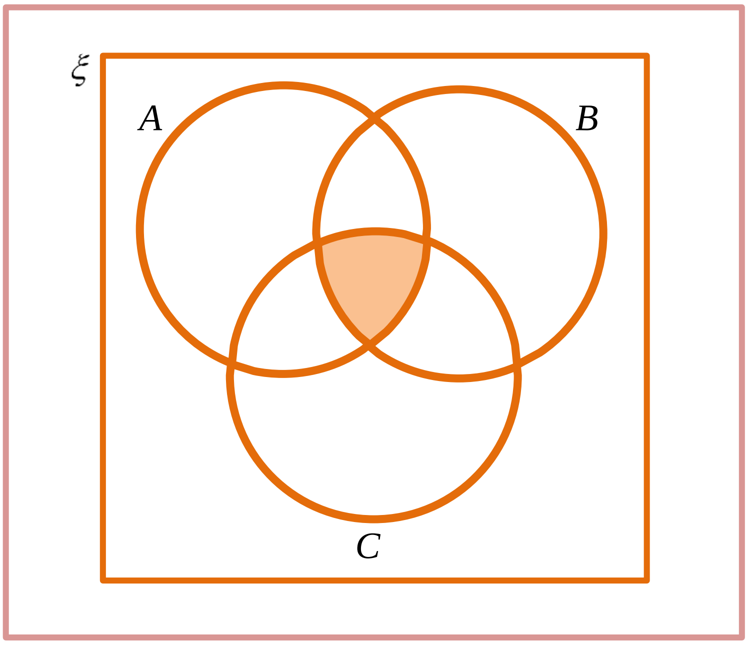
Error due to repeating elements when expressing a set.

b. Intersection

The intersection of sets A and B, denoted by $A \cap B$ is the set of elements which are common to both A and B. This relationship can be illustrated by the Venn diagram below:



The intersection of three set, A, B and C denoted by $A \cap B \cap C$ is the set of elements which are common to three sets. This relationship can be illustrated by the Venn diagram below:

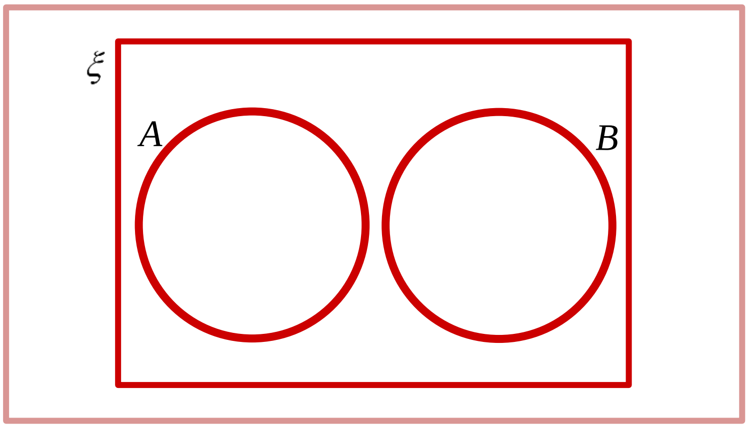


If $A = \{a, b, c\}$, $B = \{b, c, d\}$,

then $A \cap B = \{b, c\}$ and $n(A \cap B) = 2$

c. Disjoint

Two sets are said to be disjoint sets if they have no element in common, denoted by $A \cap B = \phi$. Equivalently, two disjoint sets are sets whose intersection is the empty set. This relationship can be illustrated by the Venn diagram below:



If $A = \{a, b, c\}$, $B = \{d, e, f\}$,

then $A \cap B = \{\}$ and $n(A \cap B) = 0$

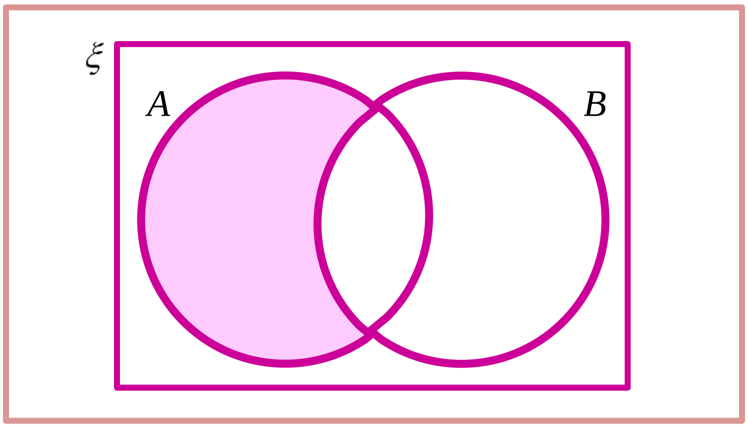
Answer written as
$A \cap B = \{0\}$ and
$n(A \cap B) = \{0\}$ or $\{\}$ or ϕ
are incorrect.

Note!

Error related to the empty set.

d. Difference

The difference between sets A and B, denoted by $A\backslash B$ or $A - B$ is the set of remaining elements in sets A after the elimination of all common elements to both A and B. Difference between A and B consists of all elements in A and not in B. This relationship can be illustrated by the Venn diagram below:



If $A = \{a, b, c\}$, $B = \{a, e, f\}$,

then $A - B = \{b, c\}$ and $n(A - B) = 2$

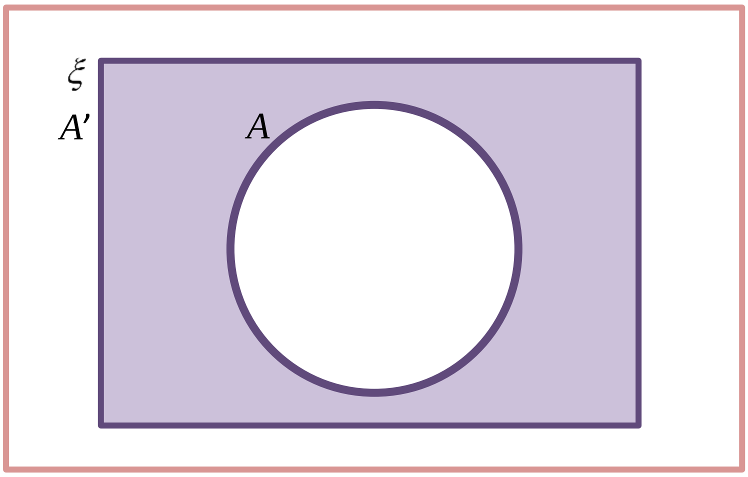
Answer written as
$A - B = \{b, c, e, f\}$ and
$n(A - B) = 4$
are incorrect.

Note!

Error related to the difference between sets.

e. Complement

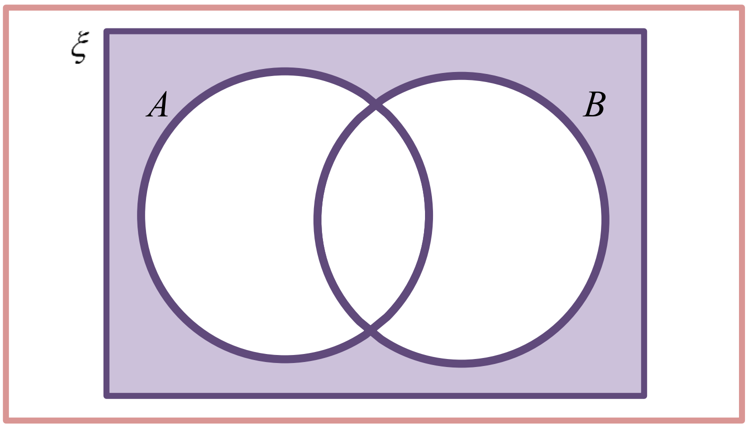
The complement of a set A written as A' is a set of all elements in the universal set ξ that are not found in A. This relationship can be illustrated by the Venn diagram below where the shaded region represents the complement, A'.

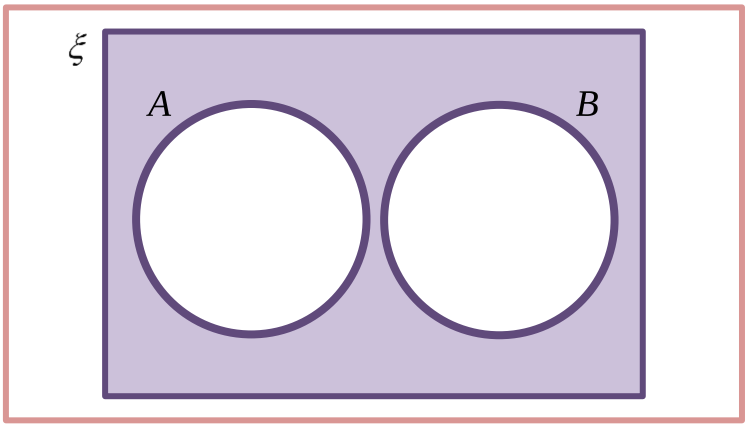


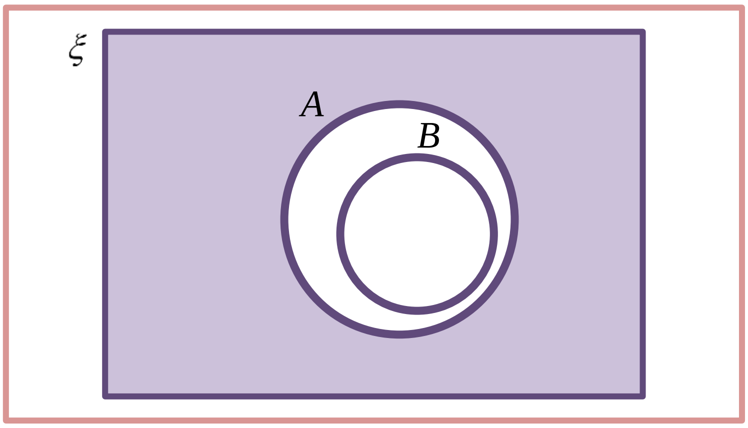
If $A = \{a, c\}$, $\xi = \{a, b, c, d, e, f, g\}$,

then $A' = \{b, d, e, f, g\}$

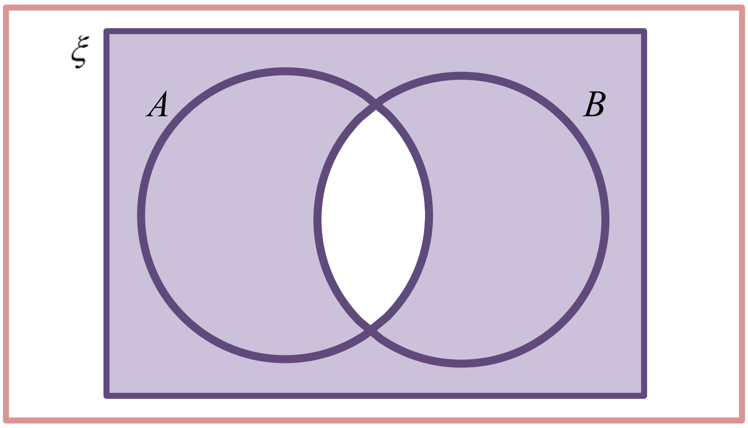
The complement of the union of two sets A and B, denoted by $(A \cup B)'$ is the set of elements in ξ which are not the element of $A \cup B$. The shaded region of each of the following Venn diagrams illustrates the relationship of $(A \cup B)'$.







Meanwhile, the complement of the intersection of two sets A and B, denoted by $(A \cap B)'$ is the set of elements in ξ which are not the element of $A \cap B$. The shaded region of the following Venn diagram illustrates the relationship of $(A \cap B)'$.



Example 3

Given that $\xi = \{x : 1 \leq x \leq 12, x \text{ is integer}\}$, set $P = \{x : x \text{ is an even number}\}$, set $Q = \{x : x \text{ is a factor of } 12\}$ and set $R = \{x : x \text{ is a multiple of } 4\}$.

a. Find $(P \cap Q)'$.

b. Find $P' \cup Q \cup R'$.

c. Determine whether set P and set R are disjoint sets or not.

d. List all the elements of $Q - R$.

e. Represent the sets ξ, P, Q and R in a Venn diagram.

Solution

$\xi = \{x : 1 \leq x \leq 12, x \text{ is integer}\}$

$= \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12\}$

> It is easier if we list down all the elements of each set before solving the problems.

$P = \{x : x \text{ is an even number}\}$

$= \{2, 4, 6, 8, 10, 12\}$

$Q = \{x : x \text{ is a factor of } 12\}$

$= \{1, 2, 3, 4, 6, 12\}$

$R = \{x : x \text{ is a multiple of } 4\}$

$= \{4, 8, 12\}$

a. $(P \cap Q)' = (\{2, 4, 6, 8, 10, 12\} \cap \{1, 2, 3, 4, 6, 12\})'$

$= \{2, 4, 6, 12\}'$

$= \{1, 3, 5, 7, 8, 9, 10, 11\}$

b. $P' \cup Q \cup R' = \{2, 4, 6, 8, 10, 12\}' \cup \{1, 2, 3, 4, 6, 12\} \cup \{4, 8, 12\}'$

$= \{1, 3, 5, 7, 9, 11\} \cup \{1, 2, 3, 4, 6, 12\} \cup \{1, 2, 3, 5, 6, 7, 9, 10, 11\}$

$= \{1, 2, 3, 4, 5, 6, 7, 9, 10, 11, 12\}$

c. $P \cap R = \{2, 4, 6, 8, 10, 12\} \cap \{4, 8, 12\}$

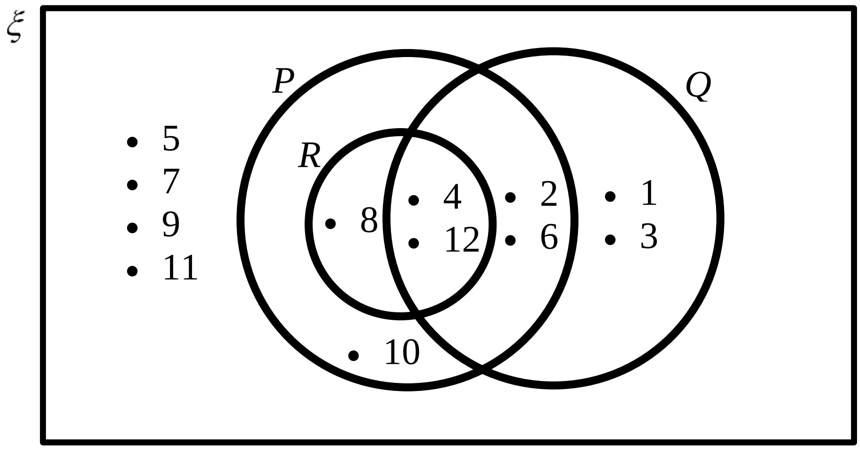
$= \{4, 8, 12\}$ where $R \subset P$

$\therefore$ P and R are not disjoint sets

d. $Q - R = \{1, 2, 3, 4, 6, 12\} - \{4, 8, 12\}$

$= \{1, 2, 3, 6,\}$

e.



Exercise 3

1. Given that the universal set $\xi = \{x : x$ is an integer, $1 \leq x \leq 10\}$, $P = \{x : x$ is an odd number$\}$ and $Q = \{x : x$ is a factor of $9\}$.

   a. Draw a clearly labeled Venn diagram to illustrate the ξ, P and Q.

   b. List the elements of the sets:

      i. P'

      ii. Q'

   c. List all the subsets of set Q.

2. Given that the universal set $\xi = \{x : x$ is an integer and $1 \leq x \leq 10\}$, $P = \{1, 3, 5, 7, 9\}$, $Q = \{2, 3, 5, 7\}$ and $R = \{2, 4, 5, 6, 7, 8, 9\}$. List the elements of each of the following sets:

   a. $P \cap Q$

   b. $P \cap Q \cap R$

   c. $Q \cap R$

   d. $P' \cap (Q \cap R)$

   e. $(P \cap Q)'$

   f. $(P \cap Q)' \cap R$

3. Given that the universal set $\xi = \{x : 1 \leq x \leq 9$ and x is an integer$\}$, $P = \{x : x$ is an odd number$\}$ and $Q = \{x : x$ is a multiple of $3\}$.

   a. List the elements of sets P and Q.

   b. Find $P \cup Q$.

   c. Represent $P \cup Q$ in a Venn diagram.

4. Given that $A = \{a, b, c, d, e, f, l\}$, $B = \{e, f\}$, $C = \{c, d, e, f, g, h, l\}$ and $\xi = A \cup B \cup C$.

a. Draw a Venn diagram to show the relationship between sets A, B and C.

b. List the elements $A' \cup B \cup C'$.

c. Find $n(A \cup B')$.

5. Given that $A = \{a, b, c, d\}$and $B = \{a, c, f, g\}$, find:

a. $A - B$

b. $B - A$

6. Given that $A = \{m, n, p, q\}$ and $B = \{m, n, r\}$, find:

a. $A - B$

b. $B - A$

7. Given that $A = \{1, 2, 3, 4\}$, $B = \{2, 4, 6, 8\}$ and $C = \{3, 4, 5, 6\}$, find:

a. $A \backslash B$

b. $B \backslash A$

c. $B \backslash C$

1.5 D'Morgan's Law

A simple way to memorize D'Morgan's Law is that each term is complemented, and then the ORs become ANDs, and the ANDs become ORs.

$$(A \cup B)' = A' \cap B'$$

$$(A \cap B)' = A' \cup B'$$

Example 4

Use D'Morgan's law on the expressions:

a. $\neg (A \wedge B \wedge C)$

b. $\overline{AB + \overline{C}D}$

Solution

a. $\neg (A \wedge B \wedge C) = \neg A \vee \neg B \vee \neg C$

b. $\overline{AB + \overline{C}D} = (\overline{AB})(\overline{\overline{C}D})$

$= (\overline{A} + \overline{B})(\overline{\overline{C}} + \overline{D})$

$= (\overline{A} + \overline{B})(C + \overline{D})$

Simplify the problems to the simplest form by using Boolean Identities.

Exercise 4

Use D'Morgan's law on the expressions:

a. $\neg (A \vee B \vee C)$

b. $\neg (E \wedge F \wedge G \wedge H)$

c. $\neg (E \vee F \vee G \vee H)$

d. $\overline{\overline{C}(\overline{A}+B)}$

Answer

Exercise 1

1. a. i. $\{M, A, L, Y, S, I\}$ ii. $\{1, 2, 4, 8\}$

   b. i. $n(P) = 6, P \neq \phi$ ii. $n(Q) = 4, Q \neq \phi$

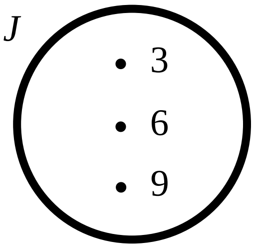
2. a. 4 b. 8

3. a. $\{\}, \{y\}, \{z\}, \{y, z\}$ b. $\{\}, \{1\}, \{2\}, \{3\}, \{1, 2\}, \{1, 3\}, \{2, 3\}, \{1, 2, 3\}$

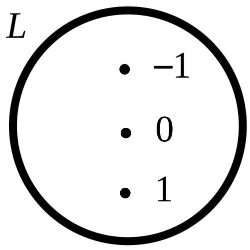
4. a. $R = S$ b. $P \neq Q$

Exercise 2

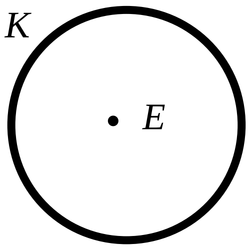
1. a.



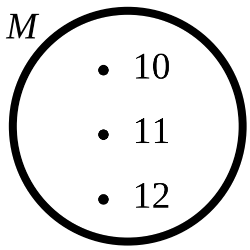
c.



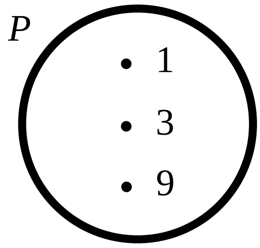
b.



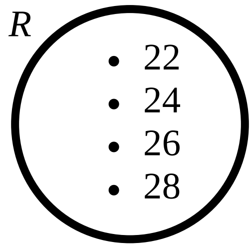
d.



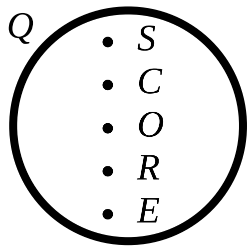
2. a.



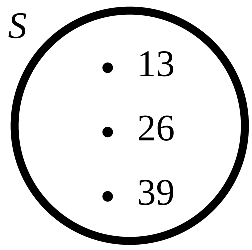
c.



b.

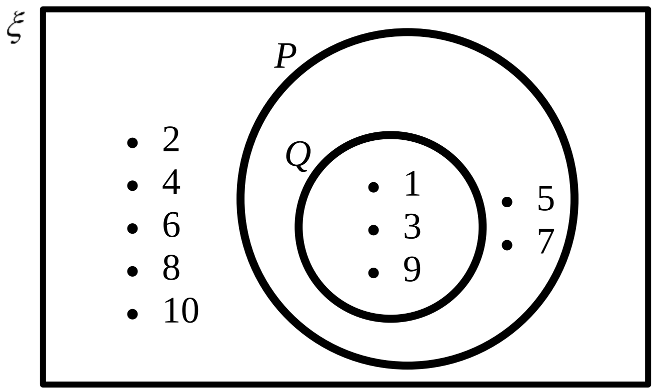


d.



Exercise 3

1. a.

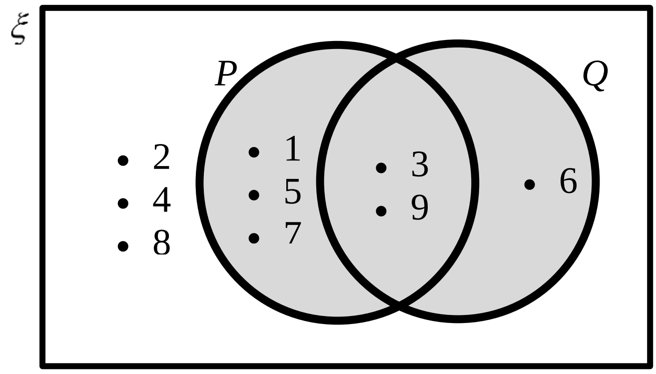


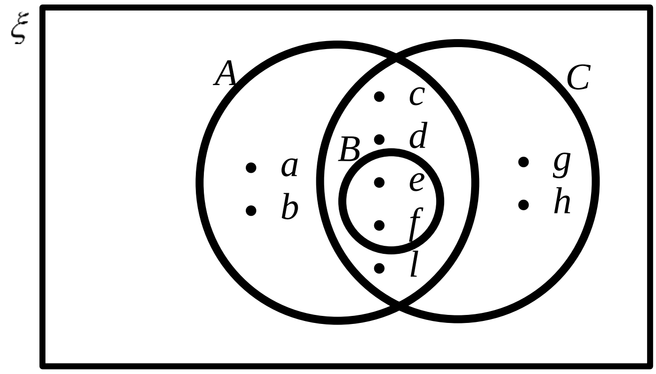
b. i. {2, 4, 6, 8, 10} ii. {2, 4, 5, 6, 7, 8, 10}

c. {}, {1}, {3}, {9}, {1, 3}, {1, 9}, {3, 9}, {1, 3, 9}

2. a. {3, 5, 7}
   b. {5, 7}
   c. {2, 5, 7}
   d. {2}
   e. {1, 2, 4, 6, 8, 9, 10}
   f. {2, 4, 6, 8, 9}

3. a. P = {1, 3, 5, 7, 9},
   Q = {3, 6, 9}
   b. {1, 3, 5, 6, 7, 9}

c. 

4. a. 

b. $\{a, b, e, f, g, h\}$

c. 9

5. a. $\{b, d\}$

b. $\{f, g\}$

6. a. $\{p, q\}$

b. $\{r\}$

7. a. $\{1, 3\}$

b. $\{6, 8\}$

c. $\{2, 8\}$

Exercise 4

1. a. $\neg A \wedge \neg B \wedge \neg C$

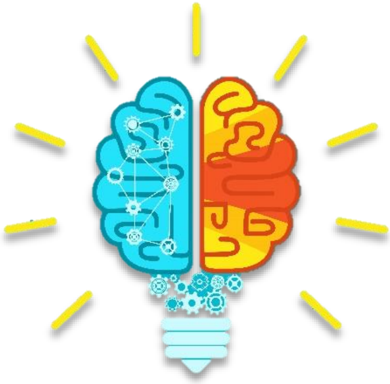
b. $\neg E \vee \neg F \vee \neg G \vee \neg H$

c. $\neg E \wedge \neg F \wedge \neg G \wedge \neg H$

d. $C + A\overline{B}$

RELATION AND THE REPRESENTATION

2.1 Definition of Relation



The relationship among the elements of the sets is the subsequent component that comes up every time sets are being discussed. Relations may occur between objects of the same set or between objects of two or more sets.

A relation is also known as a relationship between sets of values. In detail, the relation between the x-values and y-values of ordered pairs where the set of all x-values is called the domain, and the set of all y-values has termed as the range.

A relation is a type of connection occurring between objects. For example, ‘sibling of’, ‘group of’, ‘piece of’, ‘is younger than’, ‘is an antecedent of’, ‘is a division of’ are relations that exist in our daily life that fall under the category of binary relations.

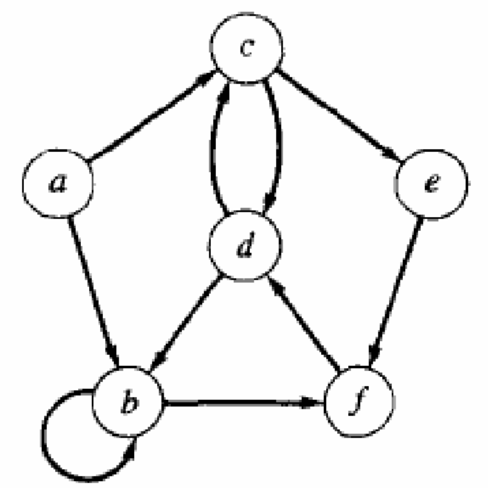
Properly, we will describe relations between elements of sets by addressing Rab or aRb for ‘a bears R to b’. When formalising relations as sets of ordered pairs regarding their elements, we will formally write $<a, b> \in R$.

2.2 Properties and Representation of Relations

There are 3 popular ways to represent a relation. Normally, we denote a relation by ordered pair called product set. A relation also can be illustrated using a directed graph or a Boolean matrix. Below are the examples of representations of a relation, R.

$$R = \{(a, b), (a, c), (b, f), (c, e), (c, d), (d, c), (d, b), (f, d), (e, f), (b, b)\}$$

Product set



Directed graph / Digraph

$$R = \begin{array}{c} \\ a \\ b \\ c \\ d \\ e \\ f \end{array} \begin{array}{c} \begin{array}{cccccc} a & b & c & d & e & f \end{array} \\ \begin{bmatrix} 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix} \end{array}$$

Boolean matrix / Binary matrix

Example 1

Given that set $A = \{2, 3, 4\}$.

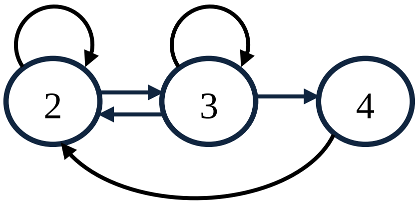
Relation, $R = \{(2, 2), (2, 3), (3, 3), (3, 2), (3, 4), (4, 2)\}$.

a. Draw digraph for the relation R.

b. Find the Boolean matrix for the relation R.

Solution

a.



The number of arrows in digraph must equal to number of ordered pairs in R.

b.

$$R = \begin{matrix} & \begin{matrix} 2 & 3 & 4 \end{matrix} \\ \begin{matrix} 2 \\ 3 \\ 4 \end{matrix} & \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 1 & 0 & 0 \end{bmatrix} \end{matrix}$$

The number of digits 1 in Boolean matrix must equal to number of ordered pairs in R.

Exercise 1

1. Given that set $A = \{1, 2, 3, 4\}$.

   Relation, $R = \{(1, 1), (1, 2), (2, 3), (3, 3), (3, 2), (3, 1), (4, 2)\}$.

   a. Draw digraph for the relation R.

   b. Find the Boolean matrix for the relation R.

2. Given that set $B = \{a, b, c\}$.

   Relation, $M = \{(a, b), (a, c), (a, a), (c, c), (c, a), (b, b)\}$.

   a. Draw digraph for the relation M.

   b. Find the Boolean matrix for the relation M.

3. Given that set $K = \{p, q, r, s, t\}$.

   Relation, $Z = \{(p, r), (r, r), (p, q), (p, p), (q, q), (r, p), (r, q), (r, s), (s, r)\}$.

   a. Draw digraph for the relation Z.

   b. Find the Boolean matrix for the relation Z.

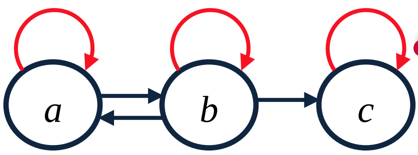
2.3 Equivalent Relation

A relation R is called an equivalent relation if it is:

- Reflexive
- Symmetric
- Transitive

a. Reflexive

A relation R on a set A is called reflexive if $(a, a) \in R$ for all $a \in A$.



Loop for **every vertex** in the **digraph.**

$$\{(a, a), (a, b), (b, b), (b, a), (b, c), (c, c)\} \text{ on } \{a, b, c\}$$

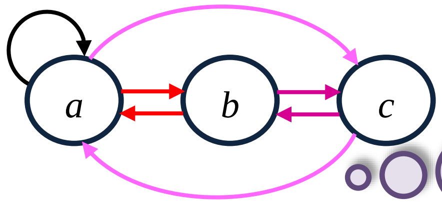
Product set must have all **repeated elements**.

$$R = \begin{array}{c} \\ a \\ b \\ c \end{array} \begin{array}{c} \begin{array}{ccc} a & b & c \end{array} \\ \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \end{array}$$

The **diagonal** elements of **Boolean matrix** always **1**.

b. Symmetric

A relation R on a set A is called symmetric if $(a, b) \in R$, then $(b, a) \in R$.



If there is an **arrow from vertex *a* to vertex *b***, then it must have an **arrow from vertex *b* to vertex *a*** in the **digraph**.

$R = \{(a, a), (a, b), (b, a), (b, c), (c, b), (a, c), (c, a)\}$

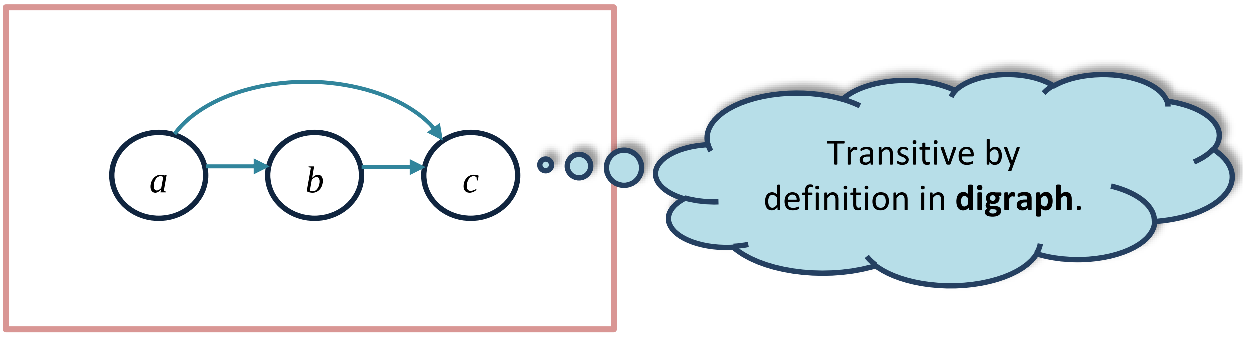
If $\mathbf{(a, b) \in R}$, then $\mathbf{(b, a) \in R}$ for all elements in **product set**.

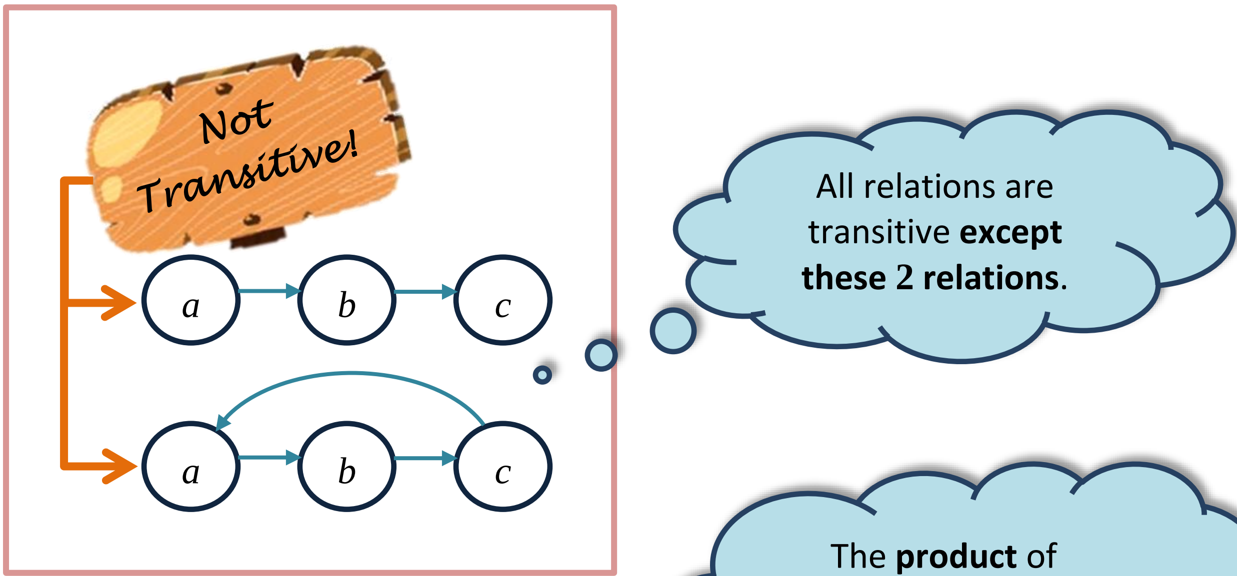
$$R = \begin{array}{c} \\ a \\ b \\ c \end{array} \begin{array}{c} \begin{array}{ccc} a & b & c \end{array} \\ \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} \end{array}$$

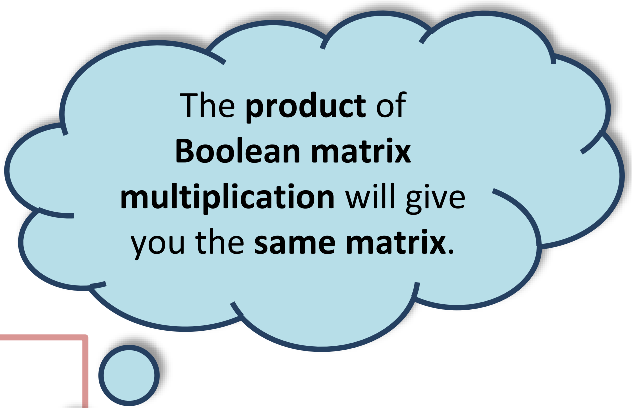
Imagine that the diagonal as a mirror. If $\mathbf{(a, b) = 1}$, then $\mathbf{(b, a) = 1}$ and If $(a, b) = 0$, then $(b, a) = 0$ for all elements in the **Boolean matrix**.

c. Transitive

A relation R on a set A is called transitive if $(a, b) \in R$ and $(b, c) \in R$, then $(a, c) \in R$. Meanwhile, R is not transitive if $(a, b) \in R$ and $(b, c) \in R$ but $(a, c) \notin R$ or a cycle. If there exists no a, b, c that follows that condition, R is still transitive.







$$\begin{bmatrix}1 & 0 & 0\\0 & 1 & 1\\0 & 0 & 1\end{bmatrix} \times \begin{bmatrix}1 & 0 & 0\\0 & 1 & 1\\0 & 0 & 1\end{bmatrix} = \begin{bmatrix}1 & 0 & 0\\0 & 1 & 1\\0 & 0 & 1\end{bmatrix}$$

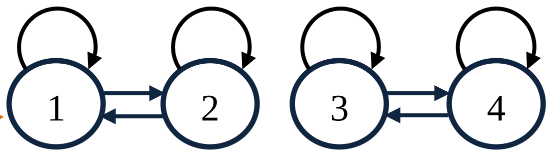
Example 2

If given set $A = \{1, 2, 3, 4\}$ and relation $R = \{(1, 1), (1, 2), (2, 1), (2, 2), (3, 3), (3, 4), (4, 3), (4, 4)\}$. Determine whether the relation is reflexive, symmetric and/or transitive. Hence, determine whether the relation is equivalent relation or not.

Solution

Note!

You may choose any **ONE** representation of R.



Digraph shows all vertices have a loop on them and all connected vertices have a pair of 2-way arrows.

$$\begin{bmatrix}1&1&0&0\\1&1&0&0\\0&0&1&1\\0&0&1&1\end{bmatrix}\times\begin{bmatrix}1&1&0&0\\1&1&0&0\\0&0&1&1\\0&0&1&1\end{bmatrix}=\begin{bmatrix}1&1&0&0\\1&1&0&0\\0&0&1&1\\0&0&1&1\end{bmatrix}$$

All of the Boolean matrices are identical.

Reflexive because digraph shows $(a, a) \in R$ for all $a \in A$.

Symmetric because digraph shows all $(a, b) \in R$ have $(b, a) \in R$.

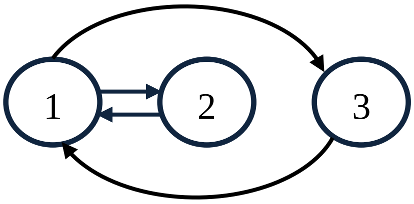
Transitive because the product of Boolean matrix multiplication is equal to the original R.

$\therefore$ Equivalent relation.

Example 3

If given set $A = \{1, 2, 3\}$ and relation $R = \{(1, 2), (2, 1), (1, 3), (3, 1)\}$. Determine whether the relation is reflexive, symmetric and/or transitive. Hence, determine whether the relation is equivalent relation or not.

Solution



Digraph shows all vertices do not have loops on them but all connected vertices have a pair of 2-way arrows.

$$\begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} \times \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}$$

All of the Boolean matrices are not identical.

Not reflexive because digraph shows $(a, a) \notin R$ for all $a \in A$.

Symmetric because digraph shows all $(a, b) \in R$ have $(b, a) \in R$.

Not transitive because the product of Boolean matrix multiplication is different to the original R.

$\therefore$ Not equivalent relation.

Exercise 2

1. Consider the following relations on $A = \{1, 2, 3\}$, determine whether these relations are reflexive, symmetric and/or transitive. Hence, determine whether these relations are equivalent relation or not. Explain your answer.

a. $R_1 = \{(1, 1), (2, 2), (3, 3), (2, 3)\}$

b. $R_2 = \{(1, 1), (2, 2), (3, 3)\}$

c. $R_3 = \{(1, 1), (2, 2), (3, 3), (2, 3), (3, 2)\}$

d. $R_4 = \{(1, 1), (2, 3), (3, 3)\}$

e. $R_5 = \{(1, 2), (2, 1), (2, 3)\}$

2. Consider the following relations on $A = \{1, 2, 3, 4\}$, determine whether these relations are reflexive, symmetric and/or transitive. Hence, determine whether these relations are equivalent relation or not. Explain your answer.

a. $R_1 = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 \end{bmatrix}$

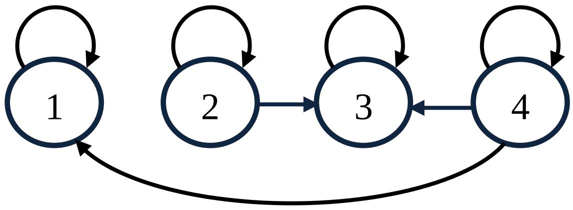
b. $R_2 = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$

c. $R_3 = \begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

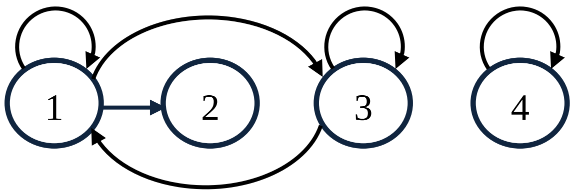
d. $R_4 = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}$

3. Consider the following relations on $A = \{1, 2, 3, 4\}$, determine whether these relations are reflexive, symmetric and/or transitive. Hence, determine whether these relations are equivalent relation or not. Explain your answer.

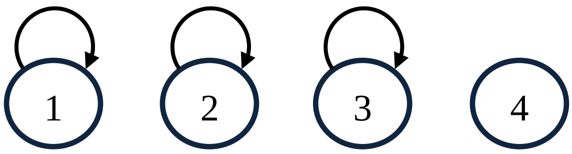
a. R_1



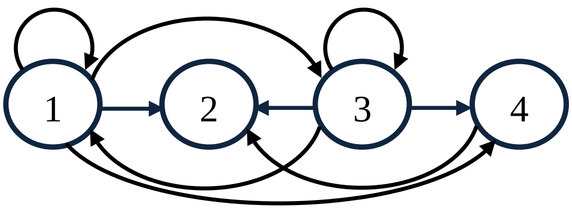
b. R_2



c. R_3



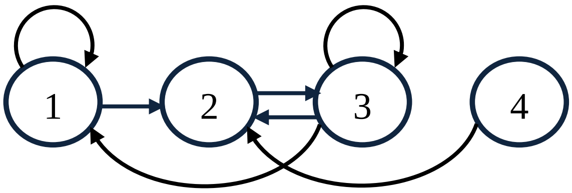
d. R_4



Answer

Exercise 1

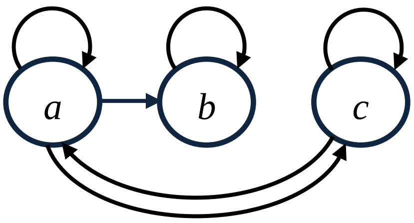
1. a.



b.

$$\begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \end{array}$$

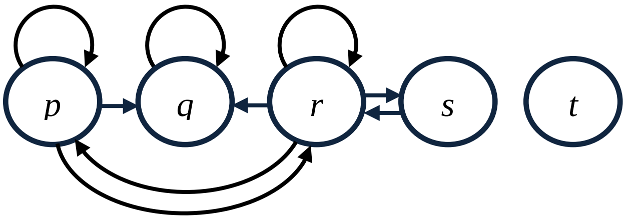
2. a.



b.

$$\begin{array}{c} \\ a \\ b \\ c \end{array} \begin{array}{c} \begin{array}{ccc} a & b & c \end{array} \\ \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \end{array}$$

3. a.



b.

$$\begin{array}{c} \\ p \\ q \\ r \\ s \\ t \end{array} \begin{array}{c} \begin{array}{ccccc} p & q & r & s & t \end{array} \\ \begin{bmatrix} 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

Exercise 2

1. a. Reflexive, not symmetric, transitive. Not equivalent relation.

   b. Reflexive, symmetric, transitive. Equivalent relation.

   c. Reflexive, symmetric, transitive. Equivalent relation.

   d. Not reflexive, not symmetric, transitive. Not equivalent relation.

   e. Not reflexive, not symmetric, not transitive. Not equivalent relation.

2. a. Reflexive, not symmetric, not transitive. Not equivalent relation.

   b. Not reflexive, not symmetric, not transitive. Not equivalent relation.

   c. Reflexive, not symmetric, not transitive. Not equivalent relation.

   d. Reflexive, symmetric, transitive. Equivalent relation.

3. a. Reflexive, not symmetric, transitive. Not equivalent relation.

   b. Not reflexive, not symmetric, not transitive. Not equivalent relation.

   c. Not reflexive, symmetric, transitive. Not equivalent relation.

   d. Not reflexive, not symmetric, not transitive. Not equivalent relation.

FUNCTIONS AND THE CONSTRUCTIONS

3.1 Definition of Functions

The relationship between two sets of numbers is called a function. A function is a relation in which every element in the input has a unique element of output (exactly one). In other words, a function is either one-to-one relation or many-to-one relation (a function f from A to B is said to be many-to-one if and only if two or more elements of A have same image in B).

Function is a subset R of $A \times B$ such that each element of A occurs as the first member of exactly one ordered pair in R. Functions from A to B in the general case are said to be A mapping into B. The element in set A is called the domain (input) and the element in set B is called codomain (output) of the function. A function from A to B is denoted as $f : A \rightarrow B$. The unique element of B which assign to an element of A is called range.

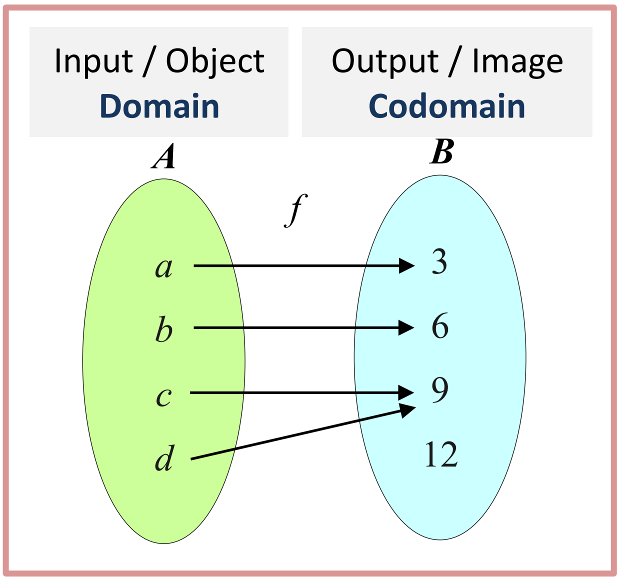
REMEMBER:
*If two element of input maps onto one element of output, that would be a relation, **not a function.***

One-to-many and many-to-many relations are types of relations that are not a function.

Functions are also called **mappings** or **transformations.**

If the range of the function is equal to the codomain, then the function is onto function (or surjection). If no element of B is assigned to more than one element of A (if ab, then $F(a)F(b)$), the function is one-to-one function (or injection). A function that has both is called a one-to-one correspondence (or bijection). Therefore, the relation f^{-1} is a function and one-to-one correspondence.

3.2 Basic Constructions of Functions



Note!
Object = a, b, c, d
Domain = $\{a, b, c, d\}$

Note!
Image = 3, 6, 9, 12
Codomain = $\{3, 6, 9, 12\}$
Range = $\{3, 6, 9\}$

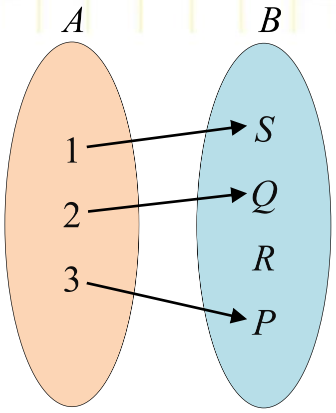
3.3 Properties of Functions

a. One-to-One Functions

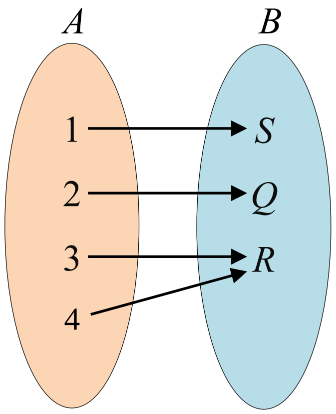
Let A and B be sets. A relation f from A to B is called a one-to-one function if each element of A is assigned / related to exactly one element of B, denoted as $f : A \to B$.

b. Onto Functions

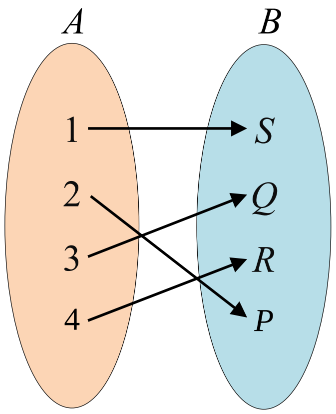
A function $f : A \to B$ is called onto if every element of B is the image of some element in A that is if B = range of f.



One-to-one (injective) but not onto function



Onto (surjective) but not one-to-one function

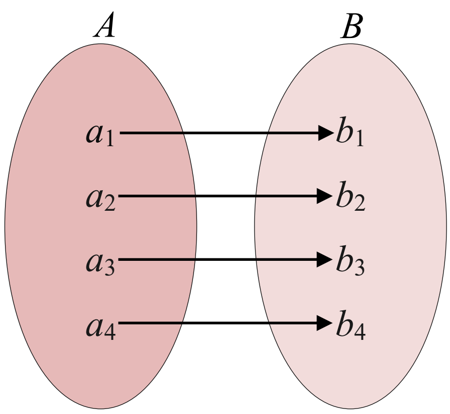


One-to-one and **onto** function

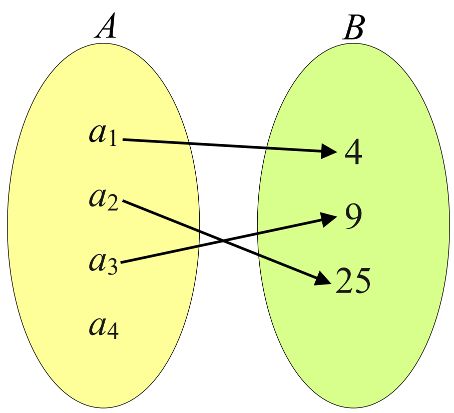
Example 1

Determine whether the following relations are one to one, onto function or both.

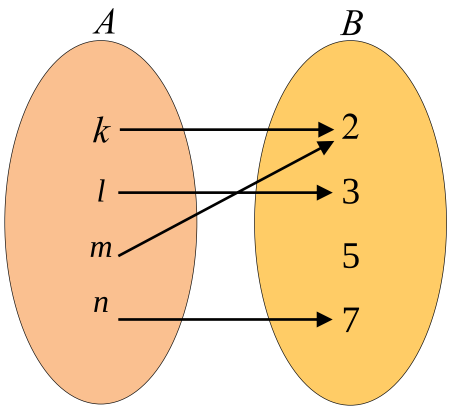
a.



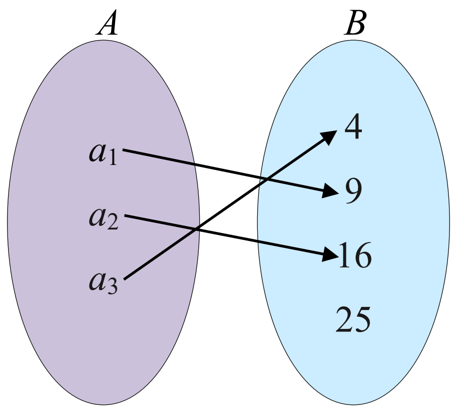
c.



b.

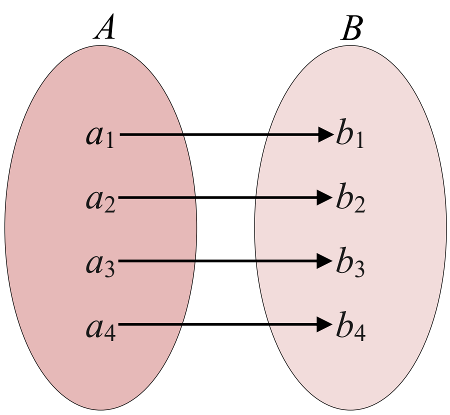


d.



Solution

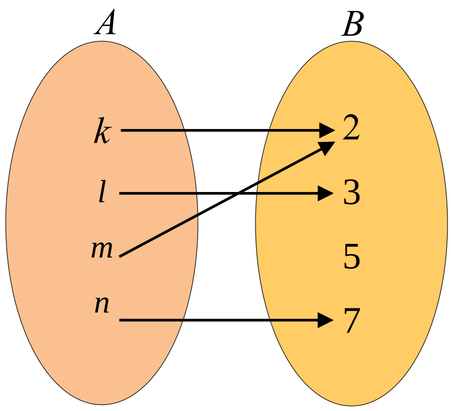
a.



One-to-one and onto function because:

- each element of A has unique element of B
- Range = Codomain

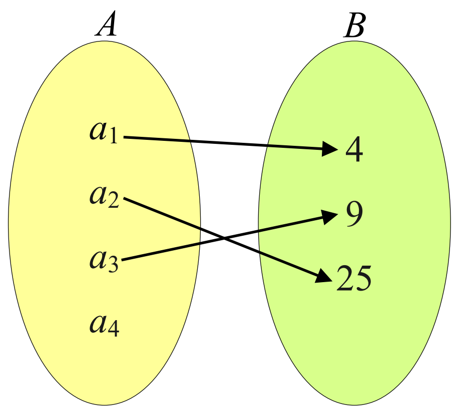
b.



Neither one-to-one nor onto function because:

- more than 1 element of A mapping into B
- Range $\neq$ Codomain

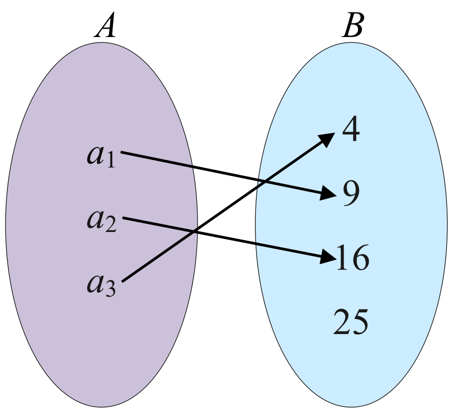
c.



Not one-to-one but onto function because:

- there is domain with no image
- Range = Codomain

d.



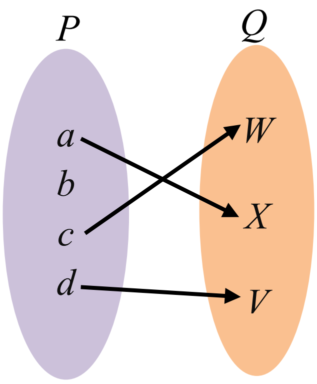
One-to-one but not onto function because:

- each element of A has unique element of B
- Range $\neq$ Codomain

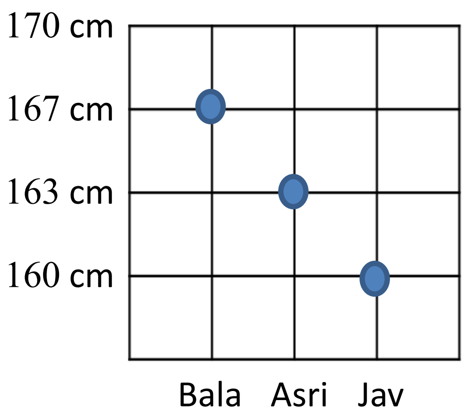
Exercise 1

1. Determine whether the following relations are one to one or onto function.

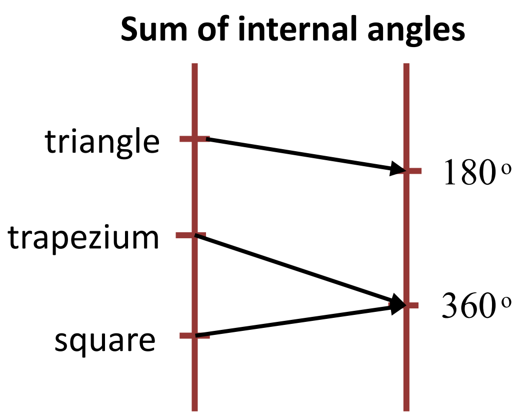
a.



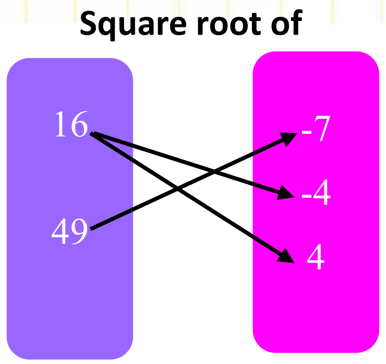
b.



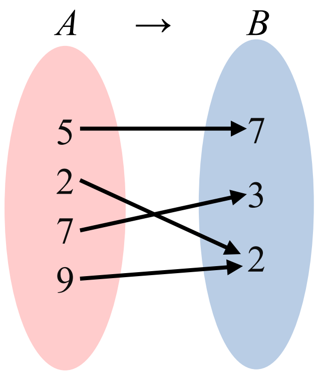
c.



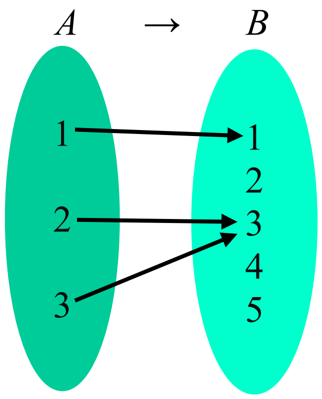
d.



e.



f.



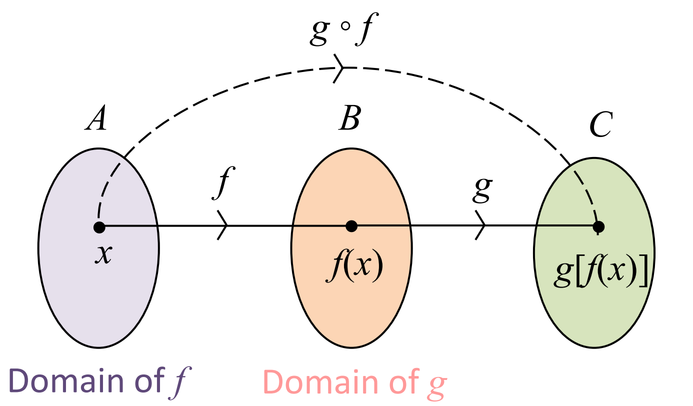
2. Given that $A = \{a, b, c, d\}$ and $B = \{1, 2, 3, 4\}$. Determine whether the following relations are one to one or onto function.

a. $R = \{(a, 1), (b, 2), (c, 3), (d, 4)\}$
b. $R = \{(a, 1), (b, 2), (b, 3), (c, 4)\}$
c. $R = \{(a, 3), (b, 4), (c, 2), (d, 1)\}$
d. $R = \{(b, 2), (a, 1), (c, 1), (d, 2)\}$
e. $R = \{(b, 1), (c, 2), (d, 3)\}$

c. Composition Functions

Composite function is a function of another function's values. The definition of composite function:

$$g \circ f = gf(x) = g[f(x)]$$
$$f \circ g = fg(x) = f[g(x)]$$
$$gf \neq fg$$
$$f^2 = ff,\ f^3 = fff, \ldots$$



*If **both** $f(x)$ and $g[f(x)]$ are defined, then x is the domain of $(g \circ f)(x)$*

Example 2

Given that $f: x \rightarrow 3x$, $g: x \rightarrow 2x + 1$. Find *fg* and *gf*.

Solution

Note!

$fg = f[g(x)]$

$= f(2x + 1)$ — Substitute $g(x)$.

$= 3(2x + 1)$ — Substitute $f(x)$ in term of $g(x)$.

$= 6x + 3$

The **object** of function ***f*** is the **image** of the function ***g***.

$gf = g[f(x)]$

$= g(3x)$ — Substitute $f(x)$.

$= 2(3x) + 1$ — Substitute $g(x)$ in term of $f(x)$.

$= 6x + 1$

The **object** of function ***g*** is the **image** of the function ***f***.

Exercise 2

1. Given that $f : x \rightarrow 2x - 3$, $g : x \rightarrow 1 + x$. Find:

   a. fg

   b. gf

   c. f^2

   d. g^2

2. Given that $f : x \rightarrow 2x + 1$, for $-2 \leq x \leq 3$ and $g : x \rightarrow 6 - 5x$. Find:

   a. gf and its range

   b. $gf(0)$

   c. the value of x if $fg(x) = 8$

3. Given that $f : x \rightarrow 5x + 1$ and $g : x \rightarrow \dfrac{x-4}{2}$. Find :

   a. the value of x if $fg(x) = 11$

   b. the value of x if $gf(x) = x$

   c. $fg(6)$

   d. $g^2(8)$

4. Given that $f(x) = 2x - 3$ and $fg(x) = 7 + 4x$, find the function $g(x)$.

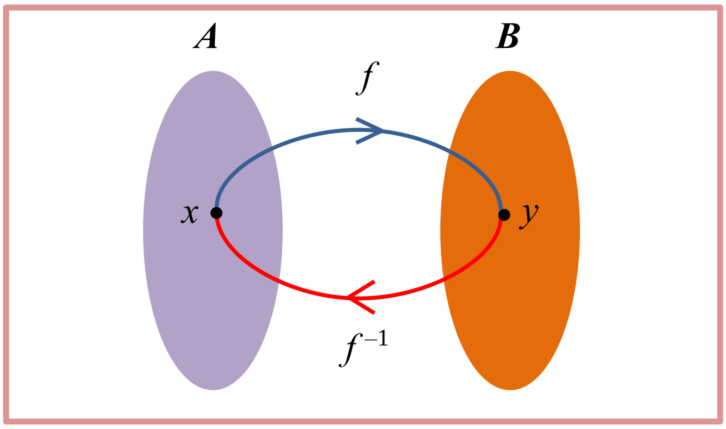
5. Given that $f(x) = 3 - 2x$ and $fg(x) = 4x - 7$, find the function $g(x)$.

6. Given that $f(x) = x - 2$ and $gf(x) = x^2 - 4x + 7$, find the function $g(x)$.

7. Given that $g(x) = \dfrac{2}{x-1}, x \neq 1$ and $fg(x) = 4x + 1$, find the function $f(x)$.

d. Inverse Functions

The inverse function, f^{-1} is the reverse correspondence of function f.



*Function **f** must be a one-to-one function for the inverse function of **f** to be defined.*

- if $f(x) = y$ then $f^{-1}(y) = x$
- $f^{-1} \neq \dfrac{1}{f}$ and $ff^{-1}(x) = f^{-1}f(x)$

Example 3

Given that $f : x \rightarrow 5 - 6x$. Determine the inverse function $f^{-1}(x)$.

Solution

Let $y = f(x)$

Use the definition of inverse function.

$$y = 5 - 6x$$

$$y - 5 = -6x$$

$$\frac{y-5}{-6} = x$$

$$x = \frac{5-y}{6}$$

Note! Solve for x in terms of y.

$$x = f^{-1}(y)$$

$$f^{-1}(y) = \frac{5-y}{6}$$

$$\therefore f^{-1}(x) = \frac{5-x}{6}$$

Note! Rewrite in terms of x.

Exercise 3

1. Given that $g : x \rightarrow 3 - \frac{2}{x}$ .Determine the inverse function g^{-1}.

2. The function h is defined as $h : x \rightarrow \frac{4}{x-5}$. Find

   a. $h^{-1}(x)$
   b. $h^{-1}(8)$

3. Given that $k : x \rightarrow \frac{5-2x}{2-x}, x \neq 2$. Find $k^{-1}(3)$.

4. Given that $f(x) = 3x - 4,\ g(x) = 5x - 6$. Find:

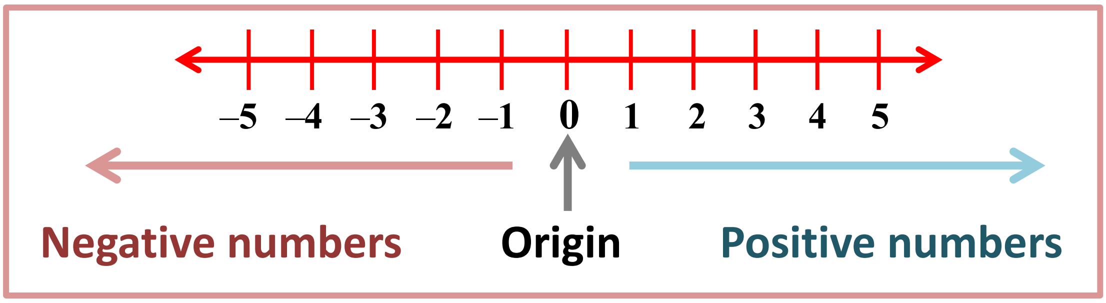
   a. $fg^{-1}(x)$
   b. the value of x such that $fg^{-1}(x) = 2$

5. Given that $f(x) = 4x - 3$, $g(x) = \frac{2x}{x+2}$. Find:

   a. $f^{-1}g(x)$
   b. the value of x such that $gf^{-1}(x) = 3$

3.4 Floor and Ceiling Functions

The floor and ceiling functions are used to get the nearest integer of an assigned number, x in a number line. These two functions often used in application involving data storage and data transmission.



a. Floor Function

The largest integer that is less than or equal to x (the integer on the left side of x in the number line)

Symbol : $\lfloor x \rfloor$

b. Ceiling Function

The smallest integer that is greater than or equal to x (the integer on the right side of x in the number line)

Symbol : $\lceil x \rceil$

c. Floor and Ceiling of Integers

The Floor and Ceiling of Integers is integers. No change.

Formula in Microsoft Excel

=FLOOR(D1,1)

D	E	F
6.01	6	7
-6.2	-7	-6
0.5	0	1
-0.5	-1	0
8	8	8

=CEILING(D1,1)

D	E	F
6.01	6	7
-6.2	-7	-6
0.5	0	1
-0.5	-1	0
8	8	8

Using the FLOOR and CEILING function to round down or up the number to a specific multiple.

Example 4

Find the floor and ceiling values for the following numbers:

a. $\lfloor 6.01 \rfloor$ and $\lceil 6.01 \rceil$

b. $\lfloor -6.2 \rfloor$ and $\lceil -6.2 \rceil$

c. $\left\lfloor \frac{1}{2} \right\rfloor$ and $\left\lceil \frac{1}{2} \right\rceil$

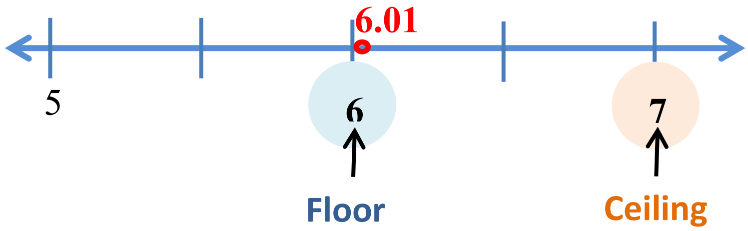
d. $\left\lfloor -\frac{1}{2} \right\rfloor$ and $\left\lceil -\frac{1}{2} \right\rceil$

e. $\lfloor 8 \rfloor$ and $\lceil 8 \rceil$

Solution

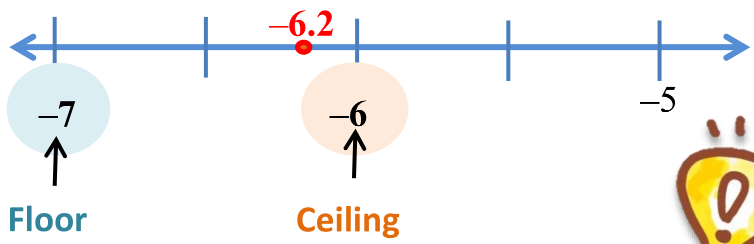
a. $\lfloor 6.01 \rfloor = 6$

$\lceil 6.01 \rceil = 7$



b. $\lfloor -6.2 \rfloor = -7$

$\lceil -6.2 \rceil = -6$



c. $\left\lfloor \frac{1}{2} \right\rfloor = 0$; $\left\lceil \frac{1}{2} \right\rceil = 1$

Change the fraction to a decimal number first and do as above.

d. $\left\lfloor -\frac{1}{2} \right\rfloor = -1$; $\left\lceil -\frac{1}{2} \right\rceil = 0$

e. $\lfloor 8 \rfloor = 8$; $\lceil 8 \rceil = 8$

The floor and ceiling functions remain the same for the integer numbers.

Exercise 4

1. Find the floor and ceiling values for the following numbers.

a. 4

b. 5.12

c. -0.05

d. $1/4$

e. $-1/4$

2. Compute $\lfloor x \rfloor$ and $\lceil x \rceil$ for each of the value of x.

a. 1.1

b. $-\frac{3}{4}$

c. -0.1

d. 2.99

e. $\frac{3}{4}$

f. $-\frac{7}{8}$

g. -1

h. 3

3. Compute $\lfloor x \rfloor$ and $\lceil x \rceil$ for each of the value of x.

a. $\left\lceil \left\lfloor \frac{1}{2} \right\rfloor + \left\lceil \frac{1}{2} \right\rceil + \frac{1}{2} \right\rceil$

b. $\lfloor 0.5 + \lceil 1.3 \rceil - \lceil -1.3 \rceil \rfloor$

c. $\left\lfloor \frac{1}{2} + \left\lceil \frac{3}{2} \right\rceil \right\rfloor$

d. $\lceil 0.5 + \lfloor 1.3 \rfloor - \lfloor -1.3 \rfloor \rceil$

e. $\left\lfloor \frac{1}{2} + \left\lceil \frac{1}{2} \right\rceil \right\rfloor \left\lfloor \frac{1}{2} \bullet \left\lfloor \frac{5}{2} \right\rfloor \right\rfloor$

Answer

Exercise 1

1. a. Neither one-to-one nor onto function.
   b. One-to-one but not onto function.
   c. Not one-to-one but onto function.
   d. Not one-to-one but onto function.
   e. Not one-to-one but onto function.
   f. Neither one-to-one nor onto function.

2. a. One-to-one and onto function.
   b. Not one-to-one but onto function.
   c. One-to-one and onto function.
   d. Neither one-to-one nor onto function.
   e. Neither one-to-one nor onto function.

Exercise 2

1. a. $2x-1$ b. $2x-2$ c. $4x-9$ d. $2+x$
2. a. $1-10x$; $[-29, 21]$ b. 1 c. $\frac{1}{2}$
3. a. 8 b. 1 c. 6 d. -1
4. $5+2x$
5. $5-2x$
6. x^2+3
7. $\frac{8}{x}+5$

Exercise 3

1. $\dfrac{2}{3-x}$

2. a. $\dfrac{4}{x}+5$ b. $\dfrac{11}{2}$

3. 1

4. a. $\dfrac{3x-2}{5}$ b. 4

5. a. $\dfrac{5x+6}{4x+8}$ b. -27

Exercise 4

1. a. 4; 4 b. 5; 6 c. –1; 0 d. 0; 1 e. –1; 0

2. a. 1; 2 b. –1; 0 c. –1; 0 d. 2; 3 e. 0; 1
   f. –1; 0 g. –1; –1 h. 3; 3

3. a. 2 b. 3 c. 2 d. 4 e. 1

REFERENCES

Anton, H., Bivens, I., & Davis, S. (2010). *Calculus : Late transcendentals*. John Wiley & Sons.

Anton, H., Irl Bivens, & Davis, S. (2005). *Calculus*. Wiley.

NCERT. (2018, April 18). *Relations and functions*. Ncert.nic.in. https://ncert.nic.in

O'Regan, G. (2016). Sets, relations and functions. *Texts in Computer Science*, 25–51. https://doi.org/10.1007/978-3-319-44561-8_2

Pumadevi, Wei, W. P., & Choy, L. C. (2008). *Success mathematics SPM*. Oxford Fajar Sdn. Bhd.

Scholars at Harvard. (2017, February 1). *Set theory, relations, and functions (II)*. Harvard.edu. http://scholar.harvard.edu

Zubairi, Y. Z., Manaf, F. A., Razak, A. F. A., & Suan, L. K. (2006). *Mathematics for STPM & Matriculation : Algebra & Trigonometry*. Thomson Learning.

e ISBN 978-967-2258-83-4