Introduction to LAPLACE TRANSFORM

Comprehensive Notes . Guided Examples . Tips . Exercises

NORSUZANA BINTI ZAKARIA
RASNA BINTI MANSUR
SAL SABILAH BINTI MOHAMAD TAHIR

"Books are mirrors: You only see in them what you already have inside you."
- Carlos Ruiz Zafón -

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POLITEKNIK MALAYSIA
SULTAN IDRIS SHAH

Cetakan Pertama/First Printing 2024

Diterbitkan di Malaysia oleh / Published in Malaysia by:
POLITEKNIK SULTAN IDRIS SHAH
Sungai Lang,
45100 Sungai Air Tawar,
Selangor, Malaysia
No. Tel: 03 3280 6200
No. Fax : 03 3280 6400
https://psis.mypolycc.edu.my/

Cataloguing-in-Publication Data

Perpustakaan Negara Malaysia

A catalogue record for this book is available from the National Library of Malaysia

eISBN 978-967-2860-89-1

preface

Introduction to Laplace Transform is specially written for Electrical Engineering students. This book introduces and consolidates basic principles and concept of Laplace Transform for students needing to a broad base for further in electrical engineering studies. It contains concise and comprehensive notes delivered using effective worked examples as well as meaningful diagrams and tips.

This book contains several noteworthy features, which include:

- Each subtopic begins with a brief outline of essential theory, definitions, formulae, laws and procedures.
- Tips – extra relevant information
- Attractive notes – highlight the most important information of the topic in such an interactive way with graphic and coloured text
- Formulae – included for convenience of reference
- Intensive examples – structured questions with full step work answers to help and guide students to solve common questions tested in examination
- Exercise questions – summative exercises at the end of every subtopics
- Answer – all answers given to encourage self-paced and self-assessed learning
- Summary - Summary of each subtopic in an interactive and clear explaination
- Sample of Polytechnic's Past Final Examination questions

We are confident that with the use of this book, students will be well on their way to success in their examinations.

Norsuzana Binti Zakaria
Rasna binti Mansur
Sal Sabilah Binti Mohamad Tahir
Mathematics, Science and Computer Department,
Politeknik Sultan Idris Shah.

acknowledgement

In the name of Allah, the Most Gracious and the Most Merciful. Alhamdulillah, all praises to Allah for the strengths and His blessing in completing this book. The aim of writing this book is to provide a module on how to learn Laplace Transform for polytechnic students taking mathematics. We hope that this book will be a good reference for them throughout the semester.

This book is not a substitute for lecturers but to complement the lectures given. No ivory is not cracked. The authors realize that the compilation of this book is far from perfect and may not be able to satisfy all parties. Therefore, any positive feedback from lecturers and students is most welcome and appreciated.

We wish to express our gratitude to En. Hairulanuar bin Rosman for providing his valuable guidance to make this work possible, the Head of Department, Puan Rasidah Binti Sapri, for her encouragement and general advice in carrying out this book and all staff of Mathematics, Science and Computer Department throughout this process. Finally, we would like to express our deepest gratitude to our husband and beloved family for their understanding and encouragement. We are very thankful for the constant support and help.

Author

Norsuzana Binti Zakaria, Rasna binti Mansur, Sal Sabilah Binti Mohamad Tahir
Mathematics, Science and Computer Department,
Politeknik Sultan Idris Shah.
Jun 2024.

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HISTORY OF
Laplace Transform

Is named after a French mathematician & astronomer Pierre-Simon Laplace (1749–1827), who used a similar transform in his work on probability theory. Laplace wrote extensively about the use of generating functions in *Essai Philosophique Sur Les Probabilités* (1814), and the integral form of the Laplace transform evolved naturally as a result.

Laplace's use of generating functions was similar to what is now known as the z-transform, and he gave little attention to the continuous variable case which was discussed by Niels Henrik Abel. The theory was further developed in the 19th and early 20th centuries by Mathias Lerch, Oliver Heaviside, and Thomas Bromwich.

Sources from https://en.wikipedia.org/wiki/Laplace_transform

In Mathematics, Laplace Transform is an integral transform that converts a function of a real variable, t to a function of a complex variable, s. The transform has many applications in science and engineering because it is a tool for solving differential equation.

What *is* LAPLACE TRANSFORM

2 WAYS of

SOLVING LAPLACE TRANSFORM

01. DEFINITION OF LAPLACE TRANSFORM

USING THE TABLE OF LAPLACE TRANSFORM **02.**

SOLVING LAPLACE TRANSFORM

Definition of Laplace Transform

Note to students:

Symbol of Laplace Transform

$$F(s) = L\{f(t)\}$$

$$F(s) = \int_{0}^{\infty} e^{-st} \, f(t) \, dt$$

Use this formula !

3 types of $f(t)$

1. Constant

eg. $f(t) = 2$, $f(t) = m$

2. Exponential Function

eg. $f(t) = e^{-3t}$, $f(t) = e^{5t}$

3. Polynomial

eg. $f(t) = 2t$, $f(t) = t^2$

EXAMPLE 1

Calculate the Laplace Transform of $\boldsymbol{f(t) = 3}$ using the definition of the Laplace Transform.

Notes:

$$F(s) = \int_0^\infty e^{-st} f(t)\, dt$$

Write down the definition of the Laplace Transform

$$F(s) = \int_0^\infty e^{-st} (3)\, dt$$

Substitute the value of f(t)

$$F(s) = \left[\frac{3e^{-st}}{-s}\right]_0^\infty$$

Perform the integration process

$$F(s) = \left[\frac{3e^{-s(\infty)}}{-s}\right] - \left[\frac{3e^{-s(0)}}{-s}\right]$$

$$F(s) = 0 - \left(-\frac{3}{s}\right)$$

$$\mathbf{F(s) = \frac{3}{s}}$$

EXAMPLE 2

Calculate the Laplace Transform of $f(t) = 2k$ using the definition of the Laplace Transform.

$$F(s) = \int_0^{\infty} e^{-st} f(t)\, dt$$

$$F(s) = \int_0^{\infty} e^{-st} (2k)\, dt$$

$$F(s) = \left[\frac{2ke^{-st}}{-s}\right]_0^{\infty}$$

$$F(s) = \left[\frac{2ke^{-s(\infty)}}{-s}\right] - \left[\frac{2ke^{-s(0)}}{-s}\right]$$

$$F(s) = 0 - \left(-\frac{2k}{s}\right)$$

$$\mathbf{F(s) = \frac{2k}{s}}$$

Notes:

Write down the definition of the Laplace Transform

Substitute the value of f(t)

Perform the integration process

EXAMPLE 3

Calculate the Laplace Transform of $f(t) = e^{2t}$ using the definition of the Laplace Transform.

$$F(s) = \int_0^\infty e^{-st} f(t)\, dt$$

Notes: Write down the definition of the Laplace Transform

$$F(s) = \int_0^\infty e^{-st} (e^{2t}) dt$$

Substitute the value of f(t)

$$F(s) = \int_0^\infty e^{-st+2t}\, dt$$

Make the function in single index form

$$F(s) = \int_0^\infty e^{-t(s-2)}\, dt$$

Factorize the index

$$F(s) = \left[\frac{e^{-t(s-2)}}{-(s-2)}\right]_0^\infty$$

Perform the integration process

$$F(s) = \left[\frac{e^{-\infty(s-2)}}{-(s-2)}\right] - \left[\frac{e^{-0(s-2)}}{-(s-2)}\right]$$

$$F(s) = 0 - \left(\frac{1}{-(s-2)}\right)$$

$$\mathbf{F(s) = \frac{1}{(s-2)}}$$

EXAMPLE 4

Calculate the Laplace Transform of $\boldsymbol{f(t) = 5t}$ using the definition of the Laplace Transform.

$$F(s) = \int_0^{\infty} e^{-st} f(t)\, dt$$

$$F(s) = \int_0^{\infty} e^{-st} (5t)\, dt$$

$$F(s) = \left[\frac{5te^{-st}}{-s} - \int \frac{5e^{-st}}{-s}\, dt\right]_0^{\infty}$$

$$F(s) = \left[\frac{5te^{-st}}{-s} - \frac{5e^{-st}}{s^2}\right]_0^{\infty}$$

$$F(s) = \left[\frac{5(\infty)e^{-s(\infty)}}{-s} - \frac{5e^{-s(\infty)}}{s^2}\right] - \left[\frac{5(0)e^{-s(0)}}{-s} - \frac{5e^{-s(0)}}{s^2}\right]$$

$$F(s) = [0 - 0] - \left[0 - \frac{5}{s^2}\right]$$

$$\mathbf{F(s) = \frac{5}{s^2}}$$

Notes:

Write down the definition of the Laplace Transform

Substitute the value of f(t)

Perform the integration process by using integration by part

$$F(s) = uv - \int v\, du$$

EXERCISE 1

"Nothing worth having comes easy. Keep trying!"

QUESTION Calculate the Laplace Transform of each of the following using the definition of the Laplace Transform.	ANSWER
1. $f(t) = 10$	$F(s) = \frac{10}{s}$
2. $f(t) = \frac{\pi}{2}$	$F(s) = \frac{\pi}{2s}$
3. $f(t) = -a$	$F(s) = \frac{-a}{s}$
4. $f(t) = e^{-3t}$	$F(s) = \frac{1}{s+3}$
5. $f(t) = e^{mt}$	$F(s) = \frac{1}{s-m}$
6. $f(t) = 3t$	$F(s) = \frac{3}{s^2}$
7. $f(t) = \frac{t}{2}$	$F(s) = \frac{1}{2s^2}$
8. $f(t) = t^2$	$F(s) = \frac{2}{s^3}$

SOLVING LAPLACE TRANSFORM

Table of Laplace Transform

$f(t)$	$F(s)$	$f(t)$	$F(s)$
a	$\frac{a}{s}$	$\sinh \omega t$	$\frac{\omega}{s^2 - \omega^2}$
at	$\frac{a}{s^2}$	$\cosh \omega t$	$\frac{s}{s^2 - \omega^2}$
t^n	$\frac{n!}{s^{n+1}}$	$t \sin \omega t$	$\frac{2\omega s}{(s^2 + \omega^2)^2}$
e^{at}	$\frac{1}{s - a}$	$t \cos \omega t$	$\frac{s^2 - \omega^2}{(s^2 + \omega^2)^2}$
e^{-at}	$\frac{1}{s + a}$	$e^{-at} \sin \omega t$	$\frac{\omega}{(s + a)^2 + \omega^2}$
te^{-at}	$\frac{1}{(s + a)^2}$	$e^{-at} \cos \omega t$	$\frac{s + a}{(s + a)^2 + \omega^2}$
$t^n e^{at}$	$\frac{1}{(s - a)^{n+1}}$	$e^{at} \sinh \omega t$	$\frac{\omega}{(s - a)^2 - \omega^2}$
$\sin \omega t$	$\frac{\omega}{s^2 + \omega^2}$	$e^{-at} \sinh \omega t$	$\frac{\omega}{(s + a)^2 - \omega^2}$
$\cos \omega t$	$\frac{s}{s^2 + \omega^2}$	$e^{-at} \cosh \omega t$	$\frac{s + a}{(s + a)^2 - \omega^2}$

HOW to use Table of Laplace Transform

Let $f(t)=2t$

1. Find f(t) in the table that matching f(t) in the question

$f(t)$	$F(s)$
at	$\frac{a}{s^2}$

matching to $f(t)=2t$ where $a=2$

2. F(s) is determined by substituting a=2.

Therefore, $F(s)=\frac{a}{s^2}$

By using the Table of Laplace Transform, find the Laplace Transform of $\boldsymbol{f(t) = -4}$.

example 1

SOLUTION

$$f(t) = -4$$

Refer to the table of Laplace Transform

$f(t)$	$F(s)$
a	$\frac{a}{s}$

$$F(s) = L\{-4\}$$

Which is $a = -4$

$$\boldsymbol{F(s) = \frac{-4}{s}}$$

By using the Table of Laplace Transform, find the Laplace Transform of $\boldsymbol{f(t) = \pi j}$.

example 2

SOLUTION

$$f(t) = \pi j$$

Refer to the table of Laplace Transform

$f(t)$	$F(s)$
a	$\frac{a}{s}$

$$F(s) = L\{\pi j\}$$

Which is $a = \pi j$

$$\boldsymbol{F(s) = \frac{\pi j}{s}}$$

By using the Table of Laplace Transform, find the Laplace Transform of $\boldsymbol{f(t) = e^{-4t}}$.

example 3

SOLUTION

$$f(t) = e^{-4t}$$

Refer to the table of Laplace Transform

$f(t)$	$F(s)$
e^{-at}	$\frac{1}{s+a}$

$$F(s) = L\{e^{-4t}\}$$

Which is $a = 4$

$$F(s) = \frac{1}{s+4}$$

example 4

By using the Table of Laplace Transform, find the Laplace Transform of $\boldsymbol{f(t) = 2t^3}$.

SOLUTION

$$f(t) = 2t^3$$

Refer to the table of Laplace Transform

$f(t)$	$F(s)$
t^n	$\frac{n!}{s^{n+1}}$

$$F(s) = L\{2t^3\}$$

$$F(s) = 2\,L\{t^3\}$$

Which is $n = 3$

$$F(s) = 2\left[\frac{3!}{s^{3+1}}\right]$$

$$\boldsymbol{F(s) = \frac{12}{s^4}}$$

By using the Table of Laplace Transform, find the Laplace Transform of $\boldsymbol{f(t) = \sin 2t}$.

example 5

SOLUTION

$$f(t) = \sin 2t$$

Refer to the table of Laplace Transform

$f(t)$	$F(s)$
$\sin \omega t$	$\dfrac{\omega}{s^2 + \omega^2}$

$$F(s) = L\,\{\sin 2t\}$$

Which is $\omega = 2$

$$F(s) = \frac{2}{s^2 + 2^2}$$

$$\boldsymbol{F(s) = \frac{2}{s^2 + 4}}$$

By using the Table of Laplace Transform, find the Laplace Transform of $\boldsymbol{f(t) = 2\cosh 5t}$.

example 6

SOLUTION

$$f(t) = 2\cosh 5t$$

Refer to the table of Laplace Transform

$f(t)$	$F(s)$
$\cosh \omega t$	$\dfrac{s}{s^2-\omega^2}$

$$F(s) = L\{2\cosh 5t\}$$

$$F(s) = 2\,L\{\cosh 5t\}$$

Which is $\omega = 5$

$$F(s) = 2\left[\frac{s}{s^2+5^2}\right]$$

$$\boldsymbol{F(s) = \frac{2s}{s^2+25}}$$

example 7

By using the Table of Laplace Transform, find the Laplace Transform of $\boldsymbol{f(t) = e^{-t}\cos 2t}$.

SOLUTION

$$f(t) = e^{-t}\cos 2t$$

Refer to the table of Laplace Transform

$f(t)$	$F(s)$
$e^{-at}\cosh \omega t$	$\dfrac{s+a}{(s+a)^2+\omega^2}$

$$F(s) = L\{e^{-t}\cos 2t\}$$

$$F(s) = \frac{s+1}{(s+1)^2+2^2}$$

Which is $a = 1, \omega = 2$

expand

$$\boldsymbol{F(s) = \frac{s+1}{s^2+2s+5}}$$

EXERCISE 2

"try and fail. But don't fail to try."

QUESTION By using the table of Laplace Transform, find the Laplace Transform for each of the following.	**ANSWER**
1. $f(t) = t^5$	$F(s) = \frac{120}{s^6}$
2. $f(t) = \cos 4t$	$F(s) = \frac{s}{s^2+16}$
3. $f(t) = e^{-3t}\sinh 2t$	$F(s) = \frac{2}{(s+3)^2-4}$
4. $f(t) = t\sin 4t$	$F(s) = \frac{8s}{(s^2+16)^2}$
5. $f(t) = t^2e^{-5t}$	$F(s) = \frac{2}{(s+5)^3}$
6. $f(t) = \frac{t^2}{e^{4t}}$	$F(s) = \frac{2}{(s+4)^3}$
7. $f(t) = \frac{\sin 5t}{e^{6t}}$	$F(s) = \frac{5}{(s+6)^2+25}$
8. $f(t) = e^{4t}\sin\frac{2t}{3}$	$F(s) = \frac{6}{9[(s-4)^2+4]}$

3 PROPERTIES of LAPLACE TRANSFORM

01. Linearity

$$L\{\alpha f_1(t) \pm \beta f_2(t)\} = \alpha F_1 \pm \beta F_2$$

First Shift Theorem 02.

$$L\{e^{at} f(t)\} = F(s-a)$$

03. Multiplication with t^n

$$L\{t^n f(t)\} = (-1)^n \frac{d^n}{ds^n}[F(s)]$$

PROPERTIES of LAPLACE TRANSFORM

01. Linearity

To find the Laplace Transform of **2 or more** functions **added** together, we simply find the transform of each **individual** function and then **add together** these transform.

general formula of linearity properties

$$L\{\alpha f_1(t) \pm \beta f_2(t)\} = \alpha F_1 \pm \beta F_2$$

Properties of Laplace Transform

Linearity properties

EXAMPLE 1

QUESTION:

Find $L\{3 - t^2 + 5t\}$

Notes:

SOLUTION:

Break the function

$$L\{3 - t^2 + 5t\} = L\{3\} - L\{t^2\} + L\{5t\}$$

$$= \frac{3}{s} - \frac{2!}{s^{2+1}} + \frac{5}{s^2}$$

Refer to the table of Laplace Transform

$$= \frac{3}{s} - \frac{2}{s^3} + \frac{5}{s^2}$$

Properties of Laplace Transform

Linearity properties

EXAMPLE 2

QUESTION:

Find $L\{2e^{-t} + 3\sinh 2t\}$

Notes:

SOLUTION:

Break the function

$$L\{2e^{-t} + 3\sinh 2t\} = L\{2e^{-t}\} + L\{3\sinh 2t\}$$

$$= 2L\{e^{-t}\} + 3L\{\sinh 2t\}$$

$$= 2\left[\frac{1}{s+1}\right] + 3\left[\frac{2}{s^2-2^2}\right]$$

Refer to the table of Laplace Transform

$$= \frac{2}{s+1} + \frac{6}{s^2-4}$$

Properties of Laplace Transform

Linearity properties

EXAMPLE 3

QUESTION:

Find $L\{\cosh 4t + \sin 2t\}$

Notes:

SOLUTION:

Break the function

$L\{\cosh 4t + \sin 2t\} = L\{\cosh 4t\} + L\{\sin 2t\}$

$= \left[\frac{s}{s^2 - 4^2}\right] + \left[\frac{2}{s^2 + 2^2}\right]$

Refer to the table of Laplace Transform

$= \frac{s}{s^2 - 16} + \frac{2}{s^2 + 4}$

Properties of Laplace Transform

Linearity properties

EXAMPLE 4

QUESTION:

Find $L\{e^{2t}[t^3 - \sin t]\}$

Notes:

SOLUTION:

Expand the function

$$L\{e^{2t}[t^3 - \sin t]\} = L\{e^{2t}t^3 - e^{2t}\sin t\}$$

Break the function

$$= L\{e^{2t}t^3\} - L\{e^{2t}\sin t\}$$

Refer to the table of Laplace Transform

$$= \frac{3!}{(s-2)^2} + \frac{1}{(s-2)^2+1^2}$$

$$= \frac{6}{s^2} + \frac{1}{s^2-4s+5}$$

EXERCISE 3

"If you never try you will never know."

QUESTION Find the Laplace Transform for each of the following.	ANSWER
1. $f(t) = 2t^3 - 5$	$F(s) = \frac{12}{s^4} - \frac{5}{s}$
2. $f(t) = -4t^2 + 5t - 2$	$F(s) = -\frac{8}{s^3} + \frac{5}{s^2} + \frac{2}{s}$
3. $f(t) = 4e^{-2t} + 3e^{2t}$	$F(s) = \frac{4}{s+2} + \frac{3}{s-2}$
4. $f(t) = 2te^t - 3t^2e^{-t}$	$F(s) = \frac{2}{(s-1)^2} - \frac{6}{(s+1)^3}$
5. $f(t) = 1 + t - 3e^{-2t}$	$F(s) = \frac{1}{s} + \frac{1}{s^2} - \frac{3}{s+2}$
6. $f(t) = t^2e^t - 4t$	$F(s) = \frac{2}{(s+1)^3} - \frac{4}{s^2}$
7. $f(t) = 2\sin 5t + \cosh t$	$F(s) = \frac{10}{s^2+25} + \frac{s}{s^2-1}$
8. $f(t) = e^{5t}\cos 2t + 3t^4$	$F(s) = \frac{2}{(s-5)^2+4} + \frac{72}{s^5}$

PROPERTIES of LAPLACE TRANSFORM

02. First Shift Theorem

The first shifting theorem provides a convenient way of calculating the Laplace Transform of functions that are of the form $g(t) = e^{at} f(t)$ where a is a constant and $f(t)$ is a given function.

general formula of first shift theorem

$$g(t) = e^{at} f(t)$$
$$L\{e^{at} f(t)\} = F(s - a)$$

Properties of Laplace Transform

First Shift Theorem

EXAMPLE 1

QUESTION:	Notes:
Compute Laplace Transform of $L\left\{e^{2t}\sin 3t\right\}$ by using the First Shift Theorem.	

SOLUTION:

$$f(t) = \sin 3t$$

Define f(t). f(t) should be function other than exponential function.

$$F(s) = \frac{3}{S^2 + 3^2} = \frac{3}{S^2 + 9}$$

Refer to the table of Laplace Transform

$$L\left\{e^{2t}\sin 3t\right\} = F(s-a) = \frac{3}{(s-a)^2 + 9}$$

$$= \frac{3}{(s-2)^2 + 9}$$

Use the first shift theorem with a = 2

$$= \frac{\mathbf{3}}{\boldsymbol{s^2 - 4s + 13}}$$

Properties of Laplace Transform

First Shift Theorem

EXAMPLE 2

QUESTION:

Compute Laplace Transform of $L\{t^2 e^{3t}\}$ by using the First Shift Theorem.

Notes:

SOLUTION:

$f(t) = t^2$

Define f(t). f(t) should be function other than exponential function.

$F(s) = \frac{2!}{s^{2+1}} = \frac{2}{s^3}$

Refer to the table of Laplace Transform

$L\{t^2 e^{3t}\} = F(s-a) = \frac{2}{(s-a)^3}$

$= \frac{2}{(s-3)^3}$

Use the first shift theorem with a = 3

Properties of Laplace Transform

First Shift Theorem

EXAMPLE 3

QUESTION:

Compute Laplace Transform of $L\{e^{-t}\cos 2t\}$ by using the First Shift Theorem.

Notes:

SOLUTION:

$f(t) = \cos 2t$

Define f(t). f(t) should be function other than exponential function.

$F(s) = \frac{s}{s^2 + 2^2} = \frac{s}{s^2 + 4}$

Refer to the table of Laplace Transform

$L\{e^{-t}\cos 2t\} = F(s - a) = \frac{s - a}{(s - a)^2 + 4}$

$= \frac{s + 1}{(s + 1)^2 + 4}$

Use the first shift theorem with a = -1

$= \frac{\mathbf{s + 1}}{\mathbf{s^2 + 2s + 5}}$

EXERCISE 4

"Don't decrease the goal. Increase the effort"

QUESTION Compute Laplace Transform for each of the following by using the First Shift Theorem.	**ANSWER**
1. $f(t) = e^{2t}\cos 2t$	$F(s) = \dfrac{s-2}{s^2-4s+8}$
2. $f(t) = t^2 e^{2t}$	$F(s) = \dfrac{2}{(s-2)^3}$
3. $f(t) = (2t+3)e^{-2t}$	$F(s) = \dfrac{3s+8}{(s+2)^2}$
4. $f(t) = te^{-t}$	$F(s) = \dfrac{1}{(s+1)^2}$
5. $f(t) = \sinh 2t\, e^{3t}$	$F(s) = \dfrac{2}{s^2-6s+5}$

PROPERTIES of LAPLACE TRANSFORM

03. Multiplication with t^n

general formula of multiplication with t^n

$$L\{t^n f(t)\} = (-1)^n \frac{d^n}{ds^n}[F(s)]$$

This proposition describes the **relationship** between the **derivative** of the Laplace Transform of a function and the **Laplace Transform** of the **function itself**.

Properties of Laplace Transform

Multiplication with t^n

EXAMPLE 1

QUESTION:

Compute Laplace Transform of $L\{te^{3t}\}$ by using the Multiplication by t^n

Notes:

SOLUTION:

$f(t) = e^{3t}$	Define f(t). f(t) should be function other than exponential function.
$F(s) = \dfrac{1}{s-3}$	Refer to the table of Laplace Transform
$L\{te^{3t}\} = (-1)^n \dfrac{d^n}{ds^n}[F(s)]$	
$= (-1)^1 \dfrac{d^1}{ds^1}\left[\dfrac{1}{s-3}\right]$	
$= -\dfrac{\mathbf{1}}{(\boldsymbol{s}-\mathbf{3})^2}$	

Properties of Laplace Transform

Multiplication with t^n

EXAMPLE 2

QUESTION:

Compute Laplace Transform of $L\{t \sin 2t\}$ by using the Multiplication by t^n

Notes:

SOLUTION:

$f(t) = \sin 2t$	Define f(t). f(t) should be function other than exponential function.
$F(s) = \dfrac{2}{s^2 + 2^2} = \dfrac{2}{s^2 + 4}$	Refer to the table of Laplace Transform
$L\{t \sin 2t\} = (-1)^n \dfrac{d^n}{ds^n}[F(s)]$	
$= (-1)^1 \dfrac{d^1}{ds^1}\left[\dfrac{2}{s^2 + 4}\right]$	
$= -\dfrac{\mathbf{4s}}{\mathbf{(s^2 + 4)^2}}$	

Properties of Laplace Transform

Multiplication with t^n

EXAMPLE 3

QUESTION:

Compute Laplace Transform of $L\{t^2e^{-3t}\}$ by using the Multiplication by t^n

Notes:

SOLUTION:

$f(t) = e^{-3t}$

Define f(t). f(t) should be function other than exponential function.

$F(s) = \dfrac{1}{s+3}$

Refer to the table of Laplace Transform

$$L\{t^2e^{-3t}\} = (-1)^n \frac{d^n}{ds^n}[F(s)]$$

$$= (-1)^2 \frac{d^2}{ds^2}\left[\frac{1}{s+3}\right]$$

$$= \frac{d}{ds}\left[-\frac{1}{(s+3)^2}\right]$$

$$= \frac{\mathbf{2}}{(\mathbf{s+3})^{\mathbf{3}}}$$

EXERCISE 5

"a little progress each day adds up to big result"

QUESTION	ANSWER
Compute Laplace Transform for each of the following by using the First Shift Theorem.	
1. $f(t) = t \sin 2t$	$F(s) = \dfrac{-4s}{(s^2+4)^2}$
2. $f(t) = te^{-t}$	$F(s) = \dfrac{1}{(s+1)^2}$
3. $f(t) = 2te^{-2t}$	$F(s) = \dfrac{-2}{(s+2)^2}$
4. $f(t) = t^2 e^{-t}$	$F(s) = \dfrac{2}{(s+1)^3}$
5. $f(t) = t^2 \sinh t$	$F(s) = \dfrac{-2(s^2-4s-1)}{(s^2-1)^3}$

In Mathematics, the Inverse Laplace Transform of a function $F(s)$ is the piecewise-continuous and exponentially-restricted real function $f(t)$.
Or in simple word we call it as a reverse process of Laplace Transform.

If $L\{f(t)\} = F(s)$, then

$$L^{-1}\{F(s)\} = f(t)$$

WHAT is INVERSE LAPLACE TRANSFORM

Example 1

Find the inverse Laplace Transform of

$$F(s) = \frac{4}{s}$$

Solution

$$F(s) = \frac{4}{s}$$

Tips

Read the table in reverse way than before which is from the right to the left.

Refer to the table of Laplace Transform

$f(t)$	$F(s)$
a	$\frac{a}{s}$

matching to $F(s) = \frac{4}{s}$ where $a = 4$

$$f(t) = L^{-1}\left\{\frac{4}{s}\right\}$$

$$\boldsymbol{f(t) = 4}$$

Example 2

Find the inverse Laplace Transform of

$$F(s) = \frac{1}{s+3}$$

Solution

$$F(s) = \frac{1}{s+3}$$

Tips: Read the table in reverse way than before which is from the right to the left.

Refer to the table of Laplace Transform

$f(t)$	$F(s)$
e^{-at}	$\frac{1}{s+a}$

matching to $F(s) = \frac{1}{s+3}$ where $a=3$

$$f(t) = L^{-1}\left\{\frac{1}{s+3}\right\}$$

$$\boldsymbol{f(t) = e^{-3t}}$$

Example 3

Find the inverse Laplace Transform of

$$F(s) = \frac{12}{s^4}$$

Solution

$$F(s) = \frac{12}{s^4}$$

Tips: Read the table in reverse way than before which is from the right to the left.

Refer to the table of Laplace Transform

$f(t)$	$F(s)$
t^n	$\frac{n!}{s^{n+1}}$

matching to $F(s) = \frac{15}{s^4}$ where $n=3$

$$f(t) = 12L^{-1}\left\{\frac{1}{s^4}\right\}$$

$$f(t) = \frac{12}{3!}L^{-1}\left\{\frac{3!}{s^{3+1}}\right\}$$

$$\boldsymbol{f(t) = 2t^3}$$

Example 4

Find the inverse Laplace Transform of

$$F(s) = \frac{5}{s^2+25}$$

Solution

$$F(s) = \frac{5}{s^2 + 25}$$

Tips: Read the table in reverse way than before which is from the right to the left.

Refer to the table of Laplace Transform

$f(t)$	$F(s)$
$\sin \omega t$	$\frac{\omega}{s^2+\omega^2}$

matching to $F(s) = \frac{5}{s^2+5^2}$ where $\omega = 5$

$$f(t) = L^{-1}\left\{\frac{5}{s^2 + 25}\right\}$$

$$f(t) = L^{-1}\left\{\frac{5}{s^2 + 5^2}\right\}$$

$$\boldsymbol{f(t) = \sin 5t}$$

Example 5

Find the inverse Laplace Transform of

$$F(s) = \frac{4}{s^3} - \frac{2}{s+5} + \frac{3}{s^2-9}$$

Solution

$$F(s) = \frac{4}{s^3} - \frac{2}{s+5} + \frac{3}{s^2-9}$$

$$f(t) = L^{-1}\left\{\frac{4}{s^3}\right\} - L^{-1}\left\{\frac{2}{s+5}\right\} + L^{-1}\left\{\frac{3}{s^2-9}\right\}$$

Transform individually

$$f(t) = \frac{4}{2!}L^{-1}\left\{\frac{2!}{s^{2+1}}\right\} - 2L^{-1}\left\{\frac{1}{s+5}\right\} + L^{-1}\left\{\frac{3}{s^2-3^2}\right\}$$

Refer to the table of Laplace Transform

$$\boldsymbol{f(t) = 2t^2 - 2e^{-5t} + \sinh 3t}$$

Example 6

Find the inverse Laplace Transform of

$$F(s) = \frac{s+1}{s^2+1}$$

Solution

$$F(s) = \frac{s+1}{s^2+1}$$

$$F(s) = \frac{s}{s^2+1} + \frac{1}{s^2+1}$$ .... Break the fraction

$$f(t) = L^{-1}\left\{\frac{s}{s^2+1}\right\} + L^{-1}\left\{\frac{1}{s^2+1}\right\}$$

$$f(t) = L^{-1}\left\{\frac{s}{s^2+1^2}\right\} + L^{-1}\left\{\frac{1}{s^2+1^2}\right\}$$

.... Refer to the table of Laplace Transform

$$\boldsymbol{f(t) = \cos t + \sin t}$$

Example 7

Find the inverse Laplace Transform of

$$F(s) = \frac{s+3}{s^2+6s+13}$$

Solution

$$F(s) = \frac{s+3}{s^2+6s+13}$$

$$F(s) = \frac{s+3}{(s+3)^2+4}$$

.... Use completing the square method to reform the quadratic function

$$F(s) = \frac{s+3}{(s+3)^2+2^2}$$

$$f(t) = L^{-1}\left\{\frac{s+3}{(s+3)^2+2^2}\right\}$$

.... Refer to the table of Laplace Transform

$$\boldsymbol{f(t) = e^{-3t}\cos 2t}$$

Example 8

Find the inverse Laplace Transform of

$$F(s) = \frac{s+3}{s^2-4s+9}$$

Solution

$$F(s) = \frac{s+3}{s^2-4s+9}$$

$$F(s) = \frac{s+3}{(s-2)^2+5}$$

.... Use completing the square method to reform the quadratic function

$$F(s) = \frac{(s-2)+5}{(s-2)^2+\sqrt{5}^2}$$

$$f(t) = L^{-1}\left\{\frac{s-2}{(s-2)^2+\sqrt{5}^2}\right\} + L^{-1}\left\{\frac{5}{(s-2)^2+\sqrt{5}^2}\right\}$$

$$f(t) = L^{-1}\left\{\frac{s-2}{(s-2)^2+\sqrt{5}^2}\right\} + \frac{5}{\sqrt{5}}L^{-1}\left\{\frac{\sqrt{5}}{(s-2)^2+\sqrt{5}^2}\right\}$$

.... Refer to the table of Laplace Transform

$$\boldsymbol{f(t) = e^{2t}\cos\sqrt{5}t + \sqrt{5}\,e^{2t}\sin\sqrt{5}t}$$

EXERCISE 6

"Success is achieved by those who try and keep trying"

QUESTION Find the inverse Laplace Transform for each of the following.	ANSWER
1. $F(s) = \frac{2}{s^4} - \frac{5}{s^3}$	$f(t) = \frac{t^3}{3} - \frac{5t^2}{2}$
2. $F(s) = \frac{1}{2s^5} - \frac{2}{5s^2}$	$f(t) = \frac{t^4}{48} - \frac{2t}{5}$
3. $F(s) = \frac{s+5}{s^2+25}$	$f(t) = \cos 5t + \sin 5t$
4. $F(s) = \frac{4}{s} + \frac{2}{s-5}$	$f(t) = 4 + 2e^{5t}$
5. $F(s) = \frac{s+3}{s^2-4}$	$f(t) = \cosh 2t + \frac{3}{2}\sinh 2t$
6. $F(s) = \frac{s+2}{(s+2)^2+9}$	$f(t) = e^{-2t}\cos 3t$
7. $F(s) = \frac{s+7}{s^2+4s+13}$	$f(t) = \frac{5}{3}e^{-2t}\sin 3t + e^{-2t}\cos 3t$

SOLVING INVERSE LAPLACE TRANSFORM using PARTIAL FRACTION

WHY?

we use Partial Fraction method in solving Inverse Laplace Transform

01 The function can't directly being transformed by using the Table of Laplace Transform.

02 Partial Fraction method could split up a complicated fraction into forms that are in the Table of Laplace Transform.

PARTIAL FRACTION method?

Partial Fraction is a technique for expressing an algebraic fraction as a sum of **simpler fractions**.

It is also can be used as an **intermediate step** to solve the Inverse Laplace Transform.

rule of PARTIAL FRACTION

LINEAR FACTOR	$\frac{P(s)}{(as+b)(cs+d)} = \frac{A}{(as+b)} + \frac{B}{(cs+d)}$
REPEATED LINEAR FACTOR	$\frac{P(s)}{(as+b)^2} = \frac{A}{(as+b)} + \frac{B}{(as+b)^2}$
QUADRATIC FACTOR	$\frac{P(s)}{(as^2+bs+c)} = \frac{(As+B)}{(as^2+bs+c)}$

EXAMPLE 1

By using partial fraction, find the inverse Laplace Transform of $\boldsymbol{F(s)} = \frac{\mathbf{3s+1}}{\mathbf{(s+1)(s-1)}}$

$$L^{-1}\left\{\frac{3s+1}{(s+1)(s-1)}\right\}$$

$$\frac{3s+1}{(s+1)(s-1)} = \frac{A}{(s+1)} + \frac{B}{(s-1)}$$

... Use partial fraction to break the fraction

$$3s+1 = A(s-1) + B(s+1)$$

When s = -1, $3(-1)+1 = A(-1-1) + B(-1+1)$

$$A = 1$$

When s = 1, $3(1)+1 = A(1-1) + B(1+1)$

$$B = 2$$

$$\frac{3s+1}{(s+1)(s-1)} = \frac{1}{(s+1)} + \frac{2}{(s-1)}$$

.... Substitute value of A and B

$$\therefore\ L^{-1}\left\{\frac{3s+1}{(s+1)(s-1)}\right\} = L^{-1}\left\{\frac{1}{(s+1)}\right\} + L^{-1}\left\{\frac{2}{(s-1)}\right\}$$

$$\therefore\ \boldsymbol{L^{-1}\left\{\frac{3s+1}{(s+1)(s-1)}\right\} = e^{-t} + 2e^{t}}$$

Refer to the table of Laplace Transform

EXAMPLE 2

By using partial fraction, find the inverse Laplace Transform of $\boldsymbol{F(s) = \frac{1}{s^2 - s}}$

$$L^{-1}\left\{\frac{1}{s^2 - s}\right\} = L^{-1}\left\{\frac{1}{s(s-1)}\right\}$$

$$\frac{1}{s(s-1)} = \frac{A}{s} + \frac{B}{(s-1)}$$ .... Use partial fraction to break the fraction

$$1 = A(s-1) + B(s)$$

When s = 0 , $1 = A(0-1) + B(0)$

$$A = -1$$

When s = 1 , $1 = A(1-1) + B(1)$

$$B = 1$$

$$\frac{1}{s(s-1)} = \frac{-1}{s} + \frac{1}{(s-1)}$$ Substitute value of A and B

$$\therefore\ L^{-1}\left\{\frac{1}{s(s-1)}\right\} = L^{-1}\left\{\frac{-1}{s}\right\} + L^{-1}\left\{\frac{1}{(s-1)}\right\}$$ .... Refer to the table of Laplace Transform

$$\therefore\ \boldsymbol{L^{-1}\left\{\frac{1}{s(s-1)}\right\} = -1 + e^t}$$

EXAMPLE 3

By using partial fraction, find the inverse Laplace Transform of $\boldsymbol{F(s)} = \frac{\boldsymbol{s^2+6s+1}}{\boldsymbol{s(s+1)^2}}$

$$L^{-1}\left\{\frac{s^2+6s+1}{s(s+1)^2}\right\} = \frac{A}{s} + \frac{B}{(s+1)} + \frac{C}{(s+1)^2}$$

Use partial fraction to break the fraction

$$s^2+6s+1 = A(s+1)^2 + B(s)(s+1) + C(s)$$

When s = 0 ,

$$0^2+6(0)+1 = A(0+1)^2 + B(0)(0+1) + C(0)$$

$$A = 1$$

When s = -1 ,

$$(-1)^2+6(-1)+1 = A(-1+1)^2 + B(-1)(-1+1) + C(-1)$$

$$C = 4$$

CONTINUE >>>

Expand the equation

$s^2 + 6s + 1 = A(s+1)^2 + B(s)(s+1) + C(s)$

$s^2 + 6s + 1 = A(s^2 + 2s + 1) + B(s^2 + s) + C(s)$

Equalize coefficient of s^2, $B = 0$

Substitute value of A, B and C

$$\frac{s^2 + 6s + 1}{s(s+1)^2} = \frac{1}{s} + \frac{0}{(s+1)} + \frac{4}{(s+1)^2}$$

$$\therefore L^{-1}\left\{\frac{s^2 + 6s + 1}{s(s+1)^2}\right\} = L^{-1}\left\{\frac{1}{s}\right\} + L^{-1}\left\{\frac{4}{(s+1)^2}\right\}$$

Refer to the table of Laplace Transform

$$= \mathbf{1 + 4te^{-t}}$$

EXAMPLE 4

By using partial fraction, find the inverse Laplace Transform of $\boldsymbol{F(s)} = \frac{\mathbf{4s^2-5s+6}}{\mathbf{(s+1)(s^2+4)}}$

$$L^{-1}\left\{\frac{4s^2-5s+6}{(s+1)(s^2+4)}\right\} = \frac{A}{(s+1)} + \frac{Bs+C}{(s^2+4)}$$

$$4s^2-5s+6 = A(s^2+4) + (Bs+C)(s+1)$$

When $s = -1$,

$$4(-1)^2-5(-1)+6 = A\left((-1)^2+4\right) + (B(-1)+C)(-1+1)$$

$$A = 3$$

Expand the equation

$$4s^2-5s+6 = A(s^2+4) + B(s^2+s) + C(s+1)$$

Equalize coefficient of s^2, $B = 1$

Equalize coefficient of s, $C = -6$

CONTINUE >>>

Substitute value of A, B and C

$$\frac{4s^2 - 5s + 6}{(s+1)(s^2+4)} = \frac{3}{(s+1)} + \frac{s-6}{(s^2+4)}$$

$$\therefore\ L^{-1}\left\{\frac{4s^2 - 5s + 6}{(s+1)(s^2+4)}\right\}$$

$$= L^{-1}\left\{\frac{3}{(s+1)}\right\} + L^{-1}\left\{\frac{s-6}{(s^2+4)}\right\}$$

$$= 3L^{-1}\left\{\frac{1}{(s+1)}\right\} + L^{-1}\left\{\frac{s}{(s^2+2^2)}\right\} - \frac{6}{2}L^{-1}\left\{\frac{2}{(s^2+2^2)}\right\}$$

Refer to the table of Laplace Transform

$$= \mathbf{3e^{-t} + \cos 2t - 3\sin 2t}$$

EXERCISE 7

"Mistakes are prof that you are trying"

QUESTION Find the inverse Laplace Transform for each of the following by using partial fraction method.	ANSWER
1. $F(s) = \frac{5s+4}{s^2+2s}$	$f(t) = 2 + \frac{3}{s+2}$
2. $F(s) = \frac{s+7}{s^2+4s+3}$	$f(t) = 3e^{-t} - \frac{2}{s+3}$
3. $F(s) = \frac{3s^2+2s+5}{(s+1)(s^2+1)}$	$f(t) = 2\sin t + 3e^{-t}$
4. $F(s) = \frac{2s^2-3s+1}{(s-3)(s^2+1)}$	$f(t) = e^{3t} + \cos t$
5. $F(s) = \frac{-8s-2}{s^3-s^2-2s}$	$f(t) = 1 + 2e^{-t} - 3e^{2t}$
6. $F(s) = \frac{s^2+3s+2}{s^3+2s^2+2s}$	$f(t) = 1 + e^{-t}\sin t$

SOLVING DIFFERENTIAL EQUATION using THE LAPLACE TRANSFORM

STEPS :

1. Re-write the equation in terms of Laplace Transform.

2. Insert the given initial conditions.

3. Rearrange the equation algebraically to give the transform of the solution.

4. Determine the inverse transform to obtain the particular solution.

THEOREM:
Transform of Derivatives

If $L\{y(t)\} = Y(s)$, then

$$L\{y'(t)\} = sY(s) - y(0)$$

$$L\{y''(t)\} = s^2Y(s) - sy(0) - y'(0)$$

example 1

By using the Laplace Transform, solve the equation $\frac{dy}{dx} - 2y = 4$, given that at $y = 1$ and $x = 0$.

Solution:

$$\frac{dy}{dx} - 2y = 4$$

$$y' - 2y = 4$$

$$L\{y'\} - 2L\{y\} = L\{4\}$$

STEP 1 : Re-write the equation in terms of Laplace Transform.

$$sY(s) - y(0) - 2Y(s) = \frac{4}{s}$$

Given $\boldsymbol{y(0) = 1}$,

$$sY(s) - 1 - 2Y(s) = \frac{4}{s}$$

STEP 2 : Insert the given initial conditions.

$$Y(s)[s-2] = \frac{4}{s} + 1$$

STEP 3 : Rearrange the equation algebraically to give the transform of the solution.

$$Y(s)[s-2] = \frac{4+s}{s}$$

$$Y(s) = \frac{4+s}{s(s-2)}$$

CONTINUE >>>

STEP 4 : Determine the inverse transform.

$$Y(s) = \frac{4+s}{s(s-2)}$$

Use the partial fraction method.

$$\frac{4+s}{s(s-2)} = \frac{A}{s} + \frac{B}{(s-2)}$$

$$4 + s = A(s-2) + Bs$$

When $s = 0, \quad A = -2$

When $s = 2, \quad B = 3$

$$\frac{4+s}{s(s-2)} = \frac{-2}{s} + \frac{3}{(s-2)}$$

$$y(t) = L^{-1}\{Y(s)\}$$

$$= L^{-1}\left\{\frac{4+s}{s(s-2)}\right\}$$

$$= L^{-1}\left\{\frac{-2}{s}\right\} + L^{-1}\left\{\frac{3}{(s-2)}\right\}$$

$$= \mathbf{-2 + 3e^{2t}}$$ ---------- Refer to the table of Laplace Transform

example 2

By using the Laplace Transform, solve the equation $\frac{dy}{dx} + y = 2e^{3t}$, given that at $y(0) = 2$.

Solution:

$$\frac{dy}{dx} + y = 2e^{3t}$$

$$y' + y = 2e^{3t}$$

$$L\{y'\} - L\{y\} = L\{2e^{3t}\}$$

STEP 1 : Re-write the equation in terms of Laplace Transform.

$$sY(s) - y(0) - Y(s) = \frac{2}{s-3}$$

Given $\boldsymbol{y(0) = 2}$,

$$sY(s) - 2 - Y(s) = \frac{2}{s-3}$$

STEP 2 : Insert the given initial conditions.

$$Y(s)[s-1] = \frac{2}{s-3} + 2$$

STEP 3 : Rearrange the equation algebraically to give the transform of the solution.

$$Y(s)[s-1] = \frac{2 + 2(s-3)}{s-3}$$

$$Y(s) = \frac{2s-4}{(s-3)(s-1)}$$

CONTINUE >>>

STEP 4 : Determine the inverse transform.

$$Y(s) = \frac{2s-4}{(s-3)(s-1)}$$

Use the partial fraction method.

$$\frac{2s-4}{(s-3)(s-1)} = \frac{A}{(s-3)} + \frac{B}{(s-1)}$$

$$2s - 4 = A(s-1) + B(s-3)$$

When $s = 3$, $\quad A = 1$

When $s = 1$, $\quad B = 1$

$$\frac{2s-4}{(s-3)(s-1)} = \frac{1}{(s-3)} + \frac{1}{(s-1)}$$

$$\begin{aligned} y(t) &= L^{-1}\{Y(s)\} \\ &= L^{-1}\left\{\frac{2s-4}{(s-3)(s-1)}\right\} \\ &= L^{-1}\left\{\frac{1}{(s-3)}\right\} + L^{-1}\left\{\frac{1}{(s-1)}\right\} \\ &= \boldsymbol{e^{3t} + e^{t}} \end{aligned}$$

---------- Refer to the table of Laplace Transform

example 3

By using the Laplace Transform, solve the equation $y' + y = \sin t$, given that at $y(0) = 1$.

Solution:

$$y' + y = \sin t$$

$$L\{y'\} + L\{y\} = L\{\sin t\}$$ ---------- STEP 1 : Re-write the equation in terms of Laplace Transform.

$$sY(s) - y(0) + Y(s) = \frac{1}{s^2+1}$$

Given $\boldsymbol{y(0) = 1}$,

$$sY(s) - 1 + Y(s) = \frac{1}{s^2+1}$$ ------ STEP 2 : Insert the given initial conditions.

$$Y(s)[s+1] = \frac{1}{s^2+1} + 1$$ ------- STEP 3 : Rearrange the equation algebraically to give the transform of the solution.

$$Y(s)[s+1] = \frac{1 + 1(s^2+1)}{s^2+1}$$

$$Y(s) = \frac{s^2+2}{(s^2+1)(s+1)}$$

CONTINUE >>>

STEP 4 : Determine the inverse transform.

$$Y(s) = \frac{s^2+2}{(s^2+1)(s+1)}$$

Use the partial fraction method.

$$\frac{s^2+2}{(s^2+1)(s+1)} = \frac{As+B}{(s^2+1)} + \frac{C}{(s+1)}$$

$$s^2+2 = (As+B)(s+1) + C(s^2+1)$$

When $s=-1, \quad C = \frac{3}{2}$

Expand the equation

$$s^2+2 = A(s^2+s) + B(s+1) + C(s^2+1)$$

Equalize coefficient of $s^2, \quad A = -\frac{1}{2}$

Equalize coefficient of $s, \quad B = \frac{1}{2}$

$$\frac{s^2+2}{(s^2+1)(s+1)} = \frac{-\frac{1}{2}s+\frac{1}{2}}{(s^2+1)} + \frac{\frac{3}{2}}{(s+1)}$$

$$y(t) = L^{-1}\{Y(s)\}$$

$$= L^{-1}\left\{\frac{s^2+2}{(s^2+1)(s+1)}\right\}$$

$$= L^{-1}\left\{\frac{-s}{2(s^2+1)}\right\} + L^{-1}\left\{\frac{1}{2(s^2+1)}\right\} + L^{-1}\left\{\frac{3}{2(s+1)}\right\}$$

$$= -\frac{\mathbf{1}}{\mathbf{2}}\boldsymbol{\cos t} + \frac{\mathbf{1}}{\mathbf{2}}\boldsymbol{\sin t} + \frac{\mathbf{3}}{\mathbf{2}}\boldsymbol{e^{-t}}$$

example 4

By using the Laplace Transform, solve the differential equation of $y'' + 4y = 5e^{-t}$, given that at $y(0) = 2$ and $y'(0) = 3$.

Solution:

$$y'' + 4y = 5e^{-t}$$

$$L\{y''\} + 4L\{y\} = L\{5e^{-t}\}$$

STEP 1 : Re-write the equation in terms of Laplace Transform.

$$s^2Y(s) - sy(0) - y'(0) + 4Y(s) = \frac{5}{s+1}$$

Given $\boldsymbol{y(0) = 2}$ and $\boldsymbol{y'(0) = 3}$,

$$s^2Y(s) - 2s - 3 + 4Y(s) = \frac{5}{s+1}$$

STEP 2 : Insert the given initial conditions.

$$Y(s)[s^2 + 4] = \frac{5}{s+1} + 2s + 3$$

$$Y(s)[s^2 + 4] = \frac{5 + (2s+3)(s+1)}{s+1}$$

STEP 3 : Rearrange the equation algebraically

$$Y(s) = \frac{2s^2 + 5s + 8}{(s+1)(s^2+4)}$$

CONTINUE >>>

STEP 4 : Determine the inverse transform.

$$Y(s) = \frac{2s^2 + 5s + 8}{(s+1)(s^2+4)}$$

Use the partial fraction method.

$$\frac{2s^2 + 5s + 8}{(s+1)(s^2+4)} = \frac{A}{(s+1)} + \frac{Bs + C}{(s^2+4)}$$

$$2s^2 + 5s + 8 = A(s^2+4) + (Bs + C)(s+1)$$

When $s = -1, \quad A = 1$

Expand the equation

$$2s^2 + 5s + 8 = A(s^2+4) + B(s^2 + s) + C(s+1)$$

Equalize coefficient of $s^2, \quad B = 1$

Equalize coefficient of $s, \quad C = 4$

$$\frac{2s^2 + 5s + 8}{(s+1)(s^2+4)} = \frac{1}{(s+1)} + \frac{s+4}{(s^2+4)}$$

$$y(t) = L^{-1}\{Y(s)\}$$

$$= L^{-1}\left\{\frac{1}{(s+1)} + \frac{s+4}{(s^2+4)}\right\}$$

$$= L^{-1}\left\{\frac{1}{(s+1)}\right\} + L^{-1}\left\{\frac{s}{(s^2+4)}\right\} + L^{-1}\left\{\frac{4}{(s^2+4)}\right\}$$

$$= L^{-1}\left\{\frac{1}{(s+1)}\right\} + L^{-1}\left\{\frac{s}{(s^2+2^2)}\right\} + \frac{4}{2}L^{-1}\left\{\frac{2}{(s^2+2^2)}\right\}$$

$$= \boldsymbol{e^{-t} + \cos 2t + 2\sin 2t}$$

example 5

By using the Laplace Transform, solve the differential equation of $y'' - 3y' + 2y = e^{-t}$, given that at $y(0) = 1$ and $y'(0) = 1$.

Solution:

$$y'' - 3y' + 2y = e^{-t}$$

$$L\{y''\} - 3L\{y'\} + 2L\{y\} = L\{e^{-t}\}$$

STEP 1 : Re-write the equation in terms of Laplace Transform.

$$s^2Y(s) - sy(0) - y'(0) - 3[sY(s) - y(0)] + 2Y(s) = \frac{1}{s+1}$$

Given $\boldsymbol{y(0) = 1}$ and $\boldsymbol{y'(0) = 1}$,

$$s^2Y(s) - s - 1 - 3sY(s) + 3 + 2Y(s) = \frac{1}{s+1}$$

STEP 2 : Insert the given initial conditions.

$$Y(s)[s^2 - 3s + 2] = \frac{1}{s+1} + s - 2$$

STEP 3 : Rearrange the equation algebraically

$$Y(s)[s^2 - 3s + 2] = \frac{1 + (s-2)(s+1)}{s+1}$$

$$Y(s) = \frac{s^2 - s - 1}{(s+1)(s^2 - 3s + 2)}$$

CONTINUE >>>

STEP 4 : Determine the inverse transform.

$$Y(s) = \frac{s^2 - s - 1}{(s+1)(s^2 - 3s + 2)} = \frac{s^2 - s - 1}{(s+1)(s-2)(s-1)}$$

Use the partial fraction method.

$$\frac{s^2 - s - 1}{(s+1)(s-2)(s-1)} = \frac{A}{(s+1)} + \frac{B}{(s-2)} + \frac{C}{(s-1)}$$

$$s^2 - s - 1 = A(s-2)(s-1) + B(s+1)(s-1) + C(s-1)(s+1)$$

When $s = -1$, $\quad A = \frac{1}{6}$

When $s = 2$, $\quad B = \frac{1}{3}$

When $s = 1$, $\quad C = -\frac{1}{2}$

$$\frac{s^2 - s - 1}{(s+1)(s-2)(s-1)} = \frac{1}{6(s+1)} + \frac{1}{3(s-2)} - \frac{1}{2(s-1)}$$

$$y(t) = L^{-1}\{Y(s)\}$$

$$= L^{-1}\left\{\frac{s^2 - s - 1}{(s+1)(s-2)(s-1)}\right\}$$

$$= L^{-1}\left\{\frac{1}{6(s+1)}\right\} + L^{-1}\left\{\frac{1}{3(s-2)}\right\} - L^{-1}\left\{\frac{1}{2(s-1)}\right\}$$

$$= \frac{1}{6}L^{-1}\left\{\frac{1}{(s+1)}\right\} + \frac{1}{3}L^{-1}\left\{\frac{1}{(s-2)}\right\} - \frac{1}{2}L^{-1}\left\{\frac{1}{(s-1)}\right\}$$

$$= \boldsymbol{\frac{1}{6}e^{-t} + \frac{1}{3}e^{2t} - \frac{1}{2}e^{t}}$$

EXERCISE 8

"always deliver more than expected"

QUESTION	ANSWER
By using the Laplace Transform solve each of the following differential equation:	
1. $\frac{dy}{dt} - 2y = 2e^{3t}$, $y(0) = 2$	$y(t) = 2e^{3t}$
2. $\frac{dy}{dx} + 5y = 2e^{-3t}$, $y(0) = \frac{1}{2}$	$y(t) = e^{-3t} - \frac{1}{2}e^{-5t}$
3. $\frac{d^2y}{dt^2} - y = 2$, $y(0) = y'(0) = 0$	$y(t) = e^{t} + e^{-t} - 2$
4. $y'' - y' + y = t$, $y(0) = y'(0) = 1$	$y(t) = 1 + t$
5. $y'' - y' + 2y = \cos 3t - 17 \sin 3t$, $y(0) = -1, y'(0) = 6$	$y(t) = 2 \sin 3t - \cos 3t$

CONFIDENTIAL
Final Examination

PAST FINAL EXAM COLLECTION
LAPLACE TRANSFORM

Student Name		Matrix No.					

INSTRUCTIONS:

EXAM	CLOSED BOOK	☐	OPEN BOOK	☑		
	SINGLE SIDED	☑	PRINTED ON BOTH SIDES	☐		
	MULTIPLE CHOICE ANSWER SHEETS	☐				
	ANSWER IN BOOKLET	☑	EXTRA BOOKLET PERMITTED: YES			
	ANSWER ON EXAM	☐				
DICTIONARY	TRANSLATION ONLY	☐	REGULAR	☐	NONE	☑
CALCULATOR	NOT PERMITTED	☐	PERMITTED (non programmable)	☑		

Do not open the examination paper until instructed

Final Examination

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Question 1

Calculate $f(t) = e^{-4t}$ by using the definition of Laplace Transform, $F(s) = \int_0^\infty e^{-st} f(t)dt$.

Question 2

Calculate the following Laplace Transforms by using the stated method:

i. $\mathcal{L}\{\cos 4t - 2t^3 + 4e^{-2t}\}$;
Table of Laplace Transform

ii. $\mathcal{L}\{e^{3t} \sinh 2t\}$;
First Shift Theorem

iii. $\mathcal{L}\{t^2 e^{6t}\}$;
Multiplication by t^n Theorem

Question 3

Solve the Inverse Laplace Transform by using the stated method:

i. $\mathrm{F(s)} = \frac{8}{s^2+36}$;
Table of Laplace Transforms.

ii. $\mathrm{F(s)} = \frac{s+4}{(s-1)(s+5)}$;
Partial Fraction Method

Answer

$$F(s) = \frac{1}{s+4}$$

i. $F(s) = \frac{s}{s^2+16} - \frac{12}{s^4} + \frac{4}{s+2}$

ii. $F(s) = \frac{2}{(s-3)^2 - 4}$

iii. $F(s) = \frac{4}{(s-6)^3}$

i. $\mathrm{f}(t) = \frac{4}{3}\, sin6t$

ii. $\mathrm{f}(t) = \frac{5}{6}\, e^{t} + \frac{1}{6}\, e^{-5t}$

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Question 4

Use the definition of Laplace Transform, $F(s) = \int_0^\infty e^{-st} f(t)dt$ to construct the Laplace Transform for $f(t) = \int_0^\infty m\, e^{-3t}$ where m is any constant.

Question 5

Apply the stated theorem to find the Laplace Transforms for the following functions:

i. $f(t) = t^5 + cosh2t$;
Linearity Theorem

ii. $g(t) = e^{3t} sin6t$;
First Shift Theorem

iii. $k(t) = te^{4t}$;
Multiplication by t^n Theorem

Answer

$$F(s) = \frac{m}{s+3}$$

i. $F(s) = \frac{120}{s^6} - \frac{s}{s^2 - 4}$

ii. $F(s) = \frac{6}{(s-3)^2 + 36}$

iii. $F(s) = \frac{1}{(s-4)^2}$

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Question 6

Solve each of the following using the specified method:

i. $\mathcal{L}^{-1}\left\{\frac{8}{(s-1)^2-4}+\frac{3s}{s^2+16}\right\};$
use the Table of Laplace Transforms.

ii. $\mathcal{L}^{-1}\left\{\frac{2}{(s+1)(s+3)}\right\};$
use the Partial Fraction Method.

Question 7

Solve each of the following using the specified method:

i. $\mathcal{L}^{-1}\left\{\frac{24}{(s+2)^2-9}+\frac{12}{s^3}\right\};$
use the Table of Laplace Transforms.

ii. $\mathcal{L}^{-1}\left\{\frac{8}{(s-3)(s+2)}\right\};$
use the Partial Fraction Method.

Answer

Question 6:

i. $f(t) = 4e^t sinh2t + 3cos4t$

ii. $f(t) = e^{-t} - e^{-3t}$

Question 7:

i. $f(t) = 8e^{-2t}sinh3t + 6t^2$

ii. $f(t) = \frac{8}{5}e^{3t} - \frac{8}{5}e^{-2t}$

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Question 8

Find the Laplace Transform for the following by using the Laplace Transform Table:

i. $f(t) = (t-3)^2$

ii. $f(t) = 7cos3t - 3sin2t$

iii. $f(t) = -t^3 + 7t^2 - 1$

iv. $f(t) = t^3(t+3)^2$

v. $f(t) = \frac{e^{5t}}{3} - 2t + 7$

Answer

i. $F(s) = \frac{2}{s^3} - \frac{6}{s^2} + \frac{9}{s}$

ii. $F(s) = \frac{7s}{s^2+9} - \frac{6}{s^2+4}$

iii. $F(s) = -\frac{6}{s^4} + \frac{14}{s^3} - \frac{1}{s}$

iv. $F(s) = \frac{120}{s^6} + \frac{144}{s^5} + \frac{54}{s^4}$

v. $F(s) = \frac{1}{3(s-5)} - \frac{2}{s^2} + \frac{7}{s}$

Question 9

Transform the functions below by using First Shift Theorem:

i. $f(t) = e^{2t}(\cosh 3t + \sinh t)$

ii. $f(t) = e^{2t}t^2$

iii. $f(t) = e^{-2t}\sinh 3t$

Answer

i. $F(s) = \frac{s}{(s-2)-9} + \frac{1}{(s-2)-1}$

ii. $F(s) = \frac{2}{(s+2)^3}$

iii. $F(s) = \frac{3}{(s+2)^2 - 9}$

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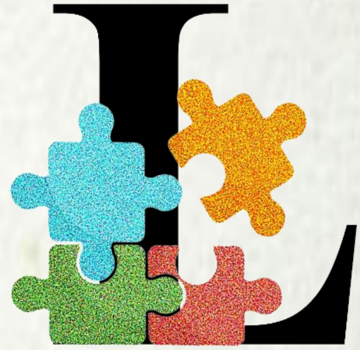
Introduction to
LAPLACE TRANSFORM



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POLITEKNIK MALAYSIA
SULTAN IDRIS SHAH

POLITEKNIK SULTAN IDRIS SHAH
SG. LANG, 45100 SG. AIR TAWAR
SELANGOR DARUL EHSAN
Tel: +603-32806200
Fax: Tel: +603-32806400
Website: http://psis.mypolycc.edu.my/

e ISBN 978-967-2860-89-1

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