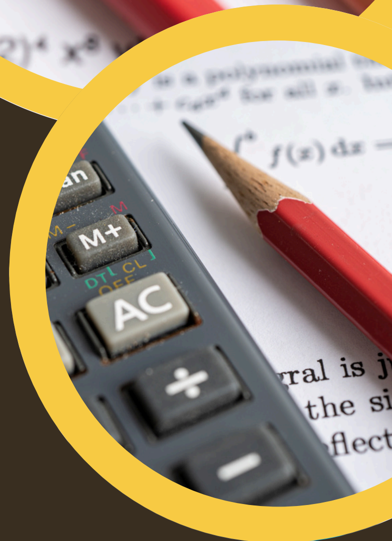

ENGINEERING MATHEMATICS I

Polytechnics student version

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PREFACE

Engineering Mathematics 1: Student Polytechnic Version is developed based on the syllabus for Polytechnic Engineering Mathematics 1. The main objectives of this book are to help students understanding for all the chapter by improving their problem-solving techniques and analytical skills. It serves as a practical tool to help students confidently solve mathematical problems and achieve excellent academic results.

The eBook is organized into five main chapters: Basic Algebra, Trigonometry, Complex Number, Matrices and Vector and Scalar. Each chapter includes a variety of worked examples and video tutorials designed to guide students in arriving at correct solutions effectively.

As a flexible learning resource, this eBook is ideal for self-paced study at any time and from any location, with no cost involved. Its straight forward and concise content make it an essential tool for students aiming to master the topic in Engineering Mathematics 1.

Azlina binti Morshidi
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ABSTRACT

E-book Engineering Mathematics 1 – Student Edition for Malaysian Polytechnics is a comprehensive reference source specifically designed to meet the needs of polytechnic students in Engineering courses. This book covers topics such as Basic Algebra, Trigonometry, Complex Numbers, Matrices and Vectors and Scalars, presented systematically and in line with the syllabus of the Department of Polytechnic Studies Malaysia (JPP).

Each chapter is presented with clear explanations, accompanied by problem-solving examples and controlled exercises to reinforce the understanding of mathematical concepts in the context of engineering. Furthermore, a visual approach and the use of modern technology in learning are also emphasized to help students master the application of mathematics in the real engineering world.

Each topic is arranged progressively, suitable for diploma-level students. Equipped with visual illustrations, concise notes, important formulas, structured question examples, and reinforcement exercises, this eBook supports students in building a strong and competitive mathematical foundation in the technical world.

This book is suitable for both self-study and as a teaching aid for lecturers and educators at Malaysian Polytechnics. Each chapter is structured with a student-friendly approach-starting with learning objectives, followed by theoretical explanations, step-by-step solution examples, and reinforcement exercises. The use of graphics, tables, and visualization methods also helps students understand abstract concepts more easily. This approach encourages active learning and supports the development of problem-solving skills and analytical thinking among TVET students.

TABLE OF CONTENTS

CHAPTER I

BASIC ALGEBRA

CHAPTER 1: BASIC ALGEBRA

1.1 Simplify basic algebra.

1.2 Solve quadratic equations.

1.3 Show procedure to find partial fractions.

1.1 Simplify basic algebra.

Definition of Algebraic Expression?

- An algebraic expression is an expression that contains one of more numbers, one or more variables (like x or y), and one or more arithmetic operations (like add, subtract, multiply and divide).
- Example of algebraic expression: $5x^2 - 6xy + 7$
- 5 and 6 are a coefficient. Numbers that used to multiply the variable,
- x and y are variables. Variables is a symbol, usually a letter that represent one of more numbers.
- 2 is an exponent (power). Exponent of a number shows how many times the numbers is to be used in a multiplication.

Simplify Basic Algebra

- Basic properties of algebra are to rewrite an algebraic expression in a simpler form.
- To simplify an algebraic expression generally means to remove symbols of grouping such as parentheses or brackets and combine like terms.
- Two or more terms of an algebraic expression can be combined only if they are like terms.
-

Like Terms	Not Like Terms
$3x$ and $-2x$	$7x$ and $8y$
$-3x^2$ and $9x^2$	$5y^2$ and $2y$
$-7x^2y^3$ and $15y^3x^2$	xy^2 and x^2y

Addition and Subtraction

a	$-2\text{m} + 8\text{m} - 9\text{m}$	b	$3x+7y+5x-2y$
c	$-7\text{p} + 8\text{q} - 9\text{q} + 7\text{r} + 2\text{q}$	d	$2x^2 + 5y^2 - 9x^2 + 7y^2$

Multiplication and Division

a	$(-a) \times (-b) = ab$ $(-a) \times b = -ab$ $a \times (-b) = -ab$ $a \times b = ab$	b	$(-a) \div (-b) = \frac{a}{b}$ $(-a) \div b = -\frac{a}{b}$ $a \div (-b) = -\frac{a}{b}$ $a \div b = \frac{a}{b}$

Simplify the following:

a	$-2mn \times 4nr$	b	$p^2q \times pq^3$
c	$24rst \div 6st$	d	$18ax^3 \div -3x$

Expanding Bracket

Can be done by multiplying the outer terms with every term inside the bracket.

$$a(x+y) = ax + ay$$

$$(a+b)(x+y) = ax + ay + bx + by$$

Expand the following algebraic expressions.

a	$5x(x-2y)$	b	$\frac{2}{5}(2a+5b-10c)$
c	$(2a-b)(3a+b)$	d	$(4x+2)(x-5)$

Consider the expansion of

$$(a+b)(a-b) = a(a) + a(-b) + b(a) + b(-b)$$

$$= a^2 - ab + ab - b^2$$

$$\therefore (a+b)(a-b) = a^2 - b^2$$

$$(a+b)(a+b) = a(a) + a(b) + b(a) + b(b)$$

$$= a^2 + ab + ab + b^2$$

$$\therefore (a+b)(a+b) = a^2 + 2ab + b^2$$

Expand the following algebraic expressions.

1.	$(t-7)^2$	2.	$(x+y)^2$
3.	$(2x-5)(2x+5)$	4.	$\left(\frac{1}{9}-x\right)\left(\frac{1}{9}+x\right)$

Factorization

1.	$\boldsymbol{ac-ab}$ $=\boldsymbol{a(c-b)}$	2.	$\boldsymbol{a^2-2ab+b^2}$ $=\boldsymbol{(a-b)^2}$
3.	$\boldsymbol{a^2+2ab+b^2}$ $=\boldsymbol{(a+b)^2}$	4.	$\boldsymbol{a^2-b^2}$ $=\boldsymbol{(a+b)(a-b)}$

Factorize the following:

1.	$\boldsymbol{xy+yx^3}$	2.	$\boldsymbol{2x^3-6x^2+14x}$
3.	$\boldsymbol{x^2+6x+8}$	4.	$\boldsymbol{16-x^2}$

EXERCISE

1.	$9y-6-3(2-5y)$	2.	$5(2x^2-6x)-3(4x^2-9)$
3.	$2a^2+3a-5a^2-a$	4.	$5y-2y[3+2(y-7)]$
5.	$3xy^2+4x^2y^2-2xy^2-(xy)^2$	6.	$4x^2+3x(-9x+6)$

Algebraic Fractions

A fraction can be simplified by multiplying or dividing the numerator and the denominator of the fraction by a common factor.

1.	$\frac{2a^2 - 2ab}{6ab - 6b^2}$	2.	$\frac{x^2 - 4}{x^2 + x - 2}$

Algebraic Fractions: Addition and Subtraction

- The procedure for adding and subtracting algebraic fractions is the same as the procedure for adding and subtracting common fractions.
- Fractions are added or subtracted by first changing into fractions with the same denominator.

Simplify the Algebraic Fraction

1.	$\frac{2}{p} + \frac{3}{q}$	2.	$\frac{1}{2x} + \frac{3}{5x}$
3.	$\frac{x}{y} + \frac{y}{x}$	4.	$x - \frac{1}{x}$
5.	$\frac{m-2}{2m^2} - \frac{3}{4m}$	6.	$\frac{p-1}{p^2} - \frac{q-1}{pq}$

7.	$\frac{7}{6a} - \frac{5-7a}{4a^2}$	8.	$\frac{x^2-x-6}{x-3} \times \frac{5}{3x+6}$

Algebraic Fractions: Multiplication and Division

- Can be done by factorizing the numerators and denominators followed by cancellations of the factors common to the numerator and denominator (if they exist), before applying multiplications to obtain the answer.
- To divide algebraic fractions, the second fraction must be inverted and then multiplied by the first fraction using the same steps in multiplication of algebraic fractions.

Simplify the algebraic fractions:

1.	$\frac{2x}{y} \times \frac{x}{6y}$	2.	$\frac{3x^2}{2y} \times \frac{y}{y-2}$
3.	$\frac{x-4}{x+3} \div \frac{2(x-4)}{3}$	4.	$(2x-x^2) \div \frac{2-x}{3}$

Solve the following equation:

1.	$4 - x = 5$	2.	$5x - 3 = 32$
3.	$-\frac{5}{3}x = 10$	4.	$13 - 5x = 8(x - 10)$
5.	$3x(2 + x) = x(3x - 2) - 24$	6.	$\frac{8}{5}x = 20 + 4x$

1.2 Quadratic Equations

- A quadratic equation is an equation in the form of $ax^2 + bc + c = 0$, where x represents and unknown and a, b and c represents known numbers with a not equal to 0.
- The highest order of a quadratic equation is two, so it is polynomial of degree two.
    - Quadratic expression: $ax^2 + bx + c\ (without\ equal\ sign)$
    - Quadratic equation: $ax^2 + bx + c = 0\ where\ a \neq 0\ (general\ form)$
- According to the fundamental theorem of algebra, any polynomial with a degree n has n roots. The roots can be determined using three (3) different methods:
    - Factorization
    - Quadratic formula
    - Completing the square

Factorization

To solve a quadratic equation using factorization, it must be factorized using the cross technique or can use the calculator scientific to get the final answer (Equation Function-Polynomial-Degree 2.

Solve the following quadratic equation using factorization:

1.	$x^2 - 3x - 10 = 0$
2.	$x(x - 7) = 18$
3.	$8 + 2x - x^2 = 0$
4.	$12x^2 - 13x - 35 = 0$
5.	$2x(4x + 15) = 27$

Quadratic Formula

The values of a, b and c need to be identified before substituted into the formula:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Solve the following quadratic equation quadratic formula:

1.	$2x^2 + 9x - 5 = 0$
2.	$2x^2 - 3x - 1 = 0$
3.	$4x^2 + 81 = 36x$
4.	$\frac{5}{k^2} + \frac{10}{k} + 2 = 0$

Completing the square

When using the completing the square method to solve quadratic equations, the coefficient of x^2 must always be a positive number 1.

$$\left(x+\frac{b}{2}\right)^2-\left(\frac{b}{2}\right)^2+c=0$$

Solve the following quadratic equation using completing the square:

1.	$x^2-4x-20=0$
2.	$5x^2=2-9x$

3.	$4x^2 = 12x + 11$
4.	$x(3x + 10) = 77$

1.3 Procedure to Find Partial Fraction

Proper Algebraic Fractions

- Proper fractions where the degree of the numerator is less than the degree of the denominator of fractions.

$$\frac{x^2 + 7x}{x^3 + 2x} \qquad \frac{x}{x^2 - 3}$$

- Improper fractions where the degree of numerator is greater than or equal to the degree of the denominator of fraction.

$$\frac{x^4 + 7x}{x^2 + 2} \qquad \frac{x^2}{x^2 - 3}$$

- Proper algebraic fractions will lead to proper partial fractions.
- An algebraic fraction can be decomposed as sum of two or more simpler algebraic fractions.
- The simpler algebraic fraction is called a partial fraction.
- There are three (3) types of partial fractions, depending on the factors of the denominator.
    - Denominator with single linear factors
    - Denominator with repeated linear factors
    - Denominator with a quadratic factor
        - Can be factorized.
        - Cannot be factorized.

Denominator with Linear Factors

For the fraction $\frac{P(x)}{Q(x)}$, where Q(x) is a combination of two or more linear factors. The fraction can be expressed as:

$$\frac{\boldsymbol{P(x)}}{\boldsymbol{(ax+b)(cx+d)}} = \frac{\boldsymbol{A}}{\boldsymbol{ax+b}} + \frac{\boldsymbol{B}}{\boldsymbol{cx+d}}$$

Example:

Express $\frac{x+4}{(x-2)(x-5)}$ in partial fraction.

Step 1: Split the fraction into two terms according to the factors of the denominator.

$$\frac{x+4}{(x-2)(x-5)} = \frac{A}{x-2} + \frac{B}{x-5}$$

Step 2: Equalize the denominator.

$$\frac{x+4}{(x-2)(x-5)} = \frac{A(x-5)}{(x-2)(x-5)} + \frac{B(x-2)}{(x-5)(x-2)}$$

$$\frac{x+4}{(x-2)(x-5)} = \frac{A(x-5)+B(x-2)}{(x-2)(x-5)}$$

Since the denominator for both sides are equal, take only the numerators into account.

$$x+4 = A(x-5) + B(x-2) \ldots\ldots..\ equation\ 1$$

Step 3: Let $x - 5 = 0\ and\ x - 2 = 0\ to\ find\ A\ and\ B$

$$x-5 = 0\ and\ x-2 = 0$$

$$x = 5\ and\ x = 2$$

Substitute $x = 5\ and\ x = 2\ into\ equation\ 1$

$$5+4 = A(5-5) + B(5-2)$$

$$9 = 3B$$

$$B = \frac{9}{3}$$

$$B = 3$$

$$2+4 = A(2-5) + B(2-2)$$

$$6 = -3A$$

$$A = \frac{6}{-3}$$

$$A = -2$$

Step 4: Substitute the values of A and B into the equation in Step 1:

$$\boldsymbol{\frac{x+4}{(x-2)(x-5)} = -\frac{2}{x-2} + \frac{3}{x-5}}$$

Exercise

$\boldsymbol{Express\,\frac{17x-53}{(x^2-2x-15)}\,In\ partial\ fraction}$
$\boldsymbol{Express\,\frac{125+4x-9x^2}{(x-1)(x+3)(x+4)}\,In\ partial\ fraction}$

Denominator with Repeated Linear Factors

For the fraction $\frac{P(x)}{Q(x)}$, where $Q(x)$ is a repeated linear factor, the fraction can be expressed as:

$$\frac{\boldsymbol{P(x)}}{\boldsymbol{(ax+b)^n}} = \frac{\boldsymbol{A}}{\boldsymbol{(ax+b)}} + \frac{\boldsymbol{B}}{\boldsymbol{(ax+b)^2}} + \cdots + \frac{\boldsymbol{F}}{\boldsymbol{(ax+b)^n}}$$

Example:

$$\text{Express } \frac{x}{(x-1)^2} \; in\; partial\; fraction.$$

Step 1: Split the fraction into two terms according to the factors of the denominator.

$$\frac{x}{(x-1)^2} = \frac{A}{x-1} + \frac{B}{(x-1)^2}$$

Step 2: Equalize the denominator.

$$\frac{x}{(x-1)^2} = \frac{A(x-1)}{(x-1)(x-1)} + \frac{B}{(x-1)^2}$$

$$\frac{x}{(x-1)^2} = \frac{A(x-1)+B}{(x-1)^2}$$

Since the denominator for both sides are equal, take only the numerators into account.

$$x = A(x-1) + B$$

$$x = A(x-1) + B \; \ldots \ldots . . \; equation\; 1$$

Step 3: Let $x - 1 = 0 \; therefore\; x = 1$

Substitute $x = 1 \; into\; equation\; 1$

$$1 = A(1-1) + B$$

$$1 = B$$

$$Since\; there\; are\; no\; other\; factors,\; we\; let\; x = 0\; and\; substitute\; x = 0\; into\; equation\; 1$$

$$0 = A(0-1) + B$$

$$0 = -A + B$$

$$A = B,\; A = 1$$

Step 4: Substitute the values of A and B into the equation in Step 1:

$$\frac{x}{(x-1)^2} = \frac{1}{x-1} + \frac{1}{(x-1)^2}$$

Exercise

$$\boldsymbol{Express\ \frac{10x+35}{(x+4)^2}\ In\ partial\ fraction}$$
$$\boldsymbol{Express\ \frac{6x+5}{(2x-1)^2}\ In\ partial\ fraction}$$

Denominator with Quadratic Factors

- For the fraction $\frac{P(x)}{Q(x)}$, where $Q(x)$ is a quadratic factor, $Q(x)$ must first be factorized and solved using the method used for the denominator with linear factors:

$$\boldsymbol{\frac{P(x)}{a^x + bx + c} = \frac{P(x)}{(ax+b)(cx+d)} = \frac{A}{(ax+b)} + \frac{B}{(cx+d)}}$$

- However, if $Q(x)$ cannot be factorized, the fraction can be expressed as:

$$\boldsymbol{\frac{P(x)}{a^x + bx + c} = \frac{Ax+B}{(ax^2+bx+c)}}$$

Example:

$$\text{Express } \frac{2x+3}{2x^2+7x-4} \text{ in partial fraction}$$

Step 1: Split the fraction into two terms according to the factors of the denominator.

$$\frac{2x+3}{2x^2+7x-4} = \frac{2x+3}{(2x-1)(x+4)} = \frac{A}{(2x-1)} + \frac{B}{(x+4)}$$

Step 2: Equalize the denominator

$$\frac{2x+3}{(2x-1)(x+4)} = \frac{A(x+4)}{(2x-1)(x+4)} + \frac{B(2x-1)}{(x+4)(2x-1)}$$

$$\frac{2x+3}{(2x-1)(x+4)} = \frac{A(x+4)+B(2x-1)}{(2x-1)(x+4)}$$

Since the denominator for both sides are equal, take only the numerators into account.

$$2x+3 = A(x+4) + B(2x-1)$$

$$2x+3 = A(x+4) + B(2x-1) \ldots\ldots.. \; equation\ 1$$

Step 3: Let $x + 4 = 0$ *and* $2x - 1 = 0$ *to find the values of* A *and* B

$$x + 4 = 0$$

$$x = -4$$

$$\text{Substitute } x = -4 \text{ into equation 1}$$

$$2(-4) + 3 = A[(-4)+4] + B[2(-4)-1]$$

$$-8 + 3 = -9B$$

$$-5 = -9B$$

$$B = \frac{5}{9}$$

$$2x - 1 = 0$$

$$x = \frac{1}{2}$$

Substitute $x = \frac{1}{2}$ into equation 1

$$2\left(\frac{1}{2}\right) + 3 = A\left[\left(\frac{1}{2}\right) + 4\right] + B\left[2\left(\frac{1}{2}\right) - 1\right]$$

$$4 = \frac{9}{2}A,$$

$$A = \frac{8}{9}$$

Step 4: Substitute the values of A and B into the equation in Step 1:

$$\frac{\mathbf{2x+3}}{\mathbf{2x^2+7x-4}} = \frac{\mathbf{2x+3}}{\mathbf{(2x-1)(x+4)}} = \frac{\mathbf{8}}{\mathbf{9(2x-1)}} + \frac{\mathbf{5}}{\mathbf{9(x+4)}}$$

Exercise

$$\boldsymbol{Express\ \frac{x-1}{x(x^2+1)}\ In\ partial\ fraction}$$

$$Express\,\frac{3x^2+7x+28}{x(x^2+x+7)}\,In\;partial\;fraction$$

Improper Algebraic Fractions

- Improper algebraic fractions are fractions where the degree of the numerator is larger than or equal to the degree of the denominator.
- For example:

$$\frac{x^2+1}{x^2+4} \; and \; \frac{x^2+4x-3}{x-1}$$

- This type of fraction needs to undergo a process called **long division**.
- This will result in a combination of **polynomials** and **proper algebraic fractions.**
- The **proper algebraic fraction** may then be expressed as partial fractions.
- Note: if there are any missing terms in either the numerator or denominator, just include the missing terms with a coefficient of **zero**.
- For example:

$$\frac{x^2+1}{x^2+4}=\frac{x^2+\mathbf{0}x+1}{x^2+\mathbf{0}x+4}$$

Example:

Express $\frac{x^4+1}{x^2-1}$ in partial fraction.

Step 1: Fill the missing terms and apply long division.

$$\begin{array}{r}
\boldsymbol{x^2} \qquad \qquad \mathbf{+1} \\
x^2+0x-1\;\overline{)\,x^4+0x^3+0x^2+0x+1} \\
-\quad +x^4 \; +0x^3 \; -x^2 \qquad\qquad \\
\hline
+x^2 \; +0x \; +1 \\
-\; +x^2 \; +0x \; -1 \\
\hline
+2
\end{array}$$

$$\frac{x^4+1}{x^2-1}=\boldsymbol{x^2}+\mathbf{1}+\frac{\mathbf{2}}{\boldsymbol{x^2}-\mathbf{1}}$$

Step 2: Solve for the partial fractions.

Step 2.1: Split the fraction into two terms according to the factors of the denominator.

$$\frac{2}{x^2-1}=\frac{2}{(x-1)(x+1)}=\frac{A}{x-1}+\frac{B}{x+1}$$

Step 2.2: Equalize the denominator.

$$\frac{2}{(x-1)(x+1)}=\frac{A(x+1)+B(x-1)}{(x-1)(x+1)}$$

Since the denominator for both sides are equal, take only the nominator into account.

$$2=A(x+1)+B(x-1)$$

$$2 = A(x+1) + B(x-1) \ldots\ldots \; equation\ 1$$

Step 2.3: Let $x + 1 = 0 \; and \; x - 1 = 0 \; to \; find \; the \; values \; of \; A \; and \; B$.

$$\frac{2}{(x-1)(x+1)} = \frac{A}{x-1} + \frac{B}{x+1}$$

$$x + 1 = 0$$

$$x = -1$$

Substitute $x = -1 \; into \; equation\ 1$

$$2 = A(-1+1) + B(-1-1)$$

$$2 = -2B$$

$$B = -1$$

$$x - 1 = 0$$

$$x = 1$$

Substitute $x = 1 \; into \; equation\ 1$

$$2 = A(1+1) + B(1-1)$$

$$2 = 2A$$

$$A = 1$$

Step 2.4: Substitute the values of A and B into the equation in Step 2.1

$$\frac{2}{x^2-1} = \frac{2}{(x-1)(x+1)} = \frac{1}{x-1} - \frac{1}{x+1}$$

Step 3: Rewrite the equation in Step 1

$$\boldsymbol{\frac{x^4+1}{(x^2-1)} = x^2 + 1 + \frac{2}{x^2-1} = x^2 + 1 + \frac{1}{x-1} - \frac{1}{x+1}}$$

Exercise

$$Express\ \frac{2x^3 + 3x^2 - 2}{(x+2)(x-1)}\ In\ partial\ fraction$$

$$Express\ \frac{15x - 12x^2 - 1}{8x + 6x^2 + 2}\ In\ partial\ fraction$$

Quiz Activity

SCAN ME

https://forms.gle/tjg45A6iSS9tsjR6A

2

CHAPTER 2

TRIGONOMETRY

CHAPTER 2: TRIGONOMETRY

2.1 Solve the fundamental of trigonometric functions.

2.2 Solve trigonometric equations and identities.

2.3 Apply sine and cosine rules.

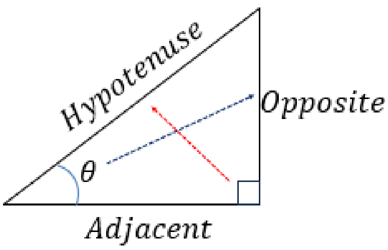
What is Trigonometry?

Trigonometry is the science of measuring triangles. The triangles is important to measure because triangles are the basic form of flat shapes, 4-sided, 5 sided, and even circles created from the combination of triangles.

2.1 Solve the fundamental of trigonometric functions.

2.1.1 Define sine, cosine, tangent, secant, cosecant and cotangent.

- Consider a right-angled triangle with θ as one of its acute angles (an angle smaller than 90°)



TRIGONOMETRIC RATIOS

$\sin\theta$, $\cos\theta$, $\tan\theta$, $\sec\theta$ $\csc\theta$, $\tan\theta$

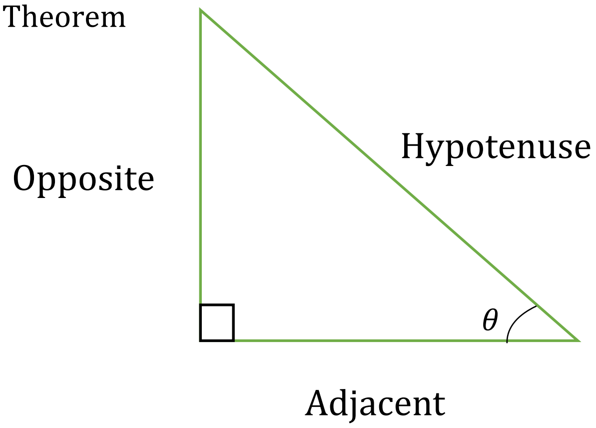
$$\sin\theta = \frac{opposite}{hypotenuse} \qquad \csc\theta = \frac{1}{\sin\theta}$$

$$\cos\theta = \frac{adjacent}{hypotenuse} \qquad \sec\theta = \frac{1}{\cos\theta}$$

$$\tan\theta = \frac{opposite}{adjacent} = \frac{\sin\theta}{\cos\theta} \quad \cot\theta = \frac{1}{\tan\theta} = \frac{\cos\theta}{\sin\theta}$$

2.1.2 Sketch the graph of sine, cosine and tangent

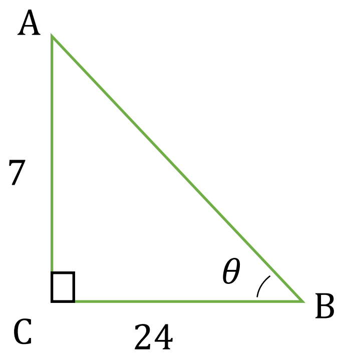
- Pythagorean Theorem



Pythagorean Theorem
$(\text{Hypotenuse})^2 = (\text{Adjacent})^2 + (\text{Opposite})^2$
$(\text{Adjacent})^2 = (\text{Hypotenuse})^2 - (\text{Opposite})^2$
$(\text{Opposite})^2 = (\text{Hypotenuse})^2 - (\text{Adjacent})^2$

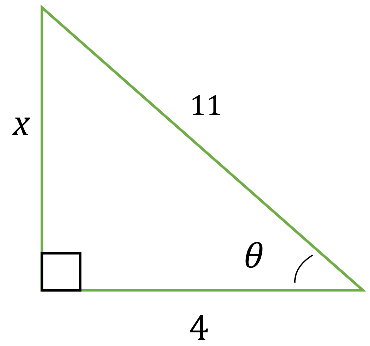
Example:

i) Consider the following right-angles triangles.



a) Find length of AB	b) Sec θ

ii) Consider a right-angled triangle with θ as one of its acute angles (an angle smaller than 90°).



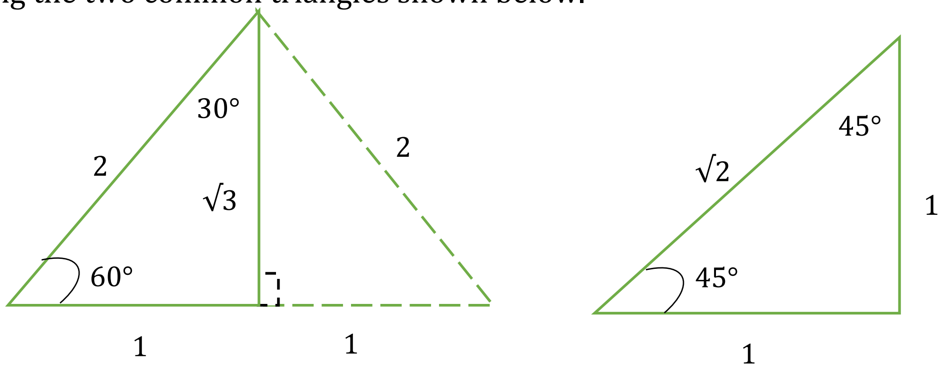
Before we start, find the value of x first:

$$x^2 = 11^2 - 4^2$$

$$x = \sqrt{105}$$

Find the values of the trigonometric ratios below.	
$a.$ $\tan\theta$	$b.\sin\theta$
$c.\cot\theta$	$d.\, cosec\ \theta$

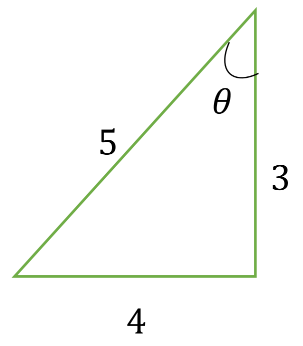
- Trigonometric ratios for special angles ($30°, 45°$ *and* $60°$) can be obtained by using the two common triangles shown below.



θ	30°	45°	60°	90°
$\sin\theta$	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{\sqrt{3}}{2}$	1
$\cos\theta$	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$	0
$\tan\theta$	$\frac{1}{\sqrt{3}}$	1	$\sqrt{3}$	$Undefined$

Exercise:

i) Based on the diagram, find:



$a. \sin\theta =$

$b. \cos\theta =$

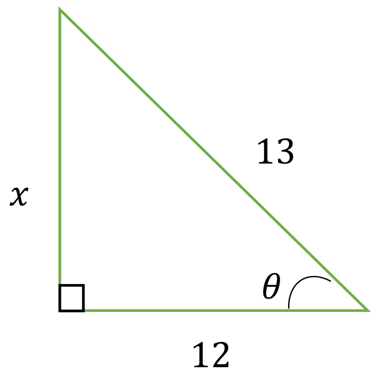
$c. \tan\theta =$

$d. \cot\theta =$

$e. cosec\ \theta =$

$f. \sec\theta =$

ii) Based on the diagram, find the value of



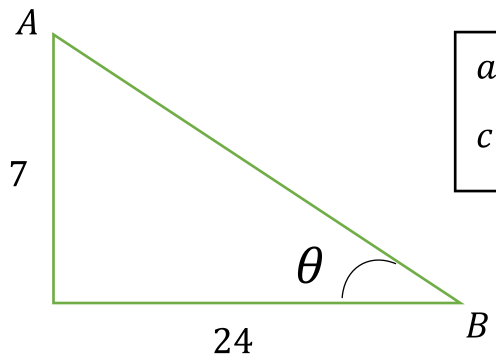
$a. x =$

$b. \tan\theta =$

$b. cosec\ \theta =$

$c. \cot\theta =$

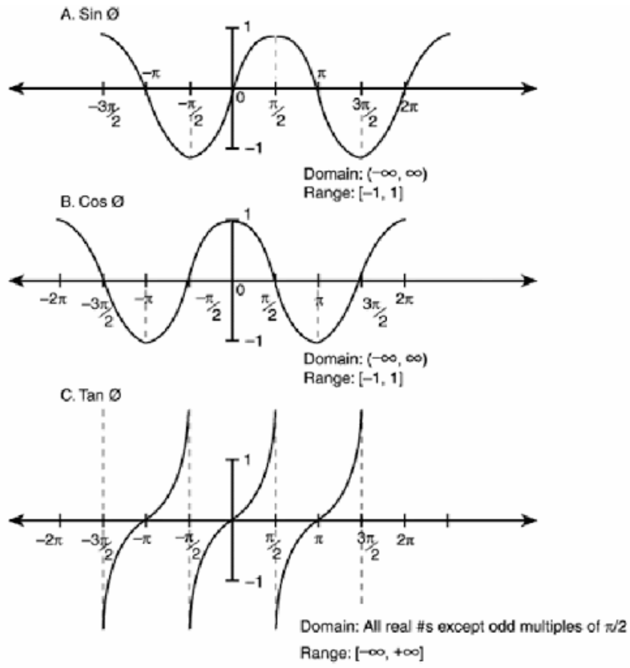
iii) Based on the diagram, find the following values:



$a.\ the\ length\ of\ AB =$

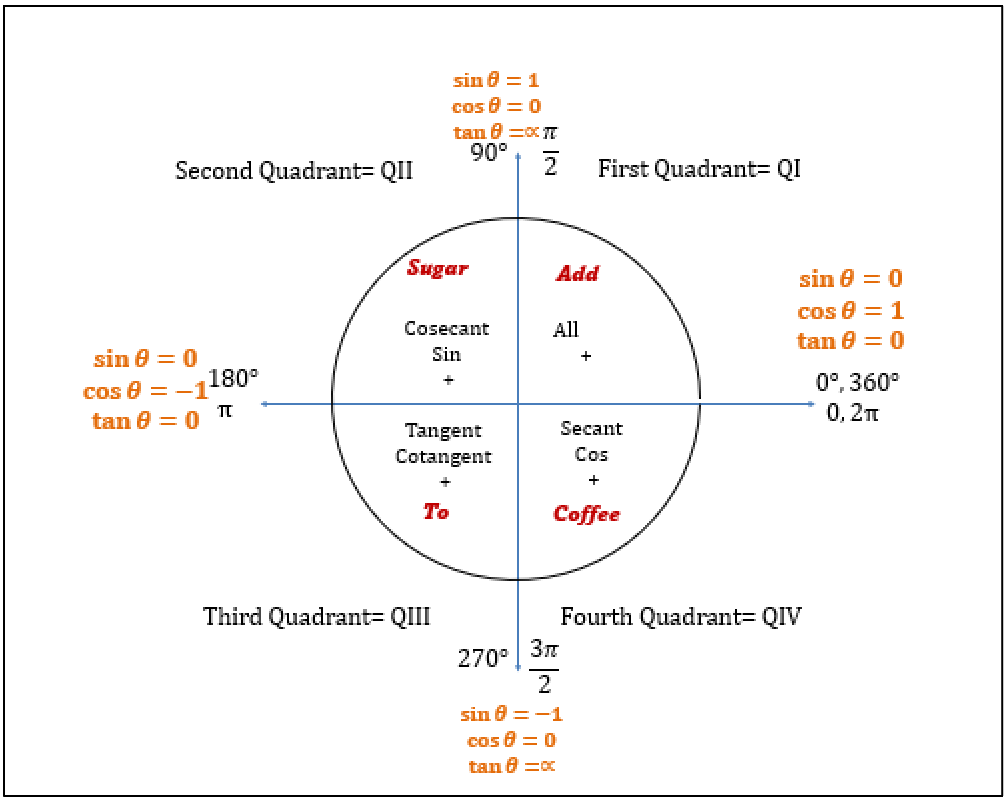
$c. \sec\theta =$

- The values of the basic trigonometric functions $\sin x, \cos x$ *and* $\tan x$ for some points in between or equivalent to -2π *to* 2π, *or equivalent to* $0°$ *to* $360°$.



2.1.3 Show the positive and negative values of trigonometric function using quadrants.

- Values of trigonometric function using quadrants.
  - Sign of Trigonometric functions is used to calculate the value of the trigonometric functions in the four quadrants. In order to do so, we need to find the reference angle, $\boldsymbol{\beta}$.

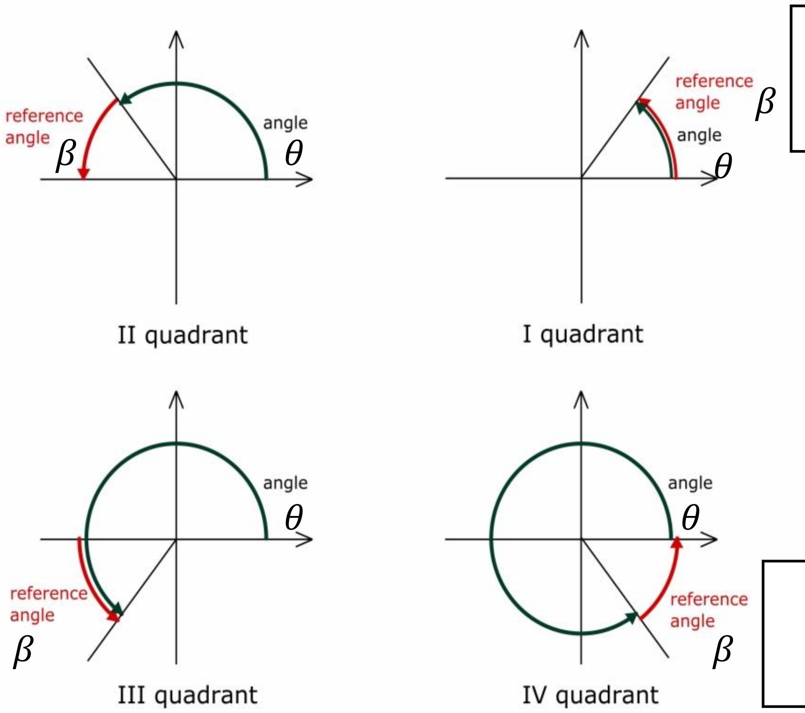


Q2

$\beta = 180 - \theta$

Q1

$\beta = \theta$



Q3

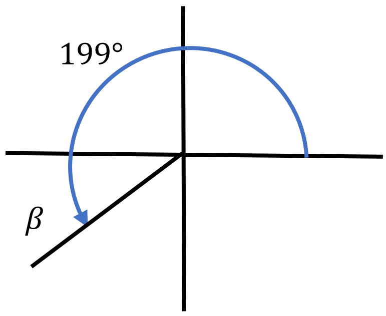
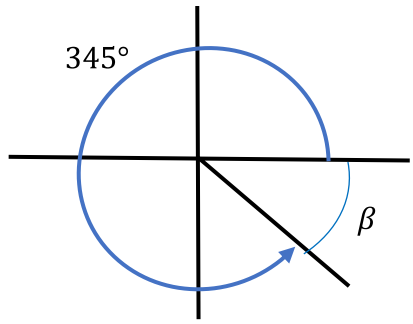
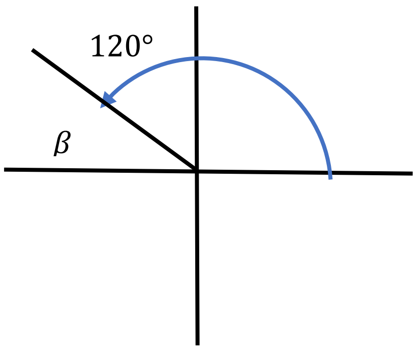
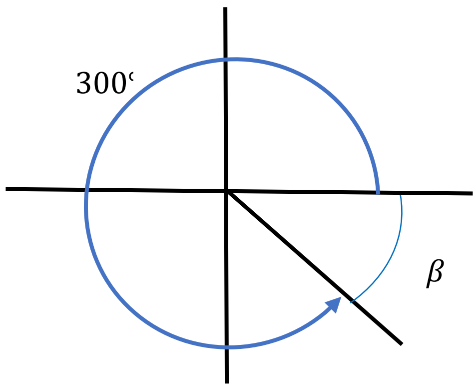
$\beta = \theta - 180$

Q14

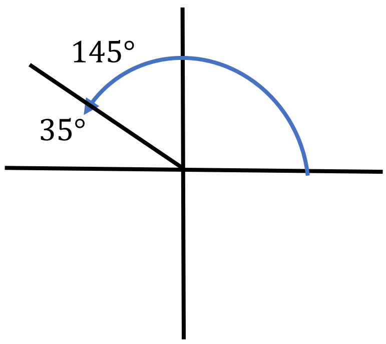
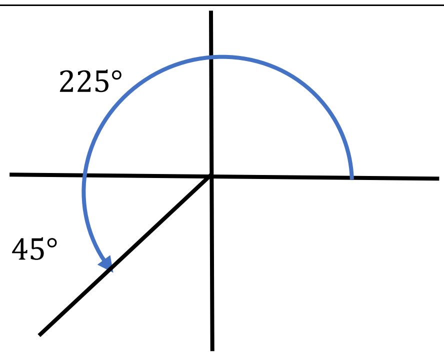
$\beta = 360 - \theta$

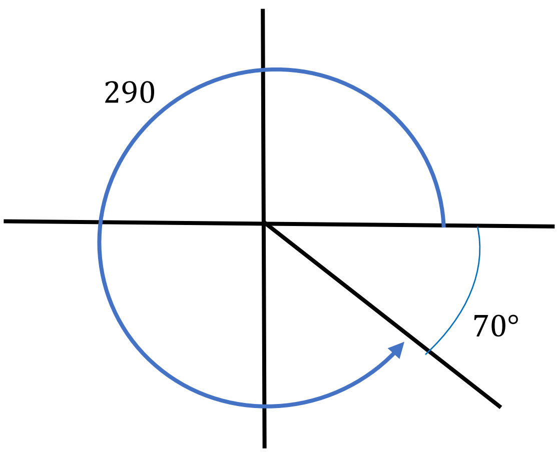
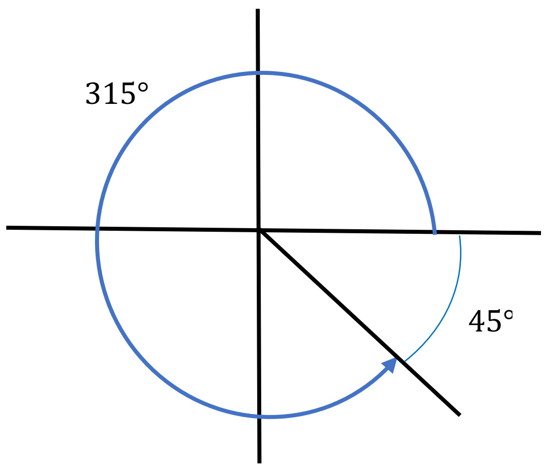
Example:

i) Find the reference angle, β for

<table>
<tr><td>a. $\theta = 199°$</td><td>b. $\theta = 345°$</td></tr>
<tr><td>
</td><td>
</td></tr>
<tr><td>c. $\theta = \frac{2}{3}\pi \, rad$</td><td>d. $\theta = \frac{5}{3}\pi \, rad$</td></tr>
<tr><td>
</td><td>
</td></tr>
</table>

ii) Find the values of

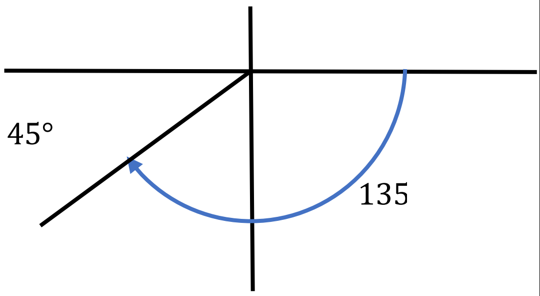
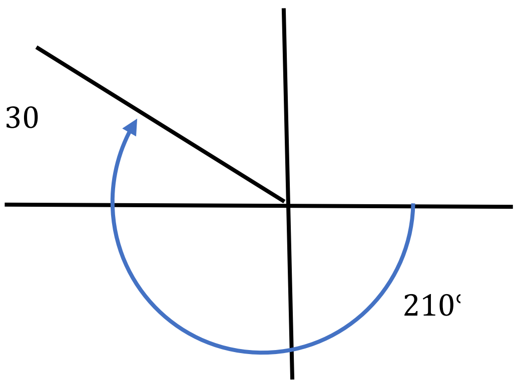
<table>
<tr><td>a. cos 145°</td><td>b. tan 225°</td></tr>
<tr><td>
</td><td>
</td></tr>
</table>

c. *cosec* 290°	d. sec 315°
	

- Negative Angle
  - The angle rotates in a clockwise direction. This angle is called a **negative angle**.

Example:

i) Find the values of

a. $\cos(-135°)$	b. $\sin(-210°)$
	

Exercise:

i) Find the values of:

$sec\ (-45°)$	$\cot\ (-385°)$

ii) Find the reference angle, β and calculate the value of trigonometric function.

$sin\ 125°$	$\tan 250°$

$cos\ 250°$	$\tan 300°$

2.1.4 Calculate the values of trigonometric functions

- Values of Trigonometric functions:

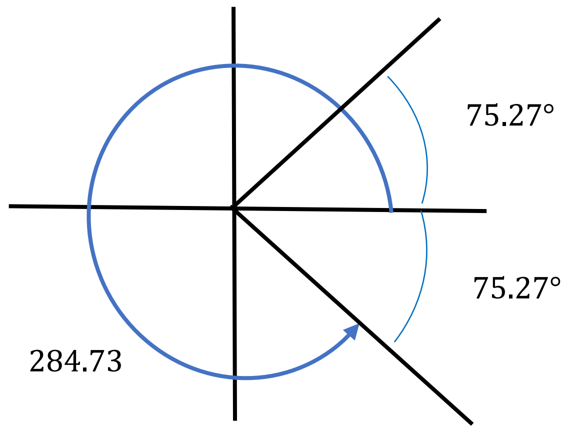
$\boldsymbol{y = sin^{-1}x}$ (**sin inverse**)
$\boldsymbol{y = cos^{-1}x}$ (**cos inverse**)
$\boldsymbol{y = tan^{-1}x\ (tan\ inverse)}$

Example:

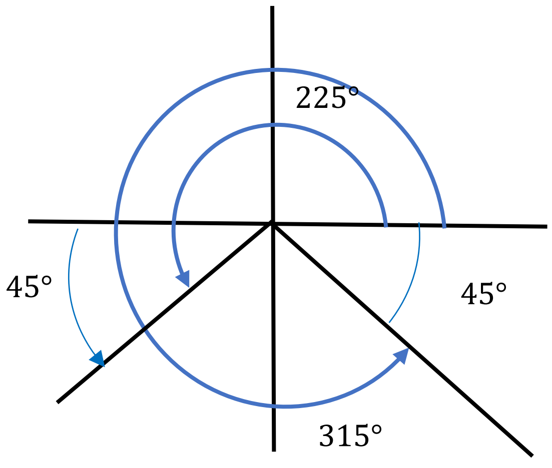
i) Find the value of θ.

$a.\, \boldsymbol{tan\ \theta = 0.864}$	$b.\, \boldsymbol{cos\ \theta = 0.799}$
c. $\boldsymbol{sec\ \theta = 1.345}$	d. $\boldsymbol{cosec\ \theta = 1.300}$

ii) Find the values of θin the range of $0° \leq \theta \leq 360°$ for:

$\boldsymbol{cos\,\theta = 0.2542}$	
Since $\cos\theta$ is positive, it shows that θ is located in quadrant I and IV. $\theta = cos^{-1}0.2542$ $\beta = 75.27°\ (reference\ angle)$ $Quadrant\ I: \theta = \beta = 75.27°$ $Quadrant\ IV: \theta = 360° - 75.27°$ $= 284.73°$ $\therefore \boldsymbol{\theta = 75.27°, 284.73°}$	

iii) Find the values of θin the range of $0° \leq \theta \leq 360°$ for:

$\boldsymbol{sin\,\theta = -cos45°}$	
$\sin\theta = -0.707$ Since $\sin\theta$ is negative, it shows that θ is located in quadrant III and IV. $\beta = sin^{-1}(0.707)$ $\theta = \beta = 45°\ (reference\ angle)$ $Quadrant\ III: \theta = 180° + 45 = 225°$ $Quadrant\ IV: \theta = 360° - 45° = 315°$ $\therefore \boldsymbol{\theta = 225°, 315°}$	

Exercise

i) Find the values of θ in the range of $0° \leq \theta \leq 360°$ for:

a. $6\,tan^2\theta - tan\theta - 1 = 0$	

b. $5\,cos^2\theta + 3\,cos\,\theta - 2 = 0$	

c. $2\cos^2\theta + \cos\theta = 0$	

2.2 Solve trigonometric equations and identities.

2.2.1 Solve trigonometric equations and identities

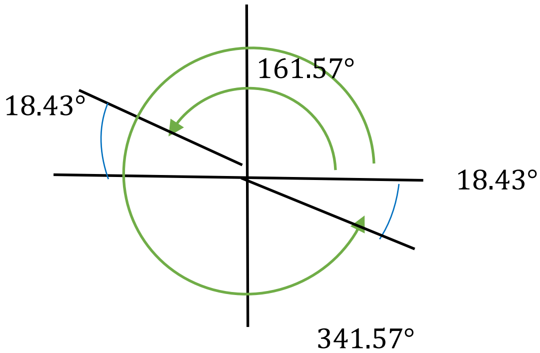
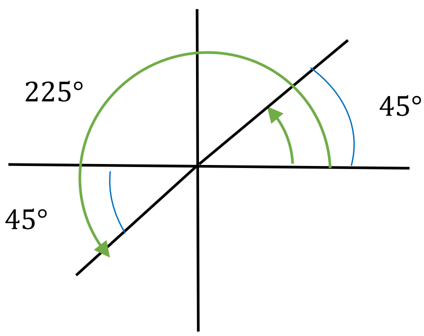
- **Trigonometric Basic Identities**

$$\boldsymbol{cos^2\theta + sin^2\theta = 1}$$

$$\boldsymbol{1 + tan^2\theta = sec^2\theta}$$

$$\boldsymbol{cot^2\theta + 1 = cosec^2\theta}$$

- Example:
  - Find the values of x in the range of $0° \leq x \leq 360°$ for:

$\boldsymbol{3\,tan\,x - cot\,x = 2}$	
$\boldsymbol{change\ cot\,x = \frac{1}{tan\,x}\,thus}$ $3\tan x - \frac{1}{\tan x} = 2$ $3tan^2x - 1 = 2\tan x$ $3tan^2x - 2\tan x - 1 = 0$ $(3tanx + 1)(\tan x - 1) = 0$ $\therefore \boldsymbol{3\,tanx + 1 = 0}\ or\ \boldsymbol{tan\,x - 1 = 0}$ $\boldsymbol{3\tan x + 1 = 0}$ $\tan x = -\frac{1}{3}(Q\ II\ and\ IV)$ $x = tan^{-1}\left(\frac{1}{2}\right) = 18.43°$ $Quadrant\ II: x = 180° - 18.43°$ $= 161.57°$ $Quadrant\ IV: x = 360° - 18.43°$ $= 341.57°$ $\boldsymbol{tan\,x - 1 = 0}$ $\tan x = 1(Q\ I\ and\ III)$ $x = tan^{-1}(1) = 45°$ $Quadrant\ I: x = \beta = 45°$ $Quadrant\ III: x = 180° + 45° = 225°$ $\therefore \boldsymbol{x = 45°, 161.57°, 225°, 341.57°}$	 

- Exercise
    - Find the values of x in the range of $0° \leq x \leq 360°$ for:

$\boldsymbol{tan^2 x = sec\, x + 1}$	
$\boldsymbol{2sin^2 x + cos\, x - 1 = 0}$	

- **Compound Angle Formula**
  a. If A and B are any two angles, $A + B$ or $A - B$ is a compound angle.
  b. The trigonometric ratios for compound angles $A \pm B$ can be expressed in terms of angle A and angle B

$$\boldsymbol{sin(A \pm B) = sinAcosB \pm cosAsinB}$$

$$\boldsymbol{cos(A \pm B) = cosAcosB \mp sinAsinB}$$

$$\boldsymbol{tan(A \pm B) = \frac{tanA \pm tanB}{1 \mp tanAtanB}}$$

- Example:

a. $\cos 80° \cos 50° + \sin 80° \sin 50°$	b. $\sin 25° \cos 15° + cos\ 25° \sin 15°$
c. $\dfrac{\tan 35° - \tan 10°}{1 + \tan 35° tan 10°}$	d. $\sin 36° \cos 54° + cos\ 36° \sin 54°$

- **Double Angle Formula**
  a. The double angle formula can be derived from the compound angle
  b. Formula by replacing with A.

$$\boldsymbol{sin2A = 2sinAcosA}$$

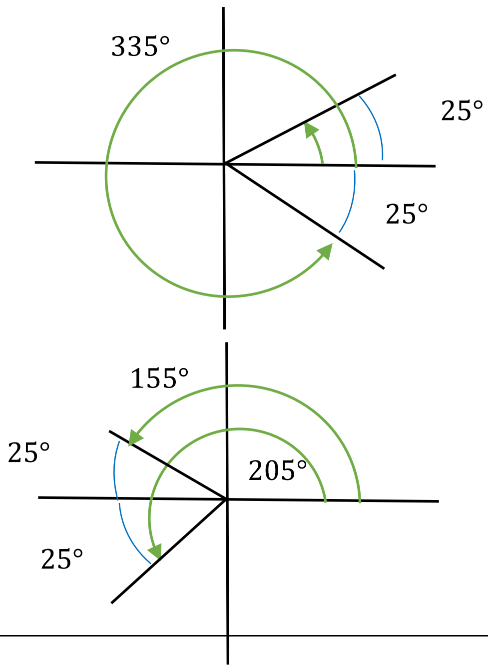
$$\boldsymbol{cos2A = cos^2A - sin^2A}$$

$$\boldsymbol{= 1 - 2sin^2A}$$

$$\boldsymbol{= 2cos^2A - 1}$$

$$\boldsymbol{tan2A = \frac{2tanA}{1 - tan^2A}}$$

- Example
    a. Find the values of x in the range of $0° \leq x \leq 360°$ for:

$\boldsymbol{cos\,2x = 0.6428}$	
$\cos 2x = 0.6428$ $2cos^2x - 1 = (0.6428)$ $2cos^2x - 1 - 0.6428 = 0$ $2cos^2x = 1.6428$ $cos^2x = \frac{1.6428}{2} = 0.8214$ $\cos x = \sqrt{0.8214}$ $\cos x = \pm 0.9063$ $\cos x = 0.9063(\,QI\ and\ IV)$ $x = 25°$ $Quadrant\ I: x = \beta = 25°$ $Quadrant\ IV: x = 360° - 25° = 335°$ $\cos x = -0.9063(\,QII\ and\ III)$ $x = 25°$ $Quadrant\ II: 180° - 25° = 155°$ $Quadrant\ III: 180° + 25° = 205°$ $\boldsymbol{x = 25°, 155°, 205°, 335°}$	

Exercise

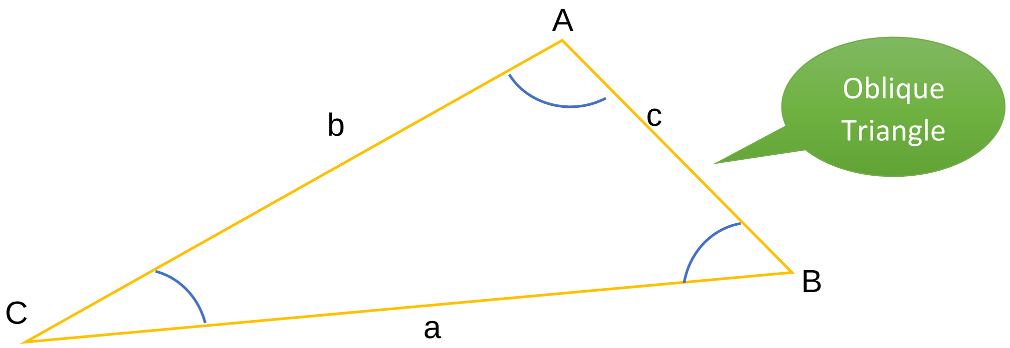
a. $\boldsymbol{sin\frac{x}{2} = 0.5736}$	
b. $\boldsymbol{cos2x + 3sinx = 2}$	

2.3 Apply sine and cosine rules.

2.3.1 Define the sine and cosine rules

- Sine Rule

Figure below shows triangle ABC. It is made up of 6 elements which has 3 sides (denoted by lower case letters a, b and c at each edge) and 3 angles (denoted by capital letters A, B and C at each of vertices).



To use the Sine Rule, we must at least have the values of :

- Two angles and one side, or
- Two sides and an excluded angle

**(The excluded angle is the angle that is not in between the two known sides.*

$$\frac{a}{Sin\ A} = \frac{b}{Sin\ B} = \frac{c}{Sin\ C}$$

- Cosine Rules

The Cosine Rule is used to solve a problem involving triangles when either

- The values of the three sides of the triangle are given or
- The value of the two sides and an included angle are given

**(The included angle is the angle in between the two given sides.*

$$\boldsymbol{a^2 = b^2 + c^2 - 2bc\cos A}$$

$$\boldsymbol{b^2 = a^2 + c^2 - 2ac\cos B}$$

$$\boldsymbol{c^2 = a^2 + b^2 - 2ab\cos C}$$

2.3.2 Calculate the area of a triangle using the formula $\frac{1}{2} \times ab \sin C$

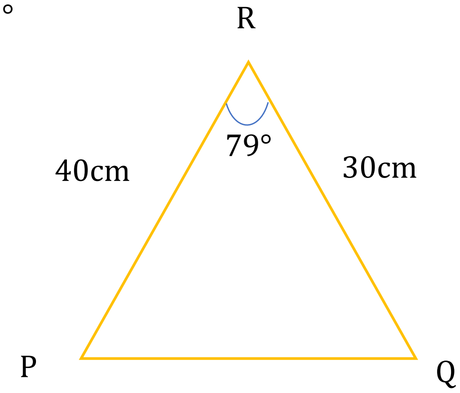
- Area of Triangle
  - The simplest way to find the area of a triangle is by using the following formula:

$$Area = \frac{1}{2} \times base \times perpendicular\ height$$

  - However, if the perpendicular height is not given, we can use the following formula:

$$Area = \frac{1}{2} \times ab \sin C \qquad Area = \frac{1}{2} \times ac \sin B \qquad Area = \frac{1}{2} \times bc \sin A$$

- Example:
  - Find the area of a triangle PQR if given $p = 30\ cm,\ q = 40\ cm\ and\ \angle R = 79°$



$$Area = \frac{1}{2}pq \sin R$$

$$= \frac{1}{2}(40)(30)\sin(79°)$$

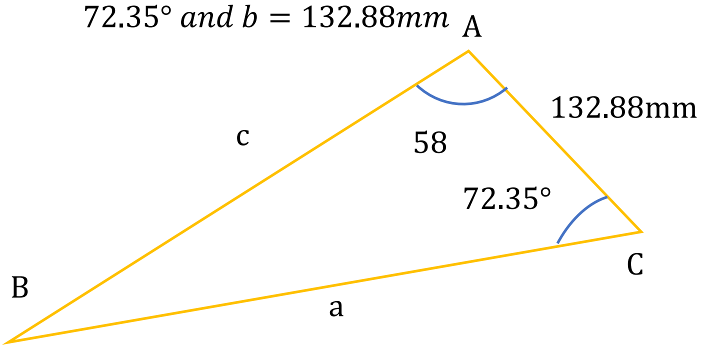
$$= 588.98cm^2$$

- Exercise
  - Find the area of the following triangle:

a) $\triangle XYZ; XY = 180cm,\ YZ = 145cm, \angle Y = 70°$	b) $\triangle ABC; AB = 75cm,\ AC = 66cm, \angle A = 62°$
c) $\triangle PQR; QR = 69cm,\ PR = 49cm, \angle R = 85°$	d) $\triangle DEF; EF = 30cm,\ DF = 40cm, \angle F = 49°$

2.3.3 Solve simple trigonometric problems using sine and cosine rules

- Example
    a. Find the values of a, c and B for triangle if $\angle A = 58°,\ \angle C = 72.35°\ and\ b = 132.88mm$



$$\angle B = 180° - 58° - 72.35° = \mathbf{49.65°}$$

$$According\ to\ the\ rule:$$

$$\frac{a}{Sin\ A} = \frac{b}{Sin\ B} = \frac{c}{Sin\ C}$$

$$\frac{a}{Sin\ 58°} = \frac{132.88}{Sin\ 49.65°} = \frac{c}{Sin\ 72.35°}$$

$$Solve\ for\ a$$

$$\frac{a}{Sin\ A} = \frac{b}{Sin\ B}$$

$$a = \frac{132.88 \times \sin 58°}{Sin\ 49.65}$$

$$= \mathbf{147.86mm}$$

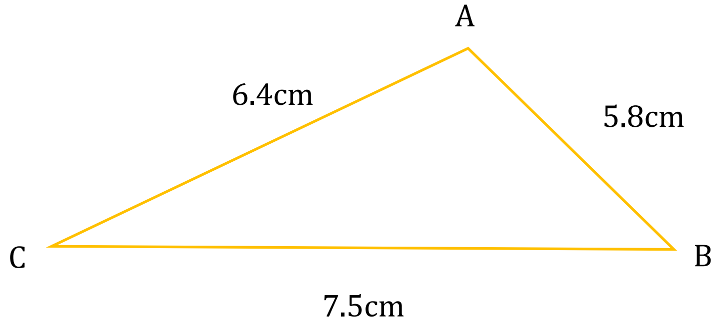
$$Solve\ for\ c$$

$$\frac{c}{Sin\ 72.35°} = \frac{132.88}{Sin\ 49.65}$$

$$c = \frac{132.88 \times \sin 72.35}{Sin\ 49.65}$$

$$= \mathbf{166.15mm}$$

b. Find all the angles A, B and C of the triangle ABC if a=7.5cm, b=6.4cm and c=5.8cm.



$$From\ the\ Cosine\ rule:$$

$$a^2 = b^2 + c^2 - 2bc\cos A,\ thus$$

$$\cos A = \frac{b^2 + c^2 - a^2}{2bc}$$

$$A = cos^{-1}(0.2472) = \mathbf{75.7°}$$

$$Solve\ for\ B$$

$$\angle B = 180° - 48.5° - 75.7°$$

$$= \mathbf{55.8°}$$

$$From\ the\ Sine\ rule:$$

$$\frac{a}{Sin\ A} = \frac{b}{Sin\ B} = \frac{c}{Sin\ C}$$

$$\frac{7.5}{Sin\ A} = \frac{6.4}{Sin\ B} = \frac{5.8}{Sin\ C}$$

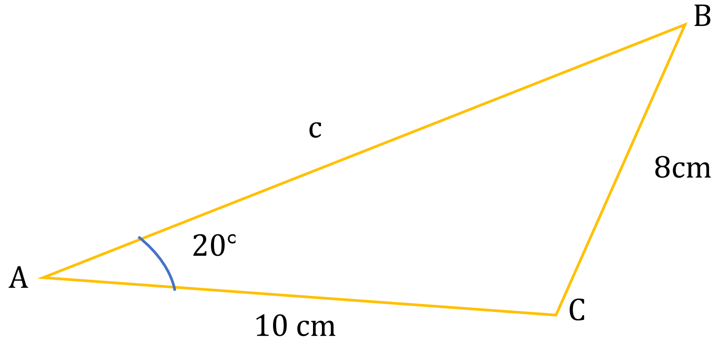
$$Solve\ for\ C$$

$$\frac{7.5}{\sin 75.7°} = \frac{5.8}{Sin\ C}$$

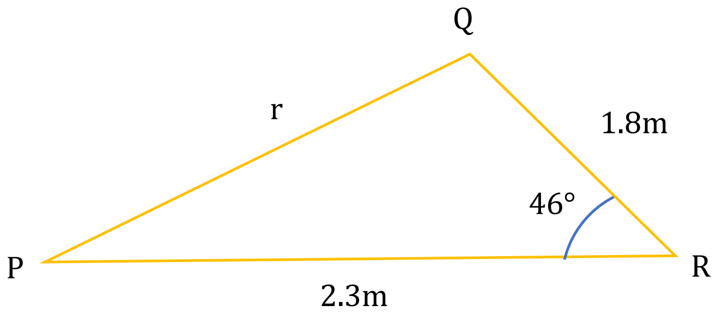
$$Sin\ C = \frac{5.8 \times \sin 75.7°}{7.5}$$

$$C = sin^{-1}(0.7494) = \mathbf{48.5°}$$

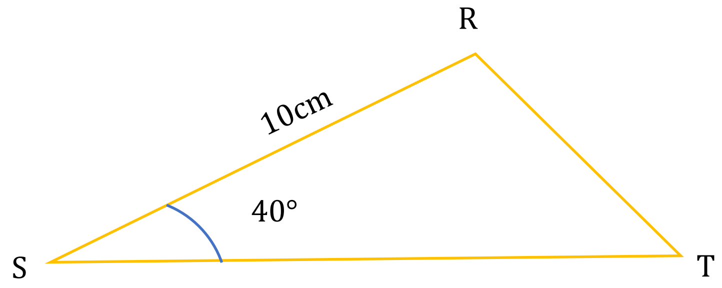
- Exercise
  a. Find the angles B and C and the side c for the triangle ABC with $\angle A = 20°,\ b = 10cm\ and\ a = 8cm$



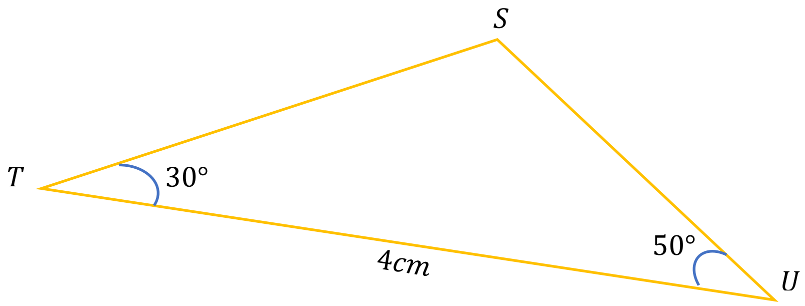
  b. Given the triangle PQR with p=1.8m, q=2.3m and $\angle R = 46°,\ find\ the\ remaining\ values\ of\ r,\ P\ and\ Q.$



c. Given a triangle RST with RS=10 cm and RT are double of RS, and angle $S = 40°$. Calculate value of ST, angle R and angle T, and then after that calculate the area of triangle.



d. Given a STU triangle with TU=4cm, $\angle T = 30° \; and \; \angle U = 50°$. Calculate the values of
   i. length of ST,
   ii. length of SU and
   iii. area of triangle.



Quiz Activity

SCAN ME

https://forms.gle/mgfLCmKyuWkSTSF97

3

CHAPTER 3

COMPLEX NUMBER

CHAPTER 3: COMPLEX NUMBER

3.1 Explain the concept of a complex number.

3.2 Demonstrate the operation of complex number.

3.3 Demonstrate graphical representation of a complex number through Argand Diagram.

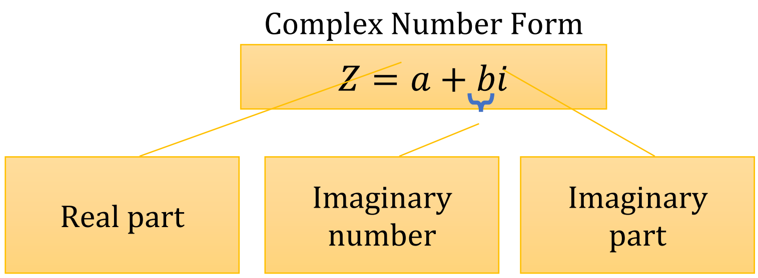
3.4 Write complex number in other form.

3.1 Explain the concept of a complex number.

What is Complex Number?

A complex number a helpful in finding the square root of negative numbers. It have applications in much scientific research, signal-processing electromagnetism, and fluid dynamics, quantum mechanics and vibration analysis.

A complex number is the sum of a real number and an imaginary number.



Imaginary numbers produce negative real numbers when squared.

Example:

$$x^2 = -36$$

$$x = \sqrt{-36} = ???$$

The imaginary unit named i is the square root of -1.

$$i^2 = -1 \qquad i = \sqrt{-1}$$

Therefore, we can solve things that need the square root of a negative number by simply using the i.

$$\sqrt{-r} = \sqrt{r} \times \sqrt{-1}$$

$$= \sqrt{r} \times i$$

$$= \sqrt{r}i$$

Examples:

1.	$\sqrt{-4}$	2.	$\sqrt{-9}$
3.	$\sqrt{-36}$	4.	$\sqrt{-64}$
5.	$\sqrt{-121}$	6.	$\sqrt{-9}\sqrt{-36}$

The first four powers of i establish an important pattern and should be memorized.

$$i^1 = i \quad i^2 = -1 \quad i^3 = -i \quad i^4 = 1$$

Tips:

$$(-1)^{odd\ number} = -1$$
$$(-1)^{even\ number} = 1$$
$$*\ If\ power\ is\ even\ number,\ divide\ with\ 2$$

$$*\ If\ power\ is\ odd\ number:$$
$$a.\ Substract\ power\ of\ i\ with\ 1$$
$$b.\ Then\ divide\ with\ 2$$

Examples:

1.	i^3	2.	i^4
3.	i^6	4.	i^8
5.	i^{15}	6.	i^{90}
7.	i^{91}	8.	$i^4 - i^{10}$
9.	$i^6 - i^{13}$	10.	$2i^3 \times 4i^8$

Simplify each of the following:

1.	$\sqrt{-49} + 5$	2.	$9 - \sqrt{-5}$
3.	$\sqrt{36} + \sqrt{-25}$	4.	$i^8 - 2i^3$
5.	$-6i^3 - 2i^4$	6.	$i + 5i^7$

7.	$\sqrt{-9} \times \sqrt{-36}$	8.	$\sqrt{(-9)(-25)}$
9.	$\sqrt{-25} \times 2i^3$	10.	$\sqrt{-64} \times 3i^6$

The form of complex number.

$$Z = a + bi$$

Complex Number	Real Part	Imaginary Part
$1 + i$	1	1
$2 + 3i$	2	3
$-2 + 7i$	-2	7
$5 - 7i$	5	-7
$3i - 9$	-9	3

3.2 Operation of a complex number.

3.2.1 Addition and Subtraction

$$a + bi + c + di = (a + c) + (b + d)i$$

1.	$\mathbf{1 + 2}\boldsymbol{i}\mathbf{ + 3 + 5}\boldsymbol{i}$	2.	$\mathbf{3 + 8}\boldsymbol{i}\mathbf{ - 3}\boldsymbol{i}$
3.	$\mathbf{6}\boldsymbol{i}\mathbf{ + 5 - }\boldsymbol{i}\mathbf{ + 8}\boldsymbol{i}$	4.	$\mathbf{-4}\boldsymbol{i}\mathbf{ - 5 - }\boldsymbol{i}\mathbf{ + 8}\boldsymbol{i}$
5.	$\mathbf{10 - 5}\boldsymbol{i}\mathbf{ - 12 + 3}\boldsymbol{i}\mathbf{ - }\boldsymbol{i}$	6.	$\mathbf{20 - 5}\boldsymbol{i}\mathbf{ - (4}\boldsymbol{i}\mathbf{ + 3)}$

3.2.2 Multiplication

$$\begin{aligned}(\boldsymbol{a} + \boldsymbol{bi})(\boldsymbol{c} + \boldsymbol{di}) &= ac + adi + bci + bdi^2 \\ &= ac + adi + bci + bd(-1) \\ &= (ac - bd) + (ad + bc)i\end{aligned}$$

1.	$\mathbf{(2 + 3}\boldsymbol{i}\mathbf{)(4 + 5}\boldsymbol{i}\mathbf{)}$	2.	$\mathbf{(6 + 7}\boldsymbol{i}\mathbf{)(3 - 2}\boldsymbol{i}\mathbf{)}$
3.	$\mathbf{(-4 + 2}\boldsymbol{i}\mathbf{)(5 - 8}\boldsymbol{i}\mathbf{)}$	4.	$\mathbf{(2 + 3}\boldsymbol{i}\mathbf{)(2 + 7}\boldsymbol{i}\mathbf{)}$

3.2.3 Division

To divide complex numbers, we write the division as a fraction, then multiply the top and the bottom of the fraction by the conjugate of the denominator.

$$\text{The conjugate of } z = c + di \text{ is } \bar{z} = c - di$$
$$\text{The conjugate of } z = c - di \text{ is } \bar{z} = c + di$$

$$\frac{a + bi}{c + di} = \frac{a + bi}{c + di} \times \frac{c - di}{c - di}$$

Example:

$$\frac{(2 + 3i)}{(4 + 5i)}$$

$$= \frac{2 + 3i}{4 + 5i} \times \frac{4 - 5i}{4 - 5i}$$

$$= \frac{8 - 10i + 12i - 15i^2}{16 - 20i + 20i - 25i^2}$$

$$= \frac{8 + 2i - 15i^2}{16 - 25i^2}$$

$$\frac{8 + 2i - 15(-1)}{16 - 25(-1)}$$

$$= \frac{8 + 2i + 15}{16 + 25}$$

$$= \frac{23 + 2i}{41} = \frac{23}{41} + \frac{2}{41}i$$

Complex number form : $a + bi$

1.	$\frac{(5+2i)}{(2-3i)}$	2.	$\frac{(2-3i)}{(4-5i)}$
3.	$\frac{(5+4i)}{(6-2i)}$	4.	$\frac{(10+3i)}{(4-3i)}$

3.2.4 Equality of Complex Number

Two complex numbers are equal if their real parts are the same and their imaginary parts are the same. The formal definition of 2 complex numbers being equal is: If and only if $a = x$ and $b = y$, two complex number $a + bi$ and $x + yi$ are equivalent, where a, b, x and y are real numbers.

$$a + bi = c + di$$

$$\therefore a = c,\ bi = di\ (based\ on\ group\ type)$$

$$If\ Z_1 = 7 - 10i\ and\ Z_2 = x - yi\ are\ equal,\ find\ the\ value\ of\ x\ and\ y$$

$$Z_1 = Z_2$$

$$7 - 10i = x - yi$$

$$\therefore 7 = x,\ 10 = y$$

$$If\ Z_1 = m + ni\ and\ Z_2 = 4(3 + i)\ are\ equal,\ find\ the\ value\ of\ m\ and\ n$$

$$Z_1 = Z_2$$

$$m + ni = 4(3 + i)$$

$$m + ni = 12 + 4i$$

$$\therefore m = 12,\ n = 4$$

1.	$If\ x + yi = (3 + i)(4 + 5i),$ $find\ the\ value\ of\ x\ and\ y$	2.	$If\ 3a + 2bi = 4(5i + 6)$ $find\ the\ value\ of\ a\ and\ b$

Exercise: Addition and Subtraction

1.	$(5+3i)+(7+5i)$	**2.**	$(5-2i)-(4-5i)$
3.	$(6-2i)+(5+4i)$	**4.**	$(4+3i)-(2-i)$
5.	$(2+3i)+(3-2i)$	**6.**	$(4-3i)-(2+2i)$
7.	$(5+2i)+(2-3i)$	**8.**	$(10+2i)-(4-3i)$
9.	$(13-2i)-(5i)$	**10.**	$(14-i)+(3i)$
11.	$If\ z=(4+2i).\ w=(2-4i),\ find$ $a.\ z+2w$ $b.\ 4w-5z$		

Exercise: Multiplication

1.	$(3+3i)(2+5i)$	**2.**	$(2+3i)(2-7i)$
3.	$(9-2i)(3-4i)$	**4.**	$(5+4i)(6-2i)$
5.	$(2+3i)(3-2i)$	**6.**	$(4-3i)(2+2i)$
7.	$(5+2i)(2-3i)$	**8.**	$(10+2i)(4-3i)$
9.	$(13-2i)(5i)$	**10.**	$(14-i)(3i)$
11.	$If\ z=(4+5i).\ w=(6-4i), find$ $a.\ z\times 2w$ $\qquad$ $b.\ 2w\times w$		

Exercise: Division

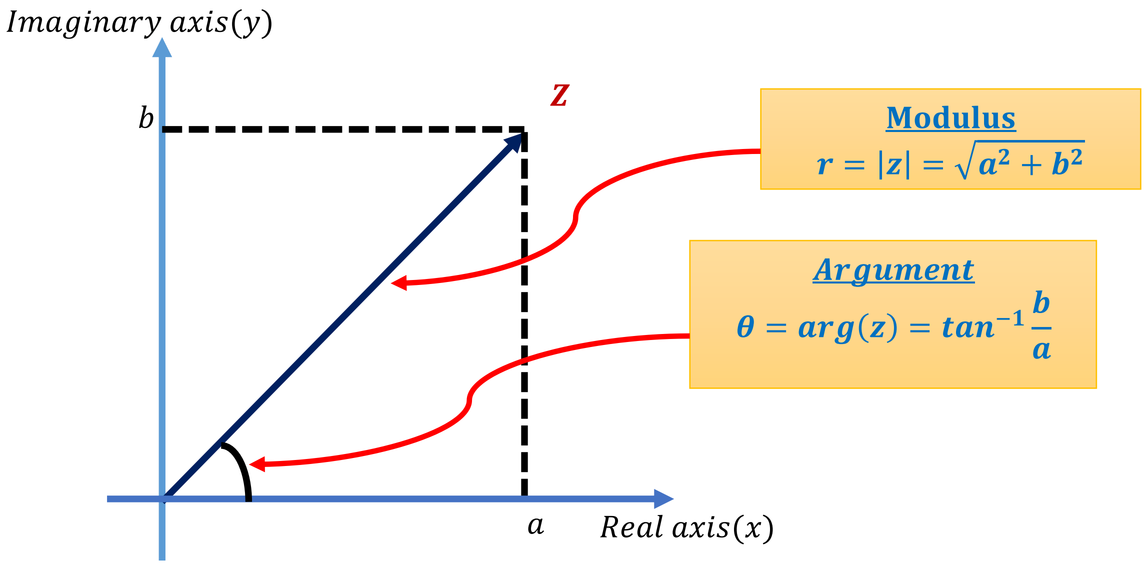
1.	$(4+2i) \div (2+5i)$	**2.**	$(5-2i) \div (3-4i)$
3.	$(9-2i) \div (3-4i)$	**4.**	$(5+4i) \div (6-2i)$
5.	$(2+3i) \div (3-2i)$	**6.**	$(4-3i) \div (2+2i)$

7.	$(5+2i)\div(2-3i)$	8.	$(10+2i)\div(4-3i)$
9.	$(13-2i)\div(5i)$	10.	$(14-i)\div(3i)$
11.	$If\ z=(4+5i).\ w=(6-4i), find$ $a.\ z\div 2w$ $b.\ 2z\div w$ $b.\ 2\bar{z}\div w$		

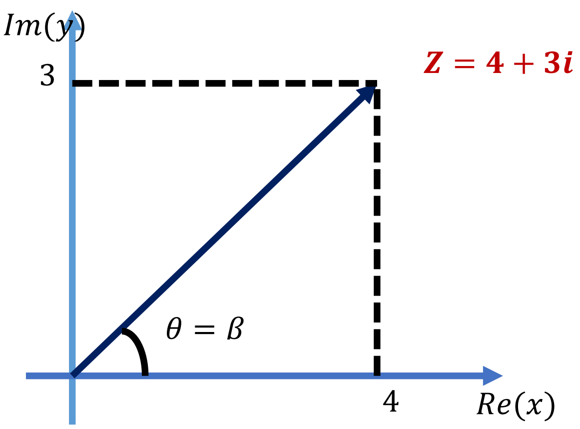
Exercise: Equality of complex number.

1.	$x + yi + 5x - 9i = 6 - 10i$	**2.**	$(2x - yi) + (y - xi) = 1 - 5i$
3.	$(2x - 1) + (x + y)i = (y - 6) + (2y - 4)i$		
4.	$7a + i(3a - b) = 14 - 6i$	**5.**	$(2 + 3i)x + y = 7 + 6i$

3.3 Graphical Representation of a Complex Number through Argand Diagram.



Represent $z = 4 + 3i$ in an Argand Diagram and find its modulus and argument.



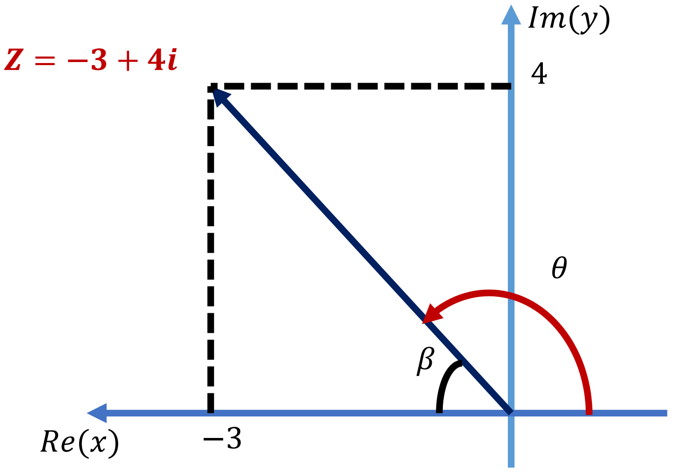
Modulus:

$$r = |Z| = \sqrt{a^2 + b^2}$$
$$= \sqrt{4^2 + 3^2}$$
$$= \sqrt{16 + 9}$$
$$= \sqrt{25}$$
$$= \mathbf{5}$$

Argument:

$$\arg(z) = \theta = tan^{-1}\frac{b}{a}$$
$$= tan^{-1}\frac{3}{4}$$
$$= \mathbf{36.87^\circ}$$

Represent $z = -3 + 4i$ in an Argand Diagram and find its modulus and argument.



Modulus:

$$r = |Z| = \sqrt{a^2 + b^2}$$
$$= \sqrt{(-3)^2 + 4^2}$$
$$= \sqrt{9 + 16}$$
$$= \sqrt{25}$$
$$= \mathbf{5}$$

Argument:

$$\arg(z) = \beta = tan^{-1}\frac{b}{a}$$
$$= tan^{-1}\frac{4}{3}$$
$$= \mathbf{53.13°}$$
$$\theta = (180° - 53.13°)$$
$$= \mathbf{126.87°}$$

Represent $z = -6 - 5i$ in an Argand Diagram and find its modulus and argument.

Represent $\boldsymbol{z = 4 - 7i}$ in an Argand Diagram and find its modulus and argument.

Given $\boldsymbol{z = 5 + 3i}$ and $\boldsymbol{Z_2 = 4 - 6i}$, represent each of the following in an Argand Diagram and find its modulus and argument.

a. $Z_1 + Z_2$ b. $Z_1 - Z_2$

Represent $\boldsymbol{z = 5 + 3i}$ and $\boldsymbol{Z_2 = 4 - 6i}$ in an Argand Diagram and find its modulus and argument.

a. $Z_1 Z_2$ b. $\frac{Z_1}{Z_2}$

3.4 Complex Number in Other Form.

Cartesian Form/Rectangular Form

$Z = a + bi$

Polar Form

$Z = |z|\angle\theta$

@

$Z = r\angle\theta$

Trigonometric Form

$Z = |z|(\cos\theta + i\sin\theta)$

@

$Z = r(\cos\theta + i\sin\theta)$

Exponential Form

$Z = |z|e^{\theta i}(\theta \text{ in radius})$

@

$Z = re^{\theta i}$

Example:

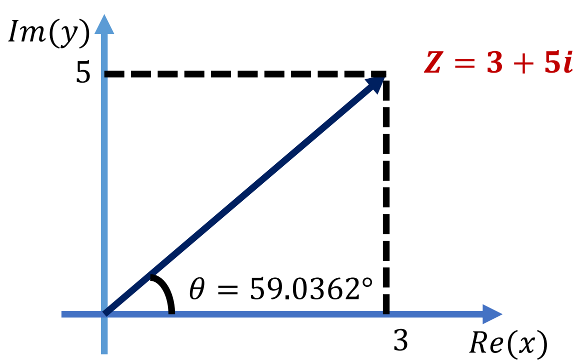
Convert $z = 3 + 5i$ into trigonometric form, polar form and exponential form.

Step 1: Find the values of modulus and argument.

$r = \sqrt{3^2 + 5^2}$
$= \sqrt{9 + 25}$
$= \sqrt{34}$

$\theta = \tan^{-1}\frac{5}{3}$
$= 59.0362°$

Step 2: Plot the complex number in Argand Diagram.



<u>Step 3:</u> Substitute the values of r and θ into the formula.

<u>Trigonometric Form</u>

$$z = r(\cos\theta + i\sin\theta)$$
$$= \sqrt{34}(\cos 59.0362° + i\sin 59.0362°)$$

<u>Polar Form</u>

$$z = r\angle\theta$$
$$= \sqrt{34}\angle 59.0362°$$

<u>Exponential Form</u>

$$\theta_0 = 59.0362 \times \frac{\pi}{180°} = 1.0304\, rad$$
$$z = re^{\theta i}$$
$$= \sqrt{34}e^{1.0304i}$$

Must change to radian

Exercise:

Convert $z = 4 + 6i$ into trigonometric form, polar form and exponential form

Convert $z = 5\angle 48°$ into trigonometric form, Cartesian form and exponential form.

Exercise 3:

1. Given that $R = 5 - 10i \text{ and } S = -8 + 2i$. Calculate the modulus and the argument. Then sketch the Argand Diagram for R+S. (8 marks)

2. Solve the following expression in an exponential form. (6 marks)

$$\frac{10(\cos 200° + i\sin 200°) \times 6(\cos 10° + i\sin 10°)}{20(\cos 70° + i\sin 70°)}$$

3. Given that $Z_1 = 10(\cos 12° + i\sin 12°)$ and $Z_2 = 20\angle 125°$. Solve $\frac{Z_2}{Z_1}$ in trigonometric form. (4 marks)

Quiz Activity

SCAN ME

https://forms.gle/kyRLGKkRavduENpG7

4

CHAPTER 4

MATRIX

CHAPTER 4: MATRICES

4.1 Construct Matrix.

4.2 Demonstrate the operation of matrices.

4.3 Demonstrate simultaneous linear equations.

4.1 Construct Matrix.

Characteristics of matrix.

- A matrix is a rectangular array of numbers.

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \dots & a_{1n} \\ a_{21} & a_{22} & a_{23} \dots & a_{2n} \\ a_{31} & a_{32} & a_{33} \dots & a_{3n} \end{bmatrix}$$

row

column

Number in any matrix called as elements and it will be arranged following by row and column.

- The **size** of the matrix is called its order, and it is denoted by rows and columns.

$$\begin{pmatrix} 6 & 7 \\ 4 & 5 \\ -9 & 3 \end{pmatrix}$$

3 x 2 matrix

$$\begin{pmatrix} 6 \\ 7 \\ 4 \end{pmatrix}$$

3 x 1 matrix

$$\begin{pmatrix} 2 & -9 & 5 \\ 0 & 3 & 6 \end{pmatrix}$$

2 x 3 matrix

$$\begin{pmatrix} 4 & 5 & 3 \end{pmatrix}$$

1 x 3 matrix

- Identity of a matrix.
  - The matrix I is called and identity matrix because IA=A and AI=A for all matrices. This is similar to the number 1, which is called the multiplication identity, because Ia=a and aI=a for all real numbers a.
  - There is no matrix that works as an identity for matrices of all dimensions. For NxN square matrices there is a matrix IN x N that works as an identity.
  - Identity matrix (I) is also a square matrix where all the main diagonal entries are 1 and all the other entries are zero

$$\begin{pmatrix}1 & 0\\0 & 1\end{pmatrix} \qquad \begin{pmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & 0 & 1\end{pmatrix}$$

- The transpose of a matrix is a new matrix whose rows are the columns of the original (which makes it columns the rows of the original). Here is a matrix and it transpose.

$$A = \begin{pmatrix}1 & 3 & 9\\4 & 6 & 5\\5 & 8 & 3\end{pmatrix}$$

Transpose of matrix A, A^T

$$A^T = \begin{pmatrix}1 & 4 & 5\\3 & 6 & 8\\9 & 5 & 3\end{pmatrix}$$

Exercise:

1.	Find the order/size of each matrix:	
	a.	$\begin{pmatrix}5 & -4\end{pmatrix}$
	b,	$\begin{pmatrix}1\\4\\6\end{pmatrix}$
	c.	$\begin{pmatrix}4 & -2\\5 & 0\\-9 & 12\end{pmatrix}$
	d.	$\begin{pmatrix}5 & 67 & 90\\43 & 5 & 7\\76 & 5 & 7\end{pmatrix}$
	e.	$\begin{pmatrix}5 & 6 & 4\\4 & 32 & 76\end{pmatrix}$

2.	Given that $A = \begin{pmatrix} 5 & 1 & 3 \\ 2 & 0 & 7 \end{pmatrix}, B = \begin{pmatrix} 9 & 4 \\ -1 & 3 \\ 1 & 0 \end{pmatrix}, C = \begin{pmatrix} 5 & 1 & 3 \\ 2 & 0 & 11 \\ 7 & 1 & 2 \end{pmatrix}$ and that $D = \begin{pmatrix} 2 & 0 & 5 \end{pmatrix}$, find:	
a.	$A_{23}, B_{31}, C_{12}, D_{13}$	
b.	A^T	
c.	B^T	
d.	C^T	
e.	D^T	

4.2 Operation of a matrix.

4.2.1 Addition

- If two matrices have the same number of rows and columns (same size), then the matrix sum can be computed.
- Example:

$$Let\ A = \begin{pmatrix} 1 & 3 & 9 \\ 4 & 6 & 5 \\ 5 & 8 & 3 \end{pmatrix}\ and\ B = \begin{pmatrix} 7 & 8 & 9 \\ 7 & 5 & 4 \\ 1 & 2 & 5 \end{pmatrix}$$

$$Addition\ A + B = \begin{pmatrix} 1 & 3 & 9 \\ 4 & 6 & 5 \\ 5 & 8 & 3 \end{pmatrix} + \begin{pmatrix} 7 & 8 & 9 \\ 7 & 5 & 4 \\ 1 & 2 & 5 \end{pmatrix} = \begin{pmatrix} 8 & 11 & 18 \\ 11 & 11 & 9 \\ 6 & 10 & 8 \end{pmatrix}$$

4.2.2 Subtraction

- If two matrices have the same number of rows and columns (same size), then the matrix can be subtracted.
- Example:

$$Let\ A = \begin{pmatrix} 1 & 3 & 9 \\ 4 & 6 & 5 \\ 5 & 8 & 3 \end{pmatrix}\ and\ B = \begin{pmatrix} 7 & 8 & 9 \\ 7 & 5 & 4 \\ 1 & 2 & 5 \end{pmatrix}$$

$$Addition\ A - B = \begin{pmatrix} 1 & 3 & 9 \\ 4 & 6 & 5 \\ 5 & 8 & 3 \end{pmatrix} - \begin{pmatrix} 7 & 8 & 9 \\ 7 & 5 & 4 \\ 1 & 2 & 5 \end{pmatrix} = \begin{pmatrix} -6 & -5 & 0 \\ -3 & 1 & 1 \\ 4 & 6 & -2 \end{pmatrix}$$

4.2.3 Multiplication

- How to multiply two given matrices? To multiply one matrix with another, we need to check first, if the number of columns of the first matrix is equal to the number of rows of the second matrix. Now multiply each element of the columns of the first matrix with each element of rows of the second matric and add them all.
- Example 1:

$$Let\ A = \begin{pmatrix} 1 & 3 & 9 \\ 4 & 6 & 5 \\ 5 & 8 & 3 \end{pmatrix} and\ B = \begin{pmatrix} 7 & 8 & 9 \\ 7 & 5 & 4 \\ 1 & 2 & 5 \end{pmatrix}$$

$$A \times B$$

$$= \begin{pmatrix} (1\times7)+(3\times7)+(9\times1) & (1\times8)+(3\times5)+(9\times2) & (1\times9)+(3\times4)+(9\times5) \\ (4\times7)+(6\times7)+(5\times1) & (4\times8)+(6\times5)+(5\times2) & (4\times9)+(6\times4)+(5\times5) \\ (5\times7)+(8\times7)+(3\times1) & (5\times8)+(8\times5)+(3\times2) & (5\times9)+(8\times4)+(3\times5) \end{pmatrix}$$

$$= \begin{pmatrix} 7+21+9 & 8+15+18 & 9+12+45 \\ 28+42+5 & 32+30+10 & 36+24+25 \\ 35+56+3 & 40+40+6 & 45+32+15 \end{pmatrix}$$

$$= \begin{pmatrix} 37 & 41 & 66 \\ 75 & 72 & 85 \\ 94 & 86 & 92 \end{pmatrix}$$

- Example 2:

$$Let\ A = \begin{pmatrix} 1 & 3 & 9 \\ 4 & 6 & 5 \end{pmatrix}\ and\ \ B = \begin{pmatrix} 8 \\ 5 \\ 2 \end{pmatrix}$$

$$A \times B$$

$$= \begin{pmatrix} (1 \times 8) + (3 \times 5) + (9 \times 2) \\ (4 \times 8) + (6 \times 5) + (5 \times 2) \end{pmatrix}$$

$$= \begin{pmatrix} 8 + 15 + 18 \\ 32 + 30 + 10 \end{pmatrix}$$

$$= \begin{pmatrix} 41 \\ 72 \end{pmatrix}$$

Exercise:

$$Let\ A = \begin{pmatrix} 6 & 8 \\ 5 & 7 \\ 8 & 3 \end{pmatrix}\ and\ B = \begin{pmatrix} 2 & 3 \\ 6 & 7 \end{pmatrix}$$

$\boldsymbol{Find\ A \times B}$
$\boldsymbol{Find\ B \times A}$

- Example 3: Scalar Multiplication

$$Let\ A = \begin{pmatrix} 1 & 3 & 9 \\ 4 & 6 & 5 \\ 5 & 8 & 3 \end{pmatrix}$$

$$y \times A$$

$$= y \times \begin{pmatrix} 1 & 3 & 9 \\ 4 & 6 & 5 \\ 5 & 8 & 3 \end{pmatrix}$$

$$= \begin{pmatrix} 1y & 3y & 9y \\ 4y & 6y & 5y \\ 5y & 8y & 3y \end{pmatrix}$$

Exercise:

$$Let\ y = 5, find\ 5A$$

4.2.4 Determinant of Matrix

- The determinant of a matrix is a special number that can be calculated from the matrix. It tells us things about the matrixes that are useful in system of linear equations, in calculus and more.
- The symbol for determinant is two vertical lines either side.

  |A| means the determinant of the matrix A
- 2 method to find determinant of matrix
  - Cofactor expansion
  - Diagonal method

Matrix:2 x 2 (Cofactor Method)

- If $A = \begin{pmatrix} a & c \\ b & d \end{pmatrix}, so\ the\ determinant\ of\ A$:

  $|A| = \begin{pmatrix} a & c \\ b & d \end{pmatrix} = ad - bc$

  Example:

  Given matrix $A = \begin{pmatrix} 5 & 3 \\ 7 & -4 \end{pmatrix}, find\ the\ determinant\ for\ A$

  $|A| = \begin{pmatrix} 5 & 3 \\ 7 & -4 \end{pmatrix} = (5 \times -4) - (7 \times 3)$

  $= -20 - 21 = -41$

Matrix: 3x 3(Cofactor Method)

- Determinant of matrix 3 x 3 are called third-order determinants.
- Consider the matrix A of order 3 x 3:

$$\boldsymbol{A} = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$$

- Expanding along the first row:

$$|\boldsymbol{A}| = a_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

$$= a_{11}[(a_{22})(a_{33}) - (a_{32})(a_{23})] - a_{12}[(a_{21})(a_{33}) - (a_{31})(a_{23})] + a_{13}[(a_{21})(a_{32}) - (a_{31})(a_{22})]$$

Example:

Find the determinant for matrix $A = \begin{pmatrix} 1 & 3 & 2 \\ 4 & 3 & 0 \\ 2 & 1 & 2 \end{pmatrix}$

Determinant of A, $|A| = 1\begin{vmatrix} 3 & 0 \\ 1 & 2 \end{vmatrix} - 3\begin{vmatrix} 4 & 0 \\ 2 & 2 \end{vmatrix} + 2\begin{vmatrix} 4 & 3 \\ 2 & 1 \end{vmatrix}$

$= 1[(3 \times 2) - (1 \times 0)] - 3[(4 \times 2) - (2 \times 0)] + 2[(4 \times 1) - (2 \times 3)]$

$= 1(6 - 0) - 3(8 - 0) + 2(4 - 6)$

$= 1(6) - 3(8) + 2(-2)$

$= 6 - 24 - 4$

$= -22$

Matrix: 3x 3(Diagonal Method)

$$\boldsymbol{A} = \begin{bmatrix} 1 & 3 & 3 \\ 2 & -3 & -2 \\ 3 & 1 & 2 \end{bmatrix}$$

$$|\boldsymbol{A}| = \begin{bmatrix} 1 & 3 & 3 \\ 2 & -3 & -2 \\ 3 & 1 & 2 \end{bmatrix} \begin{matrix} 1 & 3 \\ 2 & -3 \\ 3 & 1 \end{matrix}$$

$$= [\,(1)(-3)(2) + (3)(-2)(3) + (3)(2)(1)\,] - [\,(3)(-3)(3) + (1)(-2)(1) + (3)(2)(2)]$$

$$= [-6 - 18 + 6] - [-27 - 2 + 12]$$

$$= [-18 - (-17)]$$

$$= -18 + 17$$

$$= -1$$

Exercise:

1.	Calculate the following:	
	i.	$3\begin{pmatrix} 3 & 2 \\ -1 & 5 \\ 4 & 9 \end{pmatrix} - 2\begin{pmatrix} 5 & 0 \\ 2 & -3 \\ 8 & 4 \end{pmatrix}$
	ii.	$4\begin{pmatrix} -3 & 0 & 4 \\ 5 & 3 & 2 \\ -2 & 5 & 1 \end{pmatrix} + 5\begin{pmatrix} 2 & 7 & 3 \\ 0 & 3 & 4 \\ 6 & 2 & 6 \end{pmatrix}$

2.	Given matrix $A = \begin{pmatrix} 1 & -4 \\ 7 & 5 \end{pmatrix}$ and $B = \begin{pmatrix} 5 & 6 \\ 8 & -3 \end{pmatrix}$. Find:	
	i.	$(A + B)^T$
	ii.	$(AB)^T$
	iii.	BA + I

3.	Given matrix $A = \begin{pmatrix} 3 & 4 \\ 1 & -2 \end{pmatrix}$, $matric\ B = \begin{pmatrix} 1 & 5 & 3 \\ 4 & 0 & 7 \end{pmatrix}$ $and\ matrix\ C = \begin{pmatrix} 6 & 0 \\ 4 & -1 \\ 3 & 8 \end{pmatrix}$, Find:	
	i.	$(AB)^T$
	ii.	$2B + C^T$
	iii.	$4CB$

4	Find determinant for following matrices:	
	i.	$A = \begin{pmatrix} 3 & 5 \\ 0 & 1 \end{pmatrix}$
	ii.	$P = \begin{pmatrix} 6 & -3 \\ 5 & -1 \end{pmatrix}$
	iii	$Q = \begin{pmatrix} 12 & 7 \\ -2 & 3 \end{pmatrix}$

	iv.	$Z = \begin{pmatrix} 2 & 1 & 3 \\ 5 & 0 & 4 \\ -2 & 1 & 7 \end{pmatrix}$
	v.	$M = \begin{pmatrix} 5 & -1 & 0 \\ 6 & 8 & 11 \\ 4 & -5 & 9 \end{pmatrix}$
	vi	$N = \begin{pmatrix} 12 & 0 & -1 \\ 2 & -5 & 3 \\ 1 & 7 & 4 \end{pmatrix}$

4.3 Simultaneous Linear Equation

4.3.1: Inverse Method

How to solve linear equations by using inverse method? For example, given matrix 3 x 3 below:

$$x + 3y + 3z = 4$$
$$2x - 3y - 2z = 2$$
$$3x + y + 2z = 5$$

Steps:

1. Rewrite in form of $Ax = b$
2. Find the determinant of A.
3. Find the minor of A.
4. Find the cofactor of A.
5. Find the adjoint of A.
6. Find the inverse of A, A^{-1}.
7. Find the 3 variables by using $x = A^{-1}b$

Step 1:

Rewrite in form of Ax = b

$$\begin{bmatrix} 1 & 3 & 3 \\ 2 & -3 & -2 \\ 3 & 1 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4 \\ 2 \\ 5 \end{bmatrix}$$

Step 2:

Find the determinant of matrix A.

- The determinant of a matrix is a special number that can be calculated from the matrix. It tells us things about the matrixes that are useful in system of linear equations, in calculus and more.
- The symbol for determinant is two vertical lines either side.

|A| means the determinant of the matrix A

$$A = \begin{bmatrix} 1 & 3 & 3 \\ 2 & -3 & -2 \\ 3 & 1 & 2 \end{bmatrix}$$

$$|A| = 1\begin{bmatrix} -3 & -2 \\ 1 & 2 \end{bmatrix} - 3\begin{bmatrix} 2 & -2 \\ 3 & 2 \end{bmatrix} + 3\begin{bmatrix} 2 & -3 \\ 3 & 1 \end{bmatrix}$$

$$= 1(-6 - (-2)) - 3(4 - (-6)) + 3(2 - (-9))$$

$$= 1(-6 + 2) - 3(4 + 6) + 3(2 + 9)$$

$$= 1(-4) - 3(10) + 3(11)$$

$$= -4 - 30 + 33$$

$$= \mathbf{-1}$$

<u>Step 3:</u>

Find minor of matrix A. (M_{ij})

- If A is a square matrix, the minor for a_{ij} denoted by M_{ij} by eliminating the i_{th} row and the j_{th} column.

$$If\, A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix},\ therefore,\ Minor\ of\ A = \begin{pmatrix} M_{11} & M_{12} & M_{13} \\ M_{21} & M_{22} & M_{23} \\ M_{31} & M_{32} & M_{33} \end{pmatrix}$$

$$A = \begin{bmatrix} 1 & 3 & 3 \\ 2 & -3 & -2 \\ 3 & 1 & 2 \end{bmatrix},\ then\ the\ Minor\ of\ A\ is\ \dots..$$

$$M_{11} = \begin{bmatrix} 1 & 3 & 3 \\ 2 & -3 & -2 \\ 3 & 1 & 2 \end{bmatrix}$$

$$M_{11} = \begin{vmatrix} -3 & -2 \\ 1 & 2 \end{vmatrix} = (-3)(2) - (-2)(1)$$

$$= -6 + 2 = -4$$

$$M_{12} = \begin{bmatrix} 1 & 3 & 3 \\ 2 & -3 & -2 \\ 3 & 1 & 2 \end{bmatrix}$$

$$M_{12} = \begin{vmatrix} 2 & -2 \\ 3 & 2 \end{vmatrix} = (2)(2) - (-2)(3)$$

$$= 4 + 6 = 10$$

$$M_{13} = \begin{bmatrix} 1 & 3 & 3 \\ 2 & -3 & -2 \\ 3 & 1 & 2 \end{bmatrix}$$

$$M_{13} = \begin{vmatrix} 2 & -3 \\ 3 & 1 \end{vmatrix} = (2)(1) - (-3)(3)$$

$$= 2 + 9 = 11$$

$$M_{21} = \begin{bmatrix} 1 & 3 & 3 \\ 2 & -3 & -2 \\ 3 & 1 & 2 \end{bmatrix}$$

$$M_{21} = \begin{vmatrix} 3 & 3 \\ 1 & 2 \end{vmatrix} = (3)(2) - (3)(1)$$

$$= 6 - 3 = 3$$

$$M_{22} = \begin{bmatrix} 1 & 3 & 3 \\ 2 & -3 & -2 \\ 3 & 1 & 2 \end{bmatrix}$$

$$M_{22} = \begin{vmatrix} 1 & 3 \\ 3 & 2 \end{vmatrix} = (1)(2) - (3)(3)$$

$$= 2 - 9 = -7$$

$$M_{23} = \begin{bmatrix} 1 & 3 & 3 \\ 2 & -3 & -2 \\ 3 & 1 & 2 \end{bmatrix}$$

$$M_{23} = \begin{vmatrix} 1 & 3 \\ 3 & 1 \end{vmatrix} = (1)(1) - (3)(3)$$

$$= 1 - 9 = -8$$

$$M_{31} = \begin{bmatrix} 1 & 3 & 3 \\ 2 & -3 & -2 \\ 3 & 1 & 2 \end{bmatrix}$$

$$M_{31} = \begin{vmatrix} 3 & 3 \\ -3 & -2 \end{vmatrix} = (3)(-2) - (3)(-3)$$

$$= -6 + 9 = 3$$

$$M_{32} = \begin{bmatrix} 1 & 3 & 3 \\ 2 & -3 & -2 \\ 3 & 1 & 2 \end{bmatrix}$$

$$M_{32} = \begin{vmatrix} 1 & 3 \\ 2 & -2 \end{vmatrix} = (1)(-2) - (3)(2)$$

$$= -2 - 6 = -8$$

$$M_{33} = \begin{bmatrix} 1 & 3 & 3 \\ 2 & -3 & -2 \\ 3 & 1 & 2 \end{bmatrix}$$

$$M_{33} = \begin{vmatrix} 1 & 3 \\ 2 & -3 \end{vmatrix} = (1)(-3) - (3)(2) = -3 - 6 = -9$$

$$Therfore\ the\ Minor\ of\ A = \begin{bmatrix} M_{11} & M_{12} & M_{13} \\ M_{21} & M_{22} & M_{23} \\ M_{31} & M_{32} & M_{33} \end{bmatrix}$$

$$\boldsymbol{M_A} = \begin{bmatrix} \mathbf{-4} & \mathbf{10} & \mathbf{11} \\ \mathbf{3} & \mathbf{-7} & \mathbf{-8} \\ \mathbf{3} & \mathbf{-8} & \mathbf{-9} \end{bmatrix}$$

Step 4:

Find cofactor of matrix A, (C_{ij})

The cofactor, C_{ij} for the element a_{ij} is defined as $C_{ij} = (-1)^{i+j} M_{ij}$

$$M_A = \begin{pmatrix} \mathbf{-4} & \mathbf{10} & \mathbf{11} \\ \mathbf{3} & \mathbf{-7} & \mathbf{-8} \\ \mathbf{3} & \mathbf{-8} & \mathbf{-9} \end{pmatrix}, \ therefore,\ Cofactor\ of\ A = \begin{pmatrix} C_{11} & \boldsymbol{C_{12}} & C_{13} \\ \boldsymbol{C_{21}} & C_{22} & \boldsymbol{C_{23}} \\ C_{31} & \boldsymbol{C_{32}} & C_{33} \end{pmatrix}$$

$$C_{12} = (-1)^{1+2}M_{12}$$
$$C_{12} = (-1)^3(10)$$
$$C_{12} = (-1)(10)$$
$$C_{12} = -10$$

$$C_{21} = (-1)^{2+1}M_{21}$$
$$C_{21} = (-1)^3(3)$$
$$C_{21} = (-1)(3)$$
$$C_{21} = -3$$

$$C_{23} = (-1)^{2+3}M_{12}$$
$$C_{23} = (-1)^5(-8)$$
$$C_{23} = (-1)(-8)$$
$$C_{23} = 8$$

$$C_{32} = (-1)^{3+2}M_{12}$$
$$C_{32} = (-1)^5(-8)$$
$$C_{32} = (-1)(-8)$$
$$C_{32} = 8$$

Or we can simply apply the “Checkerboard” on the Minor Matrix

$$\begin{bmatrix} \mathbf{-4} & \mathbf{10} & \mathbf{11} \\ \mathbf{3} & \mathbf{-7} & \mathbf{-8} \\ \mathbf{3} & \mathbf{-8} & \mathbf{-9} \end{bmatrix} \qquad \begin{bmatrix} + & - & + \\ - & + & - \\ + & - & + \end{bmatrix} \qquad \begin{bmatrix} \mathbf{-4} & \mathbf{-10} & \mathbf{11} \\ \mathbf{-3} & \mathbf{-7} & \mathbf{8} \\ \mathbf{3} & \mathbf{8} & \mathbf{-9} \end{bmatrix}$$

Minor Matrix — Checkerboard — Cofactor Matrix

Therefore, the Cofactor of A =

$$\begin{bmatrix} \mathbf{-4} & \mathbf{-10} & \mathbf{11} \\ \mathbf{-3} & \mathbf{-7} & \mathbf{8} \\ \mathbf{3} & \mathbf{8} & \mathbf{-9} \end{bmatrix}$$

<u>Step 5:</u>

Find the Adjoint of matrix A, $\mathrm{A_{ij}} = (\mathrm{C_{ij}})^{\mathrm{T}}$

Adjoint of matrix A is the Transpose of Cofactor of matrix A.

$$Cofactor\ of\ A\ = \begin{bmatrix} \mathbf{-4} & \mathbf{-10} & \mathbf{11} \\ \mathbf{-3} & \mathbf{-7} & \mathbf{8} \\ \mathbf{3} & \mathbf{8} & \mathbf{-9} \end{bmatrix}, \ therefore\ the\ Adjoint\ matrix\ of\ A = \begin{bmatrix} -4 & -3 & 3 \\ -10 & -7 & 8 \\ 11 & 8 & -9 \end{bmatrix}$$

<u>Step 6:</u>

Find the Inverse of A, A^{-1}

$$A^{-1} = \frac{1}{|A|} Adjoint\ A$$

$$A^{-1} = \frac{1}{-1} \begin{bmatrix} -4 & -3 & 3 \\ -10 & -7 & 8 \\ 11 & 8 & -9 \end{bmatrix}$$

$$So,\ Inverse\ A,\ A^{-1} = \begin{bmatrix} 4 & 3 & -3 \\ 10 & 7 & -8 \\ -11 & -8 & 9 \end{bmatrix}$$

Step 7:

Find the value of x, y and z by using $x = A^{-1}b$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4 & 3 & -3 \\ 10 & 7 & -8 \\ -11 & -8 & 9 \end{bmatrix} \begin{bmatrix} 4 \\ 2 \\ 5 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} (4 \times 4) + (3 \times 2) + (-3 \times 5) \\ (10 \times 4) + (7 \times 2) + (-8 \times 5) \\ (-11 \times 4) + (-8 \times 2) + (9 \times 5) \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 16 + 6 - 15 \\ 40 + 14 - 40 \\ -44 - 16 + 45 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 7 \\ 14 \\ -15 \end{bmatrix},$$

$$\boldsymbol{x = 7,\ y = 14\ and\ z = -15}$$

4.3.2: Cramer's Rule

Solve the linear equations by using Cramer's Rule.

$$5x - y + 7z = 4$$
$$6x - 2y + 9z = 5$$
$$2x + 8y - 4z = 8$$

Steps:

1. Rewrite in form of $\mathrm{Ax} = \mathrm{b}$
2. Calculate the determinant of A, $|\mathrm{A}|$
3. Find A_1 by substituting 'b' into column 1 of A. Calculate $|\mathrm{A}_1|$.
4. Find A_2 by substituting 'b' into column 2 of A. Calculate $|\mathrm{A}_2|$.
5. Find A_3 by substituting 'b' into column 3 of A. Calculate $|\mathrm{A}_3|$.
6. Find the 3 variables using the following formula:

$$\mathbf{x} = \frac{|\mathbf{A_1}|}{|\mathbf{A}|},\ \mathbf{y} = \frac{|\mathbf{A_2}|}{|\mathbf{A}|},\ \mathbf{z} = \frac{|\mathbf{A_3}|}{|\mathbf{A}|}$$

<u>Step 1:</u>

Rewrite in form of $\mathrm{Ax} = \mathrm{b}$

$$\begin{bmatrix} 5 & -1 & 7 \\ 6 & -2 & 9 \\ 2 & 8 & -4 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4 \\ 5 \\ 8 \end{bmatrix}$$

<u>Step 2:</u>

Find the determinant of A.

$$A = \begin{bmatrix} 5 & -1 & 7 \\ 6 & -2 & 9 \\ 2 & 8 & -4 \end{bmatrix}$$

$$|A| = 5\begin{vmatrix} -2 & 9 \\ 8 & -4 \end{vmatrix} - (-1)\begin{vmatrix} 6 & 9 \\ 2 & -4 \end{vmatrix} + 7\begin{vmatrix} 6 & -2 \\ 2 & 8 \end{vmatrix}$$

$$|A| = 5(8 - 72) + 1(-24 - 18) + 7(48 + 4)$$

$$|A| = 5(-64) - 42 + 7(52)$$

$$= -320 - 42 + 364$$

$$= 2$$

Step 3:

Find A_1 by substituting 'b' into column 1 of A. Calculate $|A_1|$

$$A_1 = \begin{bmatrix} 4 & -1 & 7 \\ 5 & -2 & 9 \\ 8 & 8 & -4 \end{bmatrix}$$

$$|A_1| = 4\begin{vmatrix} -2 & 9 \\ 8 & -4 \end{vmatrix} - (-1)\begin{vmatrix} 5 & 9 \\ 8 & -4 \end{vmatrix} + 7\begin{vmatrix} 5 & -2 \\ 8 & 8 \end{vmatrix}$$

$$|A_1| = 44$$

Step 4:

Find A_2 by substituting 'b' into column 2 of A. Calculate $|A_2|$

$$A_2 = \begin{bmatrix} 5 & 4 & 7 \\ 6 & 5 & 9 \\ 2 & 8 & -4 \end{bmatrix}$$

$$|A_2| = 5\begin{vmatrix} 5 & 9 \\ 8 & -4 \end{vmatrix} - (4)\begin{vmatrix} 6 & 9 \\ 2 & -4 \end{vmatrix} + 7\begin{vmatrix} 6 & 5 \\ 2 & 8 \end{vmatrix}$$

$$|A_2| = -26$$

Step 5:

Find A_3 by substituting 'b' into column 3 of A. Calculate $|A_3|$

$$A_3 = \begin{bmatrix} 5 & -1 & 4 \\ 6 & -2 & 5 \\ 2 & 8 & 8 \end{bmatrix}$$

$$|A_3| = 5\begin{vmatrix} -2 & 5 \\ 8 & 8 \end{vmatrix} - (-1)\begin{vmatrix} 6 & 5 \\ 2 & 8 \end{vmatrix} + 4\begin{vmatrix} 6 & -2 \\ 2 & 8 \end{vmatrix}$$

$$|A_3| = -34$$

Step 6:

Find the 3 variables using the formula:

$$\boldsymbol{x = \frac{|A_1|}{|A|},\ y = \frac{|A_2|}{|A|},\ z = \frac{|A_3|}{|A|}}$$

$$\boldsymbol{x = \frac{44}{2},\ y = \frac{-26}{2},\ z = \frac{-34}{2}}$$

$$\boldsymbol{x = 22,\ y = -13,\ z = -17}$$

Exercise:

1.	Given that $\begin{pmatrix} 2 & 0 \\ 3 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 8 \\ 2y \end{pmatrix}$, find the values of x and y.	
2.	Solve the simultaneous equation by using Inverse method	
	a.	$2x - 7y = 10$ $6x - y = -12$

b.	$3x - 4y = 1$ $3x + y = 11$
c.	$x + 2y = 4$ $3x - 5y = 1$

3	Solve the simultaneous equation by using determinant method/Cramer's Rule	
a.	$2x - 7y = 10$ $6x - y = -12$	
b.	$3x - 4y = 1$ $3x + y = 11$	

4.	Solve the simultaneous equation by using determinant method/Cramer's Rule.
a.	$x + 2y - 3z = 3$ $2x - y - z = 11$ $3x + 2y + z = -5$

b.	$4x - 5y + 6z = 3$ $8x - 7y - 3z = 9$ $7x - 8y + 9z = 6$

5.	Find numerical value of $x, y\ and\ z$ by using matrix inverse method for the following equations.
a.	$2x + y + 2z = 8$ $4x + 2y - z = 1$ $x - 3y + 3z = -3$

b.	$3x + y + 2z = 3$ $2x - 3y - z = -3$ $z + 2y + x = 4$

c.	$2x + y + z = 8$ $-3y + 5x + 2z = 3$ $y + 7x + 3z = 20$

Quiz Activity

SCAN ME

https://forms.gle/84jFXMvJX5Uv3wYA7

CHAPTER 5

VECTOR AND SCALAR

CHAPTER 5: VECTOR AND SCALAR

5.1 Express Vector

5.2 Demonstrate the operation of vector.

5.3 Apply scalar (dot) product of two vectors.

5.4 Apply vectors (cross) product of two vectors.

5.1 Express Vector

Definition of Vector and Scalar

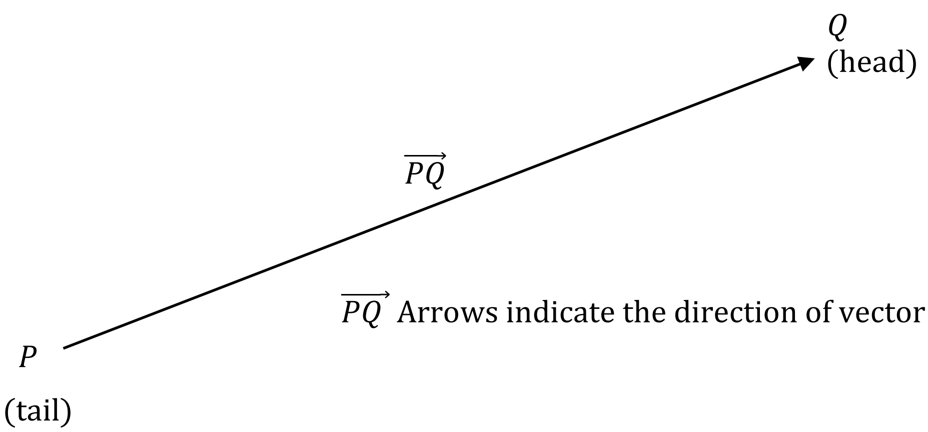
- Vector: is a physical quantity which is a positive real number, has both a magnitude and a direction.
  - Example: displacement, velocity, acceleration, force, etc.
- Scalar: is a quantity which has magnitude (numerical size) only.
  - Example: natural numbers, distance speed, temperature, volume and energy.

Vector Notation

$\overrightarrow{OA}$	$\overrightarrow{AB}$	$\boldsymbol{a}$	$\overrightarrow{a}$
$\begin{bmatrix}4\\5\end{bmatrix}$	$(4 \quad 5)$	$3i - 4j + 5k$	$2i - 3j$

Vector Representation

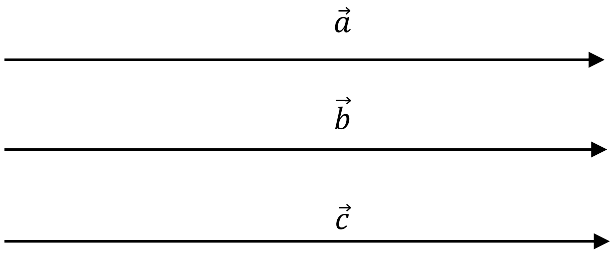
- Vectors are drawn as arrows has both magnitude and directions.
- The starting point of a vector is known as the tail and the end point is knows as the head.



$\overrightarrow{PQ}$ Arrows indicate the direction of vector from P to Q.

Equality of Vector

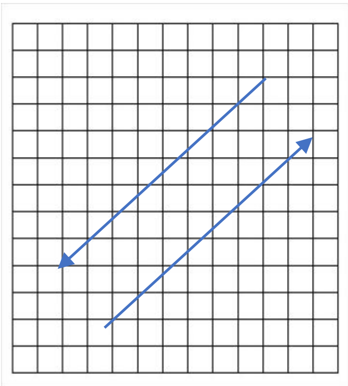
- Vectors are equal when they have same magnitude and the same direction. It may start at different position and do not need the same starting point.
- The equality of vectors (also called equivalent) is expressed in the following way:
    - $\vec{a} = \vec{b} = \vec{c}$ or $\hat{a} = \hat{b} = \hat{c}$ $(\boldsymbol{same\ direction})$



- Moreover, these three vectors also have the same value of magnitude.
    - $\therefore |\vec{a}| = |\vec{b}| = |\vec{c}|$

Negative Vector

- Negative vectors is a vector which have same magnitude but opposite to a vector.

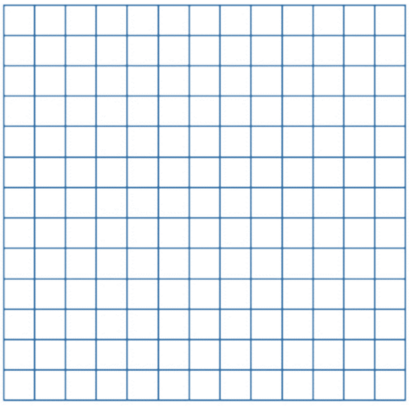


$$\overrightarrow{\boldsymbol{BA}} = -\overrightarrow{\boldsymbol{AB}}$$
$$\overrightarrow{\boldsymbol{DC}} = -\overrightarrow{\boldsymbol{CD}}$$

- If the initial and terminal points of a vector coincide, the vector has length zero, it called zero vectors and denote by 0. Zero vector is when the magnitude is zero, the direction cannot be determine or given.

Exercise:

1. Show that vector $\tilde{a} = -i + 6j$ and vector $\tilde{b} = \frac{1}{2}i - 3j$ are parallel but in opposite direction.



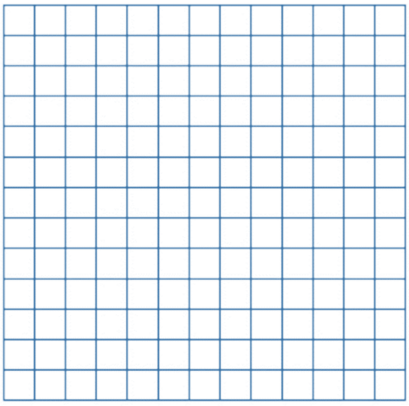
2. State the position vectors for the following points in $xi + yj$:

   i. $P(-3, -7)$

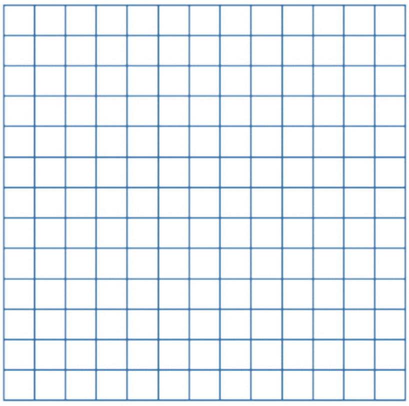
   ii. $R(6, -3, 4)$

3. Sketch the following position vector:

   i. $\vec{R} = -4i + 5j$



   ii. $\vec{Q} = -3i - 9j$



5.2 Operation of vectors.

5.2.1 Magnitude of vector

- For the vector $\overrightarrow{OP}$. The magnitude is the distance between the initial point 0 and the end point of P.
- The symbol for magnitude is $|\overrightarrow{OP}|$Example:

$$(\boldsymbol{Hypotenuse})^2 = (\boldsymbol{Adjacent})^2 + (\boldsymbol{Opposite})^2$$

$$Le\boldsymbol{For\ a\ 2 - dimensional\ vector,\ v = (x, y)}$$
$$|\overrightarrow{\boldsymbol{OP}}| = \sqrt{\boldsymbol{x}^2 + \boldsymbol{y}^2}$$

$$A\boldsymbol{For\ a\ 3 - dimensional\ vector,\ v = (x, y, z)}$$
$$|\overrightarrow{\boldsymbol{OP}}| = \sqrt{\boldsymbol{x}^2 + \boldsymbol{y}^2 + \boldsymbol{z}^2}$$

5.2.2 Unit Vector

- A unit vector is any vector with a length of one. A unit vector is often indicated with a hat in $\hat{u}$.

$$\textbf{Unit Vector, } \hat{\boldsymbol{u}} = \frac{\vec{\boldsymbol{u}}}{|\vec{\boldsymbol{u}}|}$$

$$\textbf{Where } |\vec{\boldsymbol{u}}| = |\overrightarrow{\boldsymbol{OP}}| = \sqrt{\boldsymbol{a}^2 + \boldsymbol{b}^2 + \boldsymbol{c}^2}$$
$$\hat{\boldsymbol{u}} = \frac{\vec{\boldsymbol{u}}}{|\vec{\boldsymbol{u}}|}$$
$$= \frac{\boldsymbol{a}}{|\vec{\boldsymbol{u}}|}\boldsymbol{i} + \frac{\boldsymbol{b}}{|\vec{\boldsymbol{u}}|}\boldsymbol{j} + \frac{\boldsymbol{c}}{|\vec{\boldsymbol{u}}|}\boldsymbol{k}$$

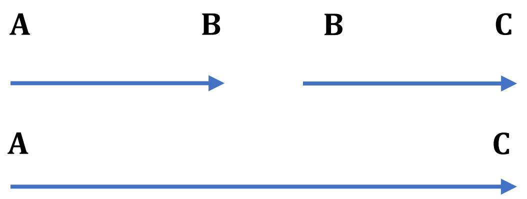
Exercise:

1.	Find the magnitude of vector $\vec{\mathrm{A}} = (-5,2)$
2.	Find the magnitude of vector $\vec{\mathrm{B}} = (5,6,3)$

3.	Given that vector $\overrightarrow{OB} = 3i - 2j + 5k\ and\ \overrightarrow{OC} = i + 2j + 4k$. Find the magnitude for $\overrightarrow{BC}$.
4.	Find the unit vector of $\overrightarrow{RS} = 8i - 2j + k$
5.	Given $\vec{A} = \begin{pmatrix} 3 \\ -2 \\ 5 \end{pmatrix}\ and\ \vec{B} = \begin{pmatrix} 2 \\ 4 \\ 7 \end{pmatrix}$, find unit vector of $\lvert\vec{A} + 3\vec{B}\rvert$

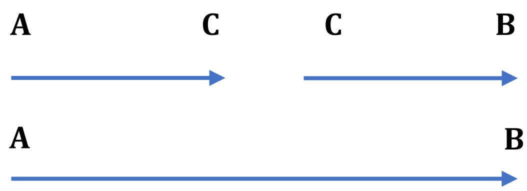
5.2.3 Addition and Subtraction of Vectors

- **Addition of Vector**



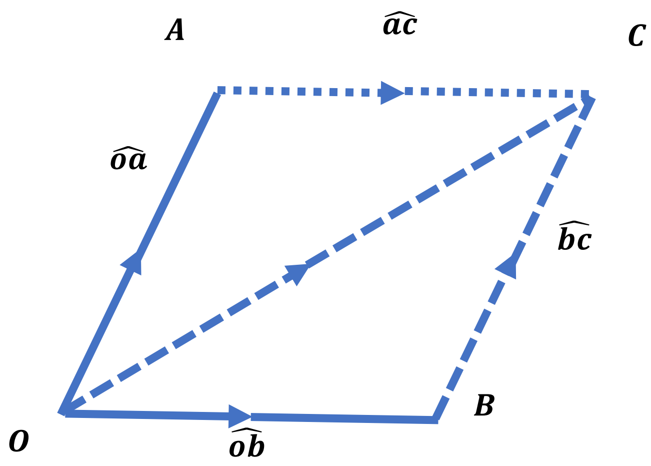
$$\overrightarrow{AB} + \overrightarrow{BC} = \overrightarrow{AC}$$

- **Subtraction of Vector**



$$\overrightarrow{AC} - \overrightarrow{BC} = \overrightarrow{AC} + \overrightarrow{CB} = \overrightarrow{AB}$$

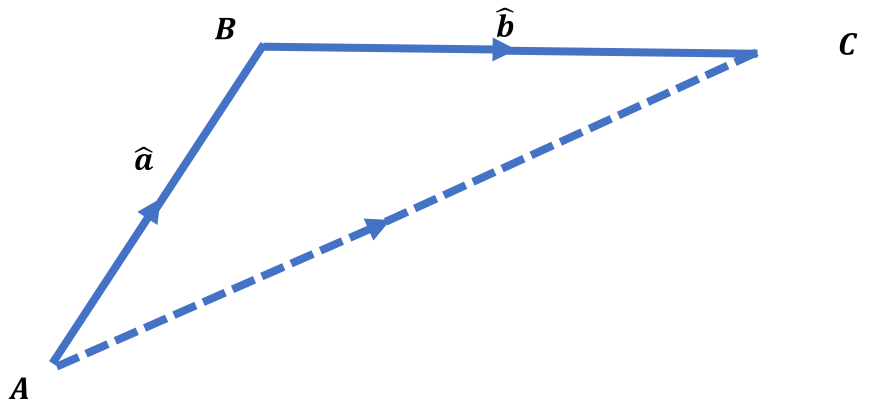
- **Parallelogram Law**



$$\overrightarrow{OA} + \overrightarrow{OB} = \overrightarrow{OC} \text{ or } \overrightarrow{OA} + \overrightarrow{AC} = \overrightarrow{OC}$$

$$\overrightarrow{OA} + \overrightarrow{AC} = \overrightarrow{OC} \text{ or } \overrightarrow{OA} - \overrightarrow{CA} = \overrightarrow{OC}$$

- **Triangle Law**



$$\overrightarrow{AB} + \overrightarrow{BC} = \overrightarrow{AC}$$

$$\overrightarrow{AB} - \overrightarrow{CB} = \overrightarrow{AC} = -\overrightarrow{CA}$$

Exercise

1.	Given vector $\vec{a} = 5i + 6j$ and $\vec{b} = 4i - 2j$. Find $\vec{a} + \vec{b}$. Draw the parallelogram for $\vec{a} + \vec{b}$
2.	Given vector $\vec{a} = 8i + 9j$ and $\vec{b} = 4i - 2j$. Find $\vec{a} - \vec{b}$. Draw the parallelogram for $\vec{a} - \vec{b}$
3.	If $\overrightarrow{OA} = 2i + 4j$ *and* $\overrightarrow{OB} = 5i - 8j$. Sketch vector $\overrightarrow{AB}$ in cartesian plane and find $\overrightarrow{AB}$.

5.2.4 Multiplication of Vector with Scalar

- When scalar is multiplied by vector then multiplication is called scalar by vector. Some basic properties of multiplication like distributive property and associative property:

$$\#\boldsymbol{k.c = kc \text{ where } k \text{ is scalar quantity and } c \text{ is vector quantity}}$$
$$\#\boldsymbol{(ab)c = a(bc)}$$
$$\boldsymbol{\text{where } a,\ b \text{ is scalar quantity and } c \text{ is vector quantity}}$$
$$\#\boldsymbol{a(b+c) = ab + ac}$$

- Example:

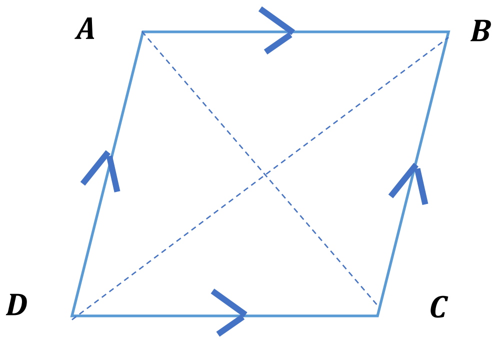
Given $\vec{A} = \begin{pmatrix} 5 \\ -3 \\ 5 \end{pmatrix}$ $and\vec{B} = \begin{pmatrix} 1 \\ 4 \\ 7 \end{pmatrix}$, $find\ \vec{A} + 3\vec{B}$

$\vec{A} + 3\vec{B} = \begin{pmatrix} 5 \\ -3 \\ 5 \end{pmatrix} + 3\begin{pmatrix} 1 \\ 4 \\ 7 \end{pmatrix}$

$= \begin{pmatrix} 5 \\ -3 \\ 5 \end{pmatrix} + \begin{pmatrix} 3 \\ 12 \\ 21 \end{pmatrix} = \begin{pmatrix} 8 \\ 11 \\ 26 \end{pmatrix}$

Exercise:

1.



ABCD is a parallelogram. Find the sum vectors below in unit of vector guide.

$i. \overrightarrow{DA} + \overrightarrow{AC}$ $ii. \overrightarrow{AD} + \overrightarrow{AB}$ $iii. \overrightarrow{AB} + \overrightarrow{CB}$

2. $2(2i + 2j + 6k) + (2i + 3j - 4k)$

3. Given $\vec{a} = -3i - 2j - k$ *and* $\vec{b} = 2i + 4j + 6k$, find $-3\vec{a} + 2\vec{b}$

4. *Given* $\vec{O} = 2i + aj - 2k$ *and* $\vec{P} = i - 2j - 2k$, *find the value of a when* $\vec{O} + \vec{P} = 3i + 4j - 4k$

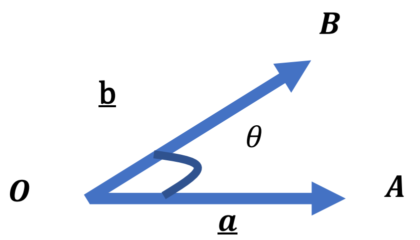
5.3 Scalar (dot) product of two vectors.

5.3.1: Define Scalar Product

If the component of vector $\vec{A} = a_1 i + a_2 j + a_3 k$ and $\vec{B} = b_1 i + b_2 j + b_3 k$ then the dot product (also called the inner product or scalar product) of two vectors is defined as:

$$\vec{A} \bullet \vec{B} = |\vec{A}||\vec{B}| \cos\theta$$

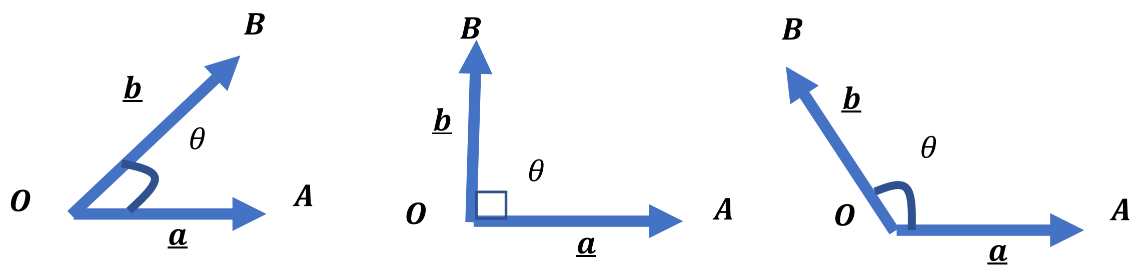
Where $|\vec{A}|$ and $|\vec{B}|$ represent the magnitudes of vectors A and B and θ is the angle between vectors A and B and $0 \leq \theta \leq \pi$.



angle between two vector can be define by

$$\cos\boldsymbol{\theta} = \frac{\vec{A} \cdot \vec{B}}{|\vec{A}||\vec{B}|}$$

If the dot product is positive the angle between the vectors is **acute**.	If the dot product is zero, the vectors are **perpendicular**.	If the dot product is negative, the angle between the vectors is **obtuse.**



5.3.2: State Properties of scalar product.

Properties of Scalar Product
1. $\vec{a} \bullet \vec{b} = \vec{b} \bullet \vec{a}$ 2. For nonzero vector a and b, $\vec{b} \bullet \vec{a} = 0$ if and only if $\vec{a}$ is perpendicular to $\vec{b}$ 3. $\vec{a} \bullet (\vec{b} + \vec{c}) = \vec{a} \bullet \vec{b} + \vec{a} \bullet \vec{c}$ 4. $\vec{a} \bullet (k\vec{b}) = (k\vec{a}) \bullet \vec{b} = k(\vec{a} \bullet \vec{b})$ 5. $\vec{a} \bullet \vec{a} = |\vec{a}|^2$ 6. $0 \bullet \vec{a} = 0$

5.3.3: Calculate the Scalar product.

- The dot or scalar product of vectors $A = a_1 i + a_2 j + a_3 k$ and $B = b_1 i + b_2 j + b_3 k$ can be written as:

$$\vec{A}\bullet\vec{B} = a_1 b_1 + a_2 b_2 + a_3 b_3$$

- Application of the scalar product:
  - If the vector is orthogonal or perpendicular, $\cos\theta = \vec{A}\bullet\vec{B} = 0$.
  - If the two vectors are parallel,

$$\cos\theta = \frac{\vec{A}\bullet\vec{B}}{|\vec{A}||\vec{B}|} = 1$$

  - If the two vectors are parallel, but in opposite direction,

$$\cos\theta = \frac{\vec{A}\bullet\vec{B}}{|\vec{A}||\vec{B}|} = -1$$

- angle between two vector can be define by

$$\cos\theta = \frac{\vec{A}\bullet\vec{B}}{|\vec{A}||\vec{B}|}$$

Example:

1. Vectors A and B are given by $\vec{A} = 2i - 3j \text{ and } \vec{B} = -4i + 2j$. Find the dot product $A\bullet B$

$$A\bullet B = (2i - 3j)\bullet(-4i + 2j)$$

$$A\bullet B = (2)\bullet(-4) + (-3)\bullet(2)$$

$$A\bullet B = -8 - 6$$

$$A\bullet B = -14$$

2. Find the angle between vector $\vec{P} = 3i + 3j - 4k$ and $\vec{Q} = 2i + 3j + 5k$

$$\vec{P}\bullet\vec{Q} = (3)(2) + (3)(3) + (-4)(5)$$

$$A\bullet B = -5$$

$$|\vec{P}| = \sqrt{3^2 + 3^2 + (-4)^2}$$

$$= \sqrt{34}$$

$$|\vec{Q}| = \sqrt{2^2 + 3^2 + 5^2}$$

$$= \sqrt{38}$$

$$\cos\boldsymbol{\theta} = \frac{\vec{P}\bullet\vec{Q}}{|\vec{P}||\vec{Q}|} = \frac{-5}{\sqrt{34}\sqrt{38}} = -0.139$$

$$\boldsymbol{\theta} = 98°$$

3. Find the value of $\vec{A}\bullet\vec{B}$ $if\ given\ \theta = 94.01°,\ |\vec{A}| = \sqrt{61}\ and\ |\vec{B}| = \sqrt{48}$

$$\cos\theta = \frac{\vec{A}\bullet\vec{B}}{|\vec{A}||\vec{B}|}$$

$$\vec{A}\bullet\vec{B} = |\vec{A}||\vec{B}|\cos\theta$$

$$= \sqrt{61}\sqrt{48}\cos 94.01$$

$$= \mathbf{-3.784}$$

Exercise:

1.	Find the value of $\vec{A}\cdot\vec{B}$ $if\ given\ \theta = 52.23°,\ |\vec{A}| = \sqrt{25}\ and\ |\vec{B}| = \sqrt{81}$
2.	Find the angle between vector $A(1,2,3) and\ B(4,-5,6)$
3.	Find the scalar product between vector $P(2,0) and\ Q(-1,3)$

4.	Find $\bar{a} \bullet \bar{b}$ *and cosine angle between* $\bar{a}$ *and* $\bar{b}$ *if* b:	
	i.	$\bar{a} = 5i - 12j, \bar{b} = i + 2j$
	ii.	$\bar{a} = 2i + 8j - k, \bar{b} = -4i + k$

5.4: Apply Vector (Cross) Product of Two Vectors

5.4.1: Calculate the Scalar product.

- The Cross Product is a vector product since it yields another vector rather than a scalar.
- As with the dot product, the cross product of two vectors contains valuable information about the two vectors themselves.
- *The cross product for two vectors*

$$\vec{A} = a_1 i + b_1 j + c_1 k$$

$$\vec{B} = b_2 i + b_2 j + c_2 k$$

5.4.2: State properties of vector product.

- Right hand rule cross product:

$$\vec{a} \times \vec{b} \text{ is perpendicular to both } a \text{ and } b$$

- For vector $\vec{a}$, $\vec{b}$, $\vec{c}$ *and scalar* k

$$\vec{a} \times \vec{b} = -(\vec{b} \times \vec{a})$$

$$(k\vec{b}) \times \vec{c} = k(\vec{b} \times \vec{c}) = \vec{b} \times (k\vec{c})$$

$$\vec{a} \times (\vec{b} + \vec{c}) = (\vec{a} \times \vec{b}) + (\vec{a} \times \vec{c})$$

- The easy way to remember the formula for the cross product by using the properties of determinants.

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

And the determinant of a 3 x 3 is

$$\begin{vmatrix} i & j & k \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix} = \begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix} i - \begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix} j + \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} k$$

$$= [b_1c_2 - b_2c_1]i - [a_1c_2 - a_2c_1]j + [a_1b_2 - a_2b_1]k$$

5.4.3: Calculate the Vector product.

- Find the cross product for $\vec{a} = (3,2,-2) and\ \vec{b} = (1,0,-5)$

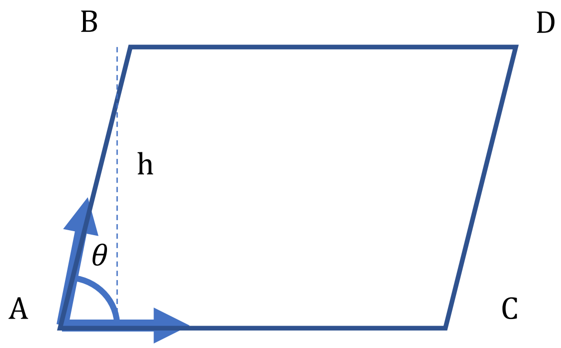
$$\begin{vmatrix} i & j & k \\ 3 & 2 & -2 \\ 1 & 0 & -5 \end{vmatrix} = \begin{vmatrix} 2 & -2 \\ 0 & -5 \end{vmatrix} i - \begin{vmatrix} 3 & -2 \\ 1 & -5 \end{vmatrix} j + \begin{vmatrix} 3 & 2 \\ 1 & 0 \end{vmatrix} k$$

$$\vec{a} \times \vec{b} = [(2)(-5) - (0)(-2)]i - [(3)(-5) - (1)(-2)]k + [(3)(0) - (1)(2)]k$$

$$= [-10 - 0]i - [-15 + 2]j + [0 - 2]k$$

$$= \mathbf{-10\boldsymbol{i} + 13\boldsymbol{j} - 2\boldsymbol{k}}$$

5.4.4: Calculate the area of triangle and parallelogram.



Area of parallelogram $= |\overrightarrow{AB}||\overrightarrow{AC}| \sin\theta = |\overrightarrow{AB} \times \overrightarrow{AC}|$

Area of triangle ABC $= \frac{1}{2}|\overrightarrow{AB} \times \overrightarrow{AC}|$

Unit vector that is perpendicular to both vector $\overrightarrow{AB}\ and\ \overrightarrow{AC}$,

$$\widehat{\overrightarrow{AB} \times \overrightarrow{AC}} = \frac{\overrightarrow{AB} \times \overrightarrow{AC}}{|\overrightarrow{AB} \times \overrightarrow{AC}|}$$

Example 1:

Calculate the area of the parallelogram spanned by the vectors $\vec{a} = (3, -3,1) and\ \vec{b} = (4,9,2)$.

The area is $|\vec{a} \times \vec{b}|$

$$\vec{a} \times \vec{b} = \begin{vmatrix} i & j & k \\ 3 & -3 & 1 \\ 4 & 9 & 2 \end{vmatrix}$$

$$= \begin{vmatrix} -3 & 1 \\ 9 & 2 \end{vmatrix} i - \begin{vmatrix} 3 & 1 \\ 4 & 2 \end{vmatrix} j + \begin{vmatrix} 3 & -3 \\ 4 & 9 \end{vmatrix}$$

$$= (-6 - 9)i - (6 - 4)j + (27 + 12)k$$

$$= -15i - 2j + 39k$$

$$Using\ the\ above\ expression\ for\ the\ cross\ product,$$

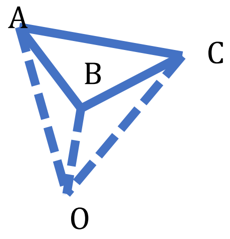
$$We\ find\ that\ the\ area\ is$$

$$|\vec{a} \times \vec{b}| = \sqrt{(-15)^2 + (-2)^2 + 39^2}$$

$$= \mathbf{5\sqrt{70}}\ \boldsymbol{units^2}$$

Example 2:

A triangle ABC has its vertices at the at the points A(3,6,3), B(3,0,9) and C(-3,6,-3). Find $\overrightarrow{BA}, \overrightarrow{BC}, \overrightarrow{BA} \times \overrightarrow{BC}$ and the area of triangle ABC.



Assume O is origin,

$$\overrightarrow{OA} = 3i + 6j + 3k$$

$$\overrightarrow{OB} = 3i + 9k$$

$$\overrightarrow{OC} = -3i + 6j - 3k$$

$$\overrightarrow{BA} = \overrightarrow{BO} + \overrightarrow{OA}$$

$$= -\overrightarrow{OB} + \overrightarrow{OA}$$

$$= -(3i + 9k) + 3i + 6j + 3k$$

$$= 6j - 6k$$

$$\overrightarrow{BC} = \overrightarrow{BO} + \overrightarrow{OC}$$

$$= -\overrightarrow{OB} + \overrightarrow{OC}$$

$$= -(3i + 9k) - 3i + 6j - 3k$$

$$= -3i - 9k - 3i + 6j - 3k$$

$$= -6i + 6j - 12k$$

$$\overrightarrow{BA} \times \overrightarrow{BC} = \begin{vmatrix} i & j & k \\ 0 & 6 & -6 \\ -6 & 6 & -12 \end{vmatrix}$$

$$= \begin{vmatrix} 6 & -6 \\ 6 & -12 \end{vmatrix} i - \begin{vmatrix} 0 & -6 \\ -6 & -12 \end{vmatrix} j + \begin{vmatrix} 0 & 6 \\ -6 & 6 \end{vmatrix} k$$

$$= [(6)(-12) - (6)(-6)]i - [(0)(-12) - (-6)(-6)]j + [(0)(6) - (-6)(6)]k$$

$$= -36i + 36j + 36k$$

$$\therefore Area\ of\ \Delta ABC = \frac{1}{2}\left|\overrightarrow{BA} \times \overrightarrow{BC}\right|$$

$$= \frac{1}{2}\sqrt{(-36)^2 + 36^2 + 36^2}$$

$$= \mathbf{31.2\ units^2}$$

Exercise:

1.	Given$\overrightarrow{AB} = i + 2j - k \; and \; \overrightarrow{BC} = -5i + j + 3k$. Find the area parallelogram that is formed by vectors $\overrightarrow{AB} \; and \; \overrightarrow{BC}$.
2	A parallelogram ABCD has its vertices at the points $A(-2,5,3), B(2,2,4) and \; C(-4,-2,4)$. Find $\overrightarrow{AB}, \overrightarrow{AC}, \overrightarrow{AB} \times \overrightarrow{AC}$ and calculate the area of parallelogram ABCD.

3.	The triangle ABC has vertices at $(2,3,4), (-2,1,0) and\ (4,0,2)$. Find the area of the triangle.
4.	Let $\vec{P} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \vec{Q} = \begin{pmatrix} -1 \\ 1 \\ 2 \end{pmatrix}\ and\ \vec{R} = \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}$. Find the area of the triangle determine by the three points.

Quiz Activity

SCAN ME

https://forms.gle/Vj68b8r3LzGcnUdx6

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