J. Math. Computer Sci., 42 (2026), 124–142

Online: ISSN 2008-949X

Journal of Mathematics and Computer Science

Journal Homepage: www.isr-publications.com/jmcs

Analysis of impulsive fractional delay differential equations with Mittag-Leffler kernel and its applications to Hutchinson's model

Wisdom Udogworen[a,b], Dodi Igobi[b], Abdulkareem Hamarsheh[c], Uko Jim[b], Hossam A Nabwey[c], Reny George[c,*]

[a]Department of Mathematics, Akwa Ibom State University, P.M.B. 1167 Uyo, Akwa Ibom State, Nigeria.

[b]University of Uyo, P.M.B. 1017 Uyo, Nigeria.

[c]Department of Mathematics, College of Science and Humanities in Alkharj, Prince Sattam Bin Abdulaziz University, Alkharj, 11942, Saudi Arabia.

Abstract

In this paper, we use the Schuader fixed point and the Banach contraction principle to study sufficient conditions for existence, uniqueness, and Lipschitz stability of solutions for fractional delay differential equations with Mittag-Leffler Kernel (without impulses) and impulsive fractional differential equations (with delay). As a result, with practical applications, the theoretical understanding of fractional differential equations containing memory effects, impulse perturbations, and delay factors is expanded. The developed concept is finally illustrated with examples and applied to impulsive fractional delay biological systems involving a single species population growth model known as, the Hutchinson's equations using the fractional order derivative with Mittag-Leffler kernel.

Keywords: Fractional Calculus, impulsive fractional calculus, delay fractional calculus, Hutchinson' model, stability, Mittag-Leffler.

2020 MSC: 34A08, 34A34, 34D20, 34N05.

©2026 All rights reserved.

1. Introduction

Fractional calculus has advanced more rapidly and attracted a lot of interest in the mathematical sciences in recent years. The Riemann-Liouville and Caputo fractional differential equation (with singular kernel), the Caputo-Fabrizio fractional derivative (with non-singular local kernel) introduced in [18] and the Atangana-Baleanu fractional derivative (with non-singular, non-local kernel) introduced in [14] are just a few of the numerous definitions of fractional derivatives.

Cauto and Fabrizio solved the singularity kernel problem in the Riemann-Liouville and Caputo derivative by introducing a new derivative with an exponential kernel. In our opinion, the primary goal of Caputo and Fabrizio's work was to figure out how to further improved the dynamics of memory-effect

*Corresponding author

Email addresses: wisdomudogworen@aksu.edu.ng (Wisdom Udogworen), dodiigobi@uniuyo.edu.ng (Dodi Igobi), abkmsaleh@yahoo.com (Abdulkareem Hamarsheh), ukojim@uniuyo.edu.ng (Uko Jim), h.mahammed@psau.edu.sa (Hossam A Nabwey), renygeorge02@yahoo.com (Reny George)

doi: 10.22436/jmcs.042.01.08

Received: 2025-10-14 *Revised:* 2025-10-31 *Accepted:* 2025-11-21

systems. However, a few problems with the Caputo-Fabrizio derivative with fractional order were noted. The Caputo-Fabrizio derivative uses a local kernel, and the anti-derivative linked to their derivative is the average of the function and its integral rather than a fractional. Since many physical problems seem to be beyond the scope of such a used kernel, this kernel problem give rise to the following question, what is the most accurate kernel which better describes the dynamics of memory-effect systems? The potential response to the question was given by Atangana and Baleanu in [14] by generalizing the exponential function in Caputo-Fabrizio derivative to mittag-Leffler function. Therefore, Atangana-Baleanu derivative with Mittag-Leffler kernel is considered as non-singular and a non-local function than the exponential function.

Impulsive delay fractional differential equations are essential for modelling most real-world problems because, like other differential equations, they are generalised to their fractional forms [30]. The fractional rate of change depends on the system's hereditary effects, which include a number of impacts from its past states and its future evolution phenomena. The dynamics of many evolutionary systems are characterised by abrupt state changes at certain times, which can be brought on by natural disasters, harvesting, or shocks [24, 38]. These procedures are prone to short-term interruptions, the duration of which is negligible in relation to the process's overall duration. These transient disruptions are frequently regarded as having occurred instantly, that is, as impulses. Numerous models in economics, biology, and medicine show impulsive consequences. Examining the qualitative behaviour of these equations is critical, particularly when modelling impulse impact problems in the real world. There are two ways to discuss impulsive delay differential equations: either we start with a delay differential and then add impulses, or we start with an impulsive differential equation and explain how one would incorporate delays into it. We start by introducing the following. For $a, b \in \mathbb{R}$, we have $a < b$ and for $S \subset \mathbb{R}^n$, define $PC([a,b],S) = \{x : [a,b] \to S | x(t^+) = x(t), \forall t \in [a,b]$. $x(t^-)$ exists in S, $\forall t \in (a,b]$ and $x(t^-) = x(t)$, for all, but at most a finite number of points $t \in (a,b]\}$, $PC([a,b),S) = \{x : [a,b) \to S | x(t^+) = x(t) \forall t \in [a,b], x(t^-)$ exists in S, $\forall t \in (a,b)$ and $x(t^-) = x(t)$ for all but at most a finite number of points $t \in (a,b)\}$, and $PC([a,b),S) = \{x : [a,\infty) \to S|, \forall c > a, x|_{[a,c]} \in PC([a,c],S)\}$. With the exception of simple jump discontinuities, where they have a left-hand limit, these function classes characterise functions that are left-continuous and right continuous on their domain, denoted by $x(t^-)$. This left-hand limit needs to be finite and fall inside the set S, unless such functions are defined on a set of the form $[a,\infty)$, they may have only a finite number of jump discontinuities. Simple jump discontinuities may be present in this situation at a countably infinite number of places, forming an expanding sequence that tends to infinity. Numerous applications of the theory of dynamical systems and ordinary differential equations can be found in chemistry, biology, biotechnologies, and population dynamics. In fact, a great deal of research has been done on integer-order biological models because of their theoretical and practical importance. On the other hand, numerous researchers have included time delays in biological models to depict feeding, maturation, resource regeneration, and reaction times. Additionally, during the past few decades, impulsive biological systems have become more and more popular. Numerous findings regarding impulsive Lotka-Voltera and related systems' dynamical behaviours can be found in [3, 21, 22].

Some biological models have used fractional calculus in the last ten years. The extension of biological models to arbitrary non-integer orders is known as fractional biological model. In fact, recent studies have demonstrated that fractional derivative formulation provides a more accurate representation of many physical systems. Only solutions that experience impulses a finite number of times over any finite interval are of interest to us; nevertheless, if defined for all $t \geqslant t_0$, they may experience impulses at a series of times tending to infinity. Two class of fractional differential equations are examined in this article:

1. fractional delay differential equations without impulse given as

$$\begin{cases} {}^{ABC}_{t_0}D^{\beta}_{t} x(t) = f(t, x_t), \quad t \in Q = [t_0, T], \quad t \neq t_\tau, \\ x_{t_0} = \psi_0; \end{cases} \tag{1.1}$$

2. and fractional delay differential equations with fixed moments of impulse effect of the type

$$\begin{cases} {}^{ABC}_{t_0}D^{\beta}_{t}x(t) = f(t, x_t), \quad t \in Q = [t_0, T], \quad t \neq t_\tau, \\ \Delta x(t_\tau) = I_\tau(t_\tau, x(t_\tau)), \quad t = t_\tau, \quad \tau = 1, 2, \ldots, k, \\ x_{t_0} = \psi_0 \in PC, \end{cases} \tag{1.2}$$

where ${}^{ABC}_{t_0}D^{\beta}_{t}x(t)$ denotes the fractional derivative with Mittag-Leffler kernel, $\Delta x(t) = x(t_\tau^+) - x(t_\tau^-)$, $t_0 < t_1 < \cdots < t_\tau < t_{k+1} = T$, $f : Q \times PC \to \mathbb{R}^n$ is piecewise continuous and $I_\tau : Q \times \mathbb{R}^n \to \mathbb{R}^n$ are continuous for $\tau = 1, \ldots, k$, where $PC = PC[[-r, 0], \mathbb{R}^n]$ denotes the space of piecewise left continuous derivative functions $\psi : [-r, 0] \to \mathbb{R}^n$ with the $\|\psi\| = \sup_{-r \leqslant s \leqslant 0} \|\psi\|$, here $r > 0$, $\|.\|$ is a norm in $\mathbb{R}^n$. $x_t \in PC$ is defined by $x_t(t + s) = \psi(s)$ for $-r \leqslant s \leqslant 0$. The set $PC([0, T]) = \{\psi \in PC : \|\psi\| < \eta, \forall \eta > 0\}$ and $PC[[t_0 - r, T], \mathbb{R}^n]$ denote the Banach space of all continuous functions from Q into $\mathbb{R}^n$ with the norm $\|x\|_\infty = \sup_{t \in [t_0 - r, T]} \|x(t)\|$.

In the qualitative theory of differential equations, stability of the solutions is one of the primary focus. We can compare how different starting points for solutions behave using stability theory. The study of stability in differential systems is a fundamental aspect of mathematical modeling with wide-ranging application across various scientific and engineering disciplines, see [19, 26–28, 34]. Examining how these model equations behave qualitatively is crucial, particularly when modeling real-world issues in the fields of biology, medicine, and economics. [1, 2, 9, 11, 15, 20, 31, 35, 36, 38–40], discuss the qualitative characteristics of existence and uniqueness of the solutions of the initial value problem for nonlinear impulsive fractional differential equation involving Caputo, Caputo-Fabrizio, and Atangana-Baleanu.

On the other hand, fixed point theorems have wide applications in studying existence, uniqueness, and stability of solutions of various differential and integral solutions (see [10, 13, 29]). In this paper, we use the Schuader fixed point and the Banach contraction principle to study sufficient conditions for existence, uniqueness, and Lipschitz stability of solutions for fractional delay differential equations with Mittag-Leffler Kernel (without impulses) and impulsive fractional differential equations (with delay). As a result, with practical applications, the theoretical understanding of fractional differential equations containing memory effects, impulse perturbations, and delay factors is expanded.

In comparison with existing studies on impulsive fractional differential equations involving Mittag-Leffler kernel, the present results on impulsive fractional delay differential equations provide a significant generalization. In most existing works, the state of the system depends only on the current value of the unknown function, by introducing a delay argument, the impulsive fractional delay model captures hereditary and memory effects that are not present in the non-delay case. This inclusion increases the analytical complexity of the problem since the solution operator must act on a functional space of histories rather than on point-wise values. Our results maintain the applicability of the standard theorems while encompassing the delayed dynamic. Hence, the impulsive fractional delay system with Mittag-Leffler kernel not only generalizes the classical impulsive fractional system but also provides a richer and more realistic description of processes influenced by past state.

2. Preliminaries

This section presents the initial findings.

Definition 2.1 ([25]). Let $n < \beta \leqslant n + 1$, $n \in \mathbb{N}$, and f be a real function defined on $t \in [a, b]$. The Riemann-Liouville fractional integral of order β is defined by

$$({}_aI^{\beta}_{t})f(t) = \frac{1}{\Gamma(\beta)} \int_a^t (t - p)^{\beta - 1} f(p)\, dp, \tag{2.1}$$

where

$$\Gamma(\beta) = \int_0^\infty t^{\beta-1} e^{-t} dt, \quad \beta > 0,$$

is the gamma function.

The Riemann-Liouville derivative of order β equivalent of equation (2.1) is defined as

$${}^{RL}_{t_0}D^{\beta}_t x(t) = \frac{1}{\Gamma(n-\beta)} D^n \int_{t_0}^t (t-p)^{n-\beta-1} x(p) dp.$$

Definition 2.2 ([20]). Let $0 < \beta \leqslant n+1$, $n \in \mathbb{N}$. The Caputo fractional derivative of order β of a function $x(t)$ is defined as

$${}^{C}_{t_0}D^{\beta}_t x(t) = \frac{1}{\Gamma(n-\beta)} \int_{t_0}^t (t-p)^{n-\beta-1} D^n x(p) dp.$$

Definition 2.3 ([25]). Let $f \in H'(a,b)$, $a < b$, $\beta \in [0,1)$, then the definition of the new (left Caputo) fractional derivative in the sense of Caputo and Fabrizio becomes

$$({}^{CFC}_{a}D^{\beta}_t f)(t) = \frac{M(\beta)}{1-\beta} \int_a^t f'(p) e^{[-\beta \frac{(t-p)}{1-\beta}]} dp,$$

where $M(\beta)$ is a normalization function such that $M(0) = M(1) = 1$.

Definition 2.4 ([4]). In the right case we have

$$({}^{CFC}_{t}D^{\beta}_b f)(t) = \frac{-M(\beta)}{1-\beta} \int_t^b f'(p) e^{[-\beta \frac{(p-t)}{1-\beta}]} dp.$$

Definition 2.5 ([7]). Let $f \in H'(a,b)$, $a < b$, $\beta \in [0,1)$, then the definition of the new (left Riemann-Liouville) fractional derivative in the sense of Caputo and Fabrizio becomes

$$({}^{CFR}_{a}D^{\beta} f)(t) = \frac{M(\beta)}{1-\beta} \frac{d}{dt} \int_a^t f(p) e^{[-\beta \frac{(t-p)}{1-\beta}]} dp.$$

The associated fractional integral is

$$({}_a{}^{CF}I^{\beta} f)(t) = \frac{1-\beta}{M(\beta)} f(t) + \frac{\beta}{M(\beta)} \int_a^t f(p) dp,$$

and in the right Riemann-Liouville fractional derivative in the sense of Caputo and Fabrizio is

$$({}^{CFR}_{b}D^{\beta} f)(t) = \frac{-M(\beta)}{1-\beta} \frac{d}{dt} \int_t^b f(p) e^{[-\beta \frac{(t-p)}{1-\beta}]} dp.$$

The associated fractional integral is

$$({}_t{}^{CF}I^{\beta}_b f)(t) = \frac{1-\beta}{M(\beta)} f(t) + \frac{\beta}{M(\beta)} \int_a^t f(p) dp.$$

Definition 2.6 ([14]). Let $f \in H'(a,b), b > a, \beta \in [0,1]$, the left Atangana-Baleanu derivative in Caputo sense is given as

$$({}^{AB}_{a}D^{\beta}_t f)(t) = \frac{M(\beta)}{1-\beta} \int_a^t f'(p) E_\beta \big[-\beta \frac{(t-p)^{\beta k}}{1-\beta}\big] dp,$$

where $M(\beta)$ has the same properties as in Caputo-Fabrizio.

Definition 2.7 ([14]). Let $f \in H^{'}(a,b), b > a, \beta \in [0,1]$, the right Atangana-Baleanu derivative in Caputo sense is given as

$$({}^{AB}_{a}D^{\beta}_{t}\, f)(t) = \frac{-M(\beta)}{1-\beta}\int_{a}^{t} f^{'}(p)E_{\beta}\big[-\beta\frac{(t-p)^{\beta k}}{1-\beta}\big]dp.$$

Definition 2.8 ([8]). Let $f \in H^{'}(a,b), b > a, \beta \in [0,1]$, the left Atangana-Baleanu derivative in Riemann-Liouville sense is given as

$$({}^{AB}_{a}D^{\beta}\, f)(t) = \frac{M(\beta)}{1-\beta}\frac{df}{dt}\int_{a}^{t} f(p)E_{\beta}\big[-\beta\frac{(t-p)^{\beta k}}{1-\beta}\big]dp.$$

Definition 2.9 ([14]). Let $f \in H^{'}(a,b), b > a, \beta \in [0,1]$, the right Atangana-Baleanu derivative in Riemann-Liouville sense is given as

$$({}^{AB}_{t}D^{\beta}_{b}\, f)(t) = \frac{-M(\beta)}{1-\beta}\frac{df}{dt}\int_{a}^{t} f(p)E_{\beta}\big[-\beta\frac{(t-p)^{\beta k}}{1-\beta}\big]dp,$$

where E_{β} is the one parametric Mittag-Leffler function given as

$$E_{\beta}(x) = \sum_{k=o}^{\infty}\frac{x^{k}}{\Gamma(k\beta+1)}.$$

Definition 2.10 ([32]). Assume that a function f is piecewise smooth over each finite interval in $[0,\infty]$ and that $|f(t)| \leqslant B^{at}$ for all $t > T$ if there are constants $B > 0$ and $T > 0$. $F(s) = \mathcal{L}f(t); s = \int_{0}^{\infty} e^{-st}f(t)dt, s \in \mathbb{C}$, then the Laplace transform $\mathcal{L}f(t); s$ of $f(t)$ exists.

Definition 2.11. If f and g are piecewise continuous on $[0,\infty)$, then the integral defines a specific product, $f*g$,

$$f*g = \int_{0}^{t} f(p)g(t-p)dp,$$

and is referred to as the f and g convolution.

Lemma 2.12 ([25]). *In the Caputo notion, the Laplace transform of the Caputo-Fabrizio derivative is provided as*

$$\mathcal{L}\{{}^{CF}_{0}D^{\beta}_{t}\, f(t)\} = \frac{pF(p)-f(0)}{p+\beta(1-p)}.$$

Lemma 2.13 ([14]). *The Laplace transform of the Atangana-Baleanu derivative in the Caputo sense is given as*

$$\mathcal{L}\{{}^{AB}_{0}D^{\beta}_{t}\, f(t)\} = \frac{M(\beta)}{1-\beta}\frac{p^{\beta-1}(pF(p)-f(0))}{p^{\beta}+\frac{\beta}{1-\beta}}.$$

Proof.

$$\mathcal{L}\{{}^{AB}_{0}D^{\beta}_{t}\, f(t)\} = \frac{M(\beta)}{1-\beta}\frac{p^{\beta-1}(pF(p)-f(0))}{p^{\beta}+\frac{\beta}{1-\beta}},$$

$$({}^{AB}_{a}D^{\beta}\, f)(t) = \frac{M(\beta)}{1-\beta}\int_{a}^{t} f^{'}(p)E_{\beta}\big[-\beta\frac{(t-p)^{\beta k}}{1-\beta}\big]dp,$$

$$\mathcal{L}\{{}^{ABC}_{0}D^{\beta}_{t}\, f(t)\} = \mathcal{L}\{\frac{1}{1-\beta}\int_{0}^{t} f^{'}(p)E_{\beta}\big[-\beta\frac{(t-p)^{\beta k}}{1-\beta}\big]dp\},$$

$$\mathcal{L}\{{}^{ABC}_{0}D^{\beta}_{t}\, f(t)\} = \frac{1}{1-\beta}\mathcal{L}\{\int_{0}^{t} f^{'}(p)E_{\beta}\big[-\beta\frac{(t-p)^{\beta k}}{1-\beta}\big]dp\}.$$

Using the convolution theorem $f * g = \int_0^t f(p)g(t-p)dp$, $\mathcal{L}\{f * g\} = \mathcal{L}\{f\} \cdot \mathcal{L}\{g\} = F(p) \cdot G(p)$. Therefore,

$$\mathcal{L}\{{}_0^{ABC}D_t^{\beta}(f(t))\} = \frac{1}{1-\beta}\mathcal{L}\left\{f'(p) * E_{\beta}\left[-\beta\frac{t^{\beta k}}{1-\beta}\right]\right\},$$
$$\mathcal{L}\left\{{}_0^{ABC}D_t^{\beta} f(t)\right\} = \frac{1}{1-\xi}\left[\mathcal{L}\{f'(p)\} * \mathcal{L}\left\{E_{\beta}\left[-\beta\frac{t^{\beta k}}{1-\beta}\right]\right\}\right],$$

we recall that

$$\mathcal{L}E_{\beta}(at^{\beta k}) = \int_0^{\infty} e^{-pt}\sum_{k=0}^{\infty}\frac{(at^{\beta})^k}{\Gamma(\beta k+1)}dt = \sum_{k=0}^{\infty}\frac{a^k}{\Gamma(\beta k+1)}\int_0^{\infty} e^{-pt}t^{\beta k}dt,$$

since

$$\mathcal{L}(t^n) = \int_0^{\infty} e^{-pt}t^n dt = \frac{\Gamma(n+1)}{p^{n+1}},$$

then this implies that

$$= \sum_{k=0}^{\infty}\frac{a^k}{\Gamma(\beta k+1)}.\mathcal{L}(t^{\beta k}) = \sum_{k=0}^{\infty}\frac{a^k}{\Gamma(\beta k+1)}.\frac{\Gamma(\beta k+1)}{p^{\beta k+1}} = \sum_{k=0}^{\infty}\frac{a^k}{p^{(\beta k+1)}} = \frac{1}{p}\sum_{k=0}^{\infty}\left(\frac{a}{p^{\beta}}\right)^k,$$

we recall that $1 + t + t^2 + t^3 + t^4 + \cdots + \sum_{k=0}^{\infty} t^k = \frac{1}{1-t}$,

$$\frac{1}{p}\sum_{k=0}^{\infty}\left(\frac{a}{p^{\beta}}\right)^k = \frac{1}{p}\frac{1}{1-\frac{a}{p^{\beta}}} = \frac{p^{\beta}}{p(p^{\beta}-a)} = \frac{p^{\beta-1}}{p^{\beta}-a}, \quad \mathcal{L}E_{\beta}(at^{\beta k}) = \frac{p^{\beta-1}}{p^{\beta}-a},$$

therefore

$$\mathcal{L}E_{\beta}\left[-\beta\frac{t^{\beta k}}{1-\beta}\right] = \frac{p^{\beta-1}}{p^{\beta}+\frac{\beta}{1-\beta}}.$$

Also $\mathcal{L}f'(p) = pF(p) - f(0)$, we have that

$$\mathcal{L}\{{}_0^{ABC}D_t^{\beta} f(t)\} = \frac{M(\beta)}{1-\beta}\left[\left(pF(p) - f(0)\right)\frac{p^{\beta-1}}{p^{\beta}+\frac{\beta}{1-\beta}}\right].$$

Taking the inverse Laplace transform of the above equation, we have

$$\mathcal{L}\{{}_0^{ABC}D_t^{\beta} f(t)\} = \mathcal{L}\{g(t)\}, \quad \frac{M(\beta)}{1-\beta}\left[\left(pF(p) - f(0)\right)\frac{p^{\beta-1}}{p^{\beta}+\frac{\beta}{1-\beta}}\right] = G(p),$$

multiplying both sides by the denominator,

$$\frac{M(\beta)}{1-\beta}p^{\beta-1}\left[\left(pF(p) - f(0)\right)\right] = G(p)\{p^{\beta} + \frac{\beta}{1-\beta}\},$$
$$pF(p) = f(0) + \frac{1-\beta}{M(\beta)}\cdot\frac{G(p)\{p^{\beta}+\frac{\beta}{1-\beta}\}}{p^{\beta-1}},$$
$$F(p) = \frac{f(0)}{p} + \frac{1-\beta}{M(\beta)}\cdot\frac{G(p)\{p^{\beta}+\frac{\beta}{1-\beta}\}}{p^{\beta}}.$$

Taking the inverse Laplace of $F(p)$,

$$f(t) = \mathcal{L}^{-1}\{\frac{1}{p}\}f(0) + \frac{1-\beta}{M(\beta)}\cdot\mathcal{L}^{-1}\left\{\frac{G(p)\{p^{\beta}+\frac{\beta}{1-\beta}\}}{p^{\beta}}\right\},$$

$$f(t) = \mathcal{L}^{-1}\{\frac{1}{p}\}f(0) + \frac{1-\beta}{M(\beta)} \cdot \left[\mathcal{L}^{-1}\{G(p)\} + \frac{\beta}{1-\beta} \cdot \mathcal{L}^{-1}\{\frac{G(p)}{p^{\beta}}\}\right].$$

We use

$$\mathcal{L}^{-1}\{\frac{1}{p} = 1\}, \quad \mathcal{L}^{-1}\{G(p)\} = g(t), \quad \mathcal{L}^{-1}\{\frac{G(p)}{p^{\beta}}\} = \frac{1}{\Gamma(\beta)}\int_0^t (t-p)^{\beta-1}g(p)dp.$$

□

Definition 2.14 ([6]). The Atangana-Baleanu-Caputo fractional derivative with non-local kernel's fractional integral is provided as

$${}^{AB}_{0}I^{\beta}_{t}(f(t)) = \frac{1-\beta}{M(\beta)}f(t) + \frac{\beta}{M(\beta)\Gamma(\beta)}\int_0^t f(p)(t-p)^{\beta-1}dp.$$

When β is zero we recover the initial function and if also β is 1, we obtain the ordinary integral.

Proposition 2.15 ([5]). *For $x(t) \in [a, b]$ and $n \leqslant \beta \leqslant n+1$, we have the following equations:*

$$\left({}^{ABC}_{a}D^{\beta} \cdot {}^{AB}_{a}I^{\beta}\, x\right)(t) = x(t), \quad \left({}^{AB}_{a}I^{\beta}\ {}^{ABC}_{a}D^{\beta}\, x\right)(t) = x(t) - \sum_{k=0}^{n-1}\frac{x^{(k)}(a)}{k!}(t-a)^{k}.$$

Proposition 2.16 ([5]). *For $x(t) \in [a, b]$ and $n \leqslant \alpha \leqslant n+1$, we have the following equations:*

$$\left({}^{ABR}D^{\beta}_{b} \cdot {}^{AB}I^{\beta}_{b}\, x\right)(t) = x(t), \quad \left({}^{AB}I^{\beta}_{b}\ {}^{ABR}D^{\beta}_{b}\, x\right)(t) = x(t) - \sum_{k=0}^{n-1}\frac{(-1)^{k}x^{(k)}(b)}{k!}(b-t)^{k}.$$

Theorem 2.17 ([14]). *The Atangana-Baleanu in Riemann and Caputo sense possess the Lipschitz condition, that is to say, for a given couple function f and h, the following inequalities can be established as*

$$\|{}^{ABC}_{0}D^{\beta}_{t}\, f(t) - {}^{ABC}_{0}D^{\beta}_{t}\, h(t)\| \leqslant H\|f(t) - h(t)\|.$$

And also

$$\|{}^{ABR}_{0}D^{\beta}_{t}\, f(t) - {}^{ABR}_{0}D^{\beta}_{t}\, h(t)\| \leqslant H\|f(t) - h(t)\|.$$

Definition 2.18 ([17]). A solution $x(t)$ of a fractional differential equation with initial condition x_0 is said to be stable if for $\epsilon > 0$ there exists $\delta = \delta(\epsilon, T)$ such that for any other initial condition y_0 satisfying $\|x_0 - y_0\| < \delta$. The corresponding solution $y(t)$ exists on $[t_0, T]$ and satisfies $\|x(t) - y(t)\| < \epsilon$.

Definition 2.19 ([38]). A solution $x(t)$ of a fractional differential equation with initial condition x_0 is said to be Lipschitz stable if there exists $C > 0$ such that for any other initial condition y_0, the corresponding solution $y(t)$ satisfies

$$\|x(t) - y(t)\| \leqslant C\|x_0 - y_0\|.$$

Definition 2.20 ([16]). A solution $x(t)$ of a fractional differential equation with initial condition x_0 is said to be uniformly stable on $[t_0, T]$ if for $\epsilon > 0$, there exists $\delta > o$ independent of the initial time such that for any other initial condition y_0 satisfying $\|x_0 - y_0\| < \delta$. The corresponding solution $y(t)$ satisfies $\|x(t) - y(t)\| < \epsilon$.

Theorem 2.21 ([37]). *Let $h \in H^{'}(0, 1)$, then h is a solution to the fractional initial value problem*

$$\begin{cases} {}^{ABC}_{t_0}D^{\beta}_{t}x(t) = f(t, x(t)), \quad t \in (0, 1), \quad \beta \in (0, 1), \\ x(0) = u, \end{cases}$$

if and only if, it is a solution to the integral equation

$$x(t) = u + \frac{1-\beta}{M(\beta)}f(t, x(t)) + \frac{\beta}{M(\beta)\Gamma(\beta)}\int_0^t f(p, x(p))(t-p)^{\beta-1}dp.$$

Theorem 2.22 ([37]). *Consider the fractional initial value problem given as*

$$T_\beta(x)(t) = f(t, x(t)), \quad t \in [0, T], \ \beta \in (0, 1),$$

and $x(0) = u$. *If* $f(t, x(t))$ *is Lipschitz function with Lipschitz constant*

$$\frac{(1-\beta)\Gamma(\beta)+1}{\Gamma(\beta)M(\beta)} < 1,$$

then the fractional initial value problem has a unique solution.

Remark 2.23 ([38]). Let $0 < \beta < 1$ and $f : [t_0, T] \times C \to \mathbb{R}^n$ be continuous, then the initial value problem (1.1) is comparable to the following: fractional integral Volterra with memory

$$\begin{cases} x_0 = \psi_0, \\ x(t) = \psi_0(0) + \frac{1-\beta}{M(\beta)} f(t, x_t) + \frac{\beta}{M(\beta)\Gamma(\beta)} \int_{t_0}^{t} (t-p)^{\beta-1} f(p, x_p)\, dp, \ t \in [t_0, T]. \end{cases} \tag{2.2}$$

In other words, all solutions of (1.1) are also solutions of (2.2), and vice versa.

Remark 2.24 ([38]). A function $x \in PC[[t_0 - r, T], \mathbb{R}^n]$ is said to be a solution of (1.2) if x satisfies the equations ${}^{AB}_{t_0}D^{\beta}_{t} x(t) = f(t, x_t)$ for almost everywhere $t \in Q$; the equations $x(t_\tau^+) = x(t_\tau^-) + I_\tau(t_\tau, x(t_\tau^-))$ and the condition $x_{t_0} = \psi_0$ such that

$$x(t) = \begin{cases} \psi_0(0) + \frac{1-\beta}{M(\beta)} f(t, x_t) + \frac{\beta}{M(\beta)\Gamma(\beta)} \int_{t_0}^{t} (t-p)^{\beta-1} f(p, x_p)\, dsp, \ t \in [t_0, T], \\ \psi_0(0) + \frac{1-\beta}{M(\beta)} f(t, x_t) + \frac{\beta}{M(\beta)\Gamma(\beta)} \int_{t_0}^{t} (t-p)^{\beta-1} f(p, x_p)\, dp \\ \quad + \sum_{\tau=1}^{k} I_\tau(t_\tau, x(t_\tau)), \ t \in (t_\tau, t_{\tau+1}], \ \tau = 1 \ldots, k. \end{cases} \tag{2.3}$$

Lemma 2.25. *A function* $x \in PC[[t_0 - r, T], \mathbb{R}^n]$, *is a solution of the IVP* (1.2) *if and only if* $x \in PC[[t_0 - r, T], \mathbb{R}^n]$ *satisfies the fractional integral equations* (2.3).

Proof. Assume x satisfies the IVP (2.2), for $t \in (t_0, t_1]$ such that

$$x(t) = \psi_0(0) + \frac{1-\beta}{M(\beta)} f(t, x_t) + \frac{\beta}{M(\beta)\Gamma(\beta)} \int_{t_0}^{t} (t-p)^{\beta-1} f(p, x_p)\, dp.$$

If $t \in (t_1, t_2]$,

$$x(t) = \psi_0(0) + I_1(t_1^-, x(t_1^-)) + \frac{1-\beta}{M(\beta)} f(t, x_t) + \frac{\beta}{M(\beta)\Gamma(\beta)} \int_{t_0}^{t} (t-p)^{\beta-1} f(p, x_p)\, dp.$$

If $t \in [t_2, t_3]$,

$$x(t) = \psi_0(0) + \frac{1-\beta}{M(\beta)} f(t, x_t) + \frac{\beta}{M(\beta)\Gamma(\beta)} \int_{t_0}^{t} (t-p)^{\beta-1} f(p, x_p)\, dp + \sum_{\tau=1}^{2} I_\tau(t_\tau^-, x(t_\tau^-)).$$

If $t \in (t_3, t_4]$,

$$x(t) = \psi_0(0) + \frac{1-\beta}{M(\beta)} f(t, x_t) + \frac{\beta}{M(\beta)\Gamma(\beta)} \int_{t_0}^{t} (t-p)^{\beta-1} f(p, x_p)\, dp + \sum_{\tau=1}^{3} I_\tau(t_\tau^-, x(t_\tau^-)).$$

If $t \in (t_\tau, T]$,

$$x(t) = \psi_0(0) + \frac{1-\beta}{M(\beta)} f(t, x_t) + \frac{\beta}{M(\beta)\Gamma(\beta)} \int_{t_0}^{t} (t-p)^{\beta-1} f(p, x_p)\, dp + \sum_{\tau=1}^{p} I_\tau(t_\tau^-, x(t_\tau^-)),$$

we obtained equations (2.3) for $x \in PC[[t_0 - r, T], \mathbb{R}^n]$. □

Theorem 2.26 (Arzela-Ascoli theorem for PC-type, [1]). *Let X be a Banach space and $x \in PC(Q,X)$, if the following conditions are satisfied:*

1. *x is uniformly bounded subset of $PC(Q,X)$;*
2. *x is equicontinuous in $(t_\tau, t_{\tau+1})$,*

then x is relatively compact subset of $PC(Q,X)$.

Theorem 2.27 (Schauder's Fixed point theorem, [2]). *Let X be a Banach space and W be a bounded, closed, and convex subset of X and $T : W \to W$ be a compact and continuous map. Then T has at least one fixed point in W.*

Theorem 2.28 (Banach contraction mapping principle, [40]). *Let N be a non-empty complete metric space and let $T : N \to N$ be a contraction map, then T has a unique fixed point in N, that is, there exists $x_0 \in N$ such that $Tx_0 = x_0$.*

3. Main results

3.1. Fractional delay differential equations without impulse

Theorem 3.1. *Assume the following conditions are satisfied.*

(D_1) *The function $f : Q \times C \to \mathbb{R}^n$ is continuous.*
(D_1) *There exist positive constants μ such that $\|f(t,x)\| \leqslant \mu$, for each $t \in Q$ and $x \in \mathbb{R}$.*

Then the IVP (1.1) has a solution on Q.

We transform the Atangana-Baleanu delay integral equation (2.2) into a fixed point problem by defining an operator T, such that the mapping $C(Q,\mathbb{R}) \to C(Q,\mathbb{R})$,

$$T(x(t)) = \begin{cases} \psi_0, & t \in [-r,0], \\ \psi_0(0) + \frac{1-\beta}{M(\beta)} f(t,x_t) + \frac{\beta}{M(\beta)\Gamma(\beta} \int_{t_0}^{t} (t-p)^{\beta-1} f(p,x_p)\,dp, & t \in [t_0,T]. \end{cases} \tag{3.1}$$

Proof. Using Schauder fixed point theorem we prove that T defined by (3.1) has a fixed point. The proof will be given in steps.

Step 1. $T : C(Q,\mathbb{R}) \to C(Q,\mathbb{R})$,

$$\begin{aligned} \|T(x(t))\| &\leqslant \|\psi_0(0)\| + \frac{1-\beta}{M(\beta)}\|f(t,x_t)\| + \frac{\beta}{M(\beta)\Gamma(\beta)} \int_{t_0}^{t} (t-p)^{\beta-1}\|f(p,x_p)\|\,dp, \\ &\leqslant \|\psi_0(0)\| + \frac{1-\beta}{M(\beta)}\mu + \frac{\beta\mu}{M(\beta)\Gamma(\beta)} \int_{t_0}^{t} (t-p)^{\beta-1}\,dp, \\ \|T(x(t))\|_\infty &\leqslant \|\psi(0)\| + \frac{1-\beta}{M(\beta)}\mu + \frac{\mu T^\beta}{M(\beta)\Gamma(\beta)} = \rho, \\ T &: C([t_0,T]) \to C([t_0,T]), \end{aligned}$$

Step 2. Let $\{\psi^n\}$ be a sequence such that $x^n(t) \to x(t)$ on $C([t_0,T])$ and $f(t,x^n(t)) \to f(t,x(t))$ and

$$\begin{aligned} \|T(x^n(t)) - T(x(t))\| &\leqslant \frac{1-\beta}{M(\beta)}\|f(t,x_t^n) - f(t,x_t)\| + \frac{\beta}{M(\beta)\Gamma(\beta)} \int_{t_0}^{t} (t-p)^{\beta-1}\|f(p,x_p^n) - f(p,x_p)\|\,dp \\ &\leqslant \frac{1-\beta}{M(\beta)}\|f(t,x_t^n) - f(t,x_t)\| + \frac{\beta}{M(\beta)\Gamma(\beta+1)} T^\beta \|f(.,x_.^n) \to f(t,x_.)\| \\ &\leqslant \frac{1-\beta}{M(\beta)}\|f(.,x_.^n) \to f(.,x_.,)\|_\infty + \frac{\beta}{M(\beta)\Gamma(\beta+1)} T^\beta \|f(.,x_.^n) - f(.,x_.)\|_\infty \end{aligned}$$

as $n \to \infty$, $f(.,x_.^n) \to f(.,x_.)$. Therefore $\|T(x^n(t)) - T(x(t))\|_\infty \to 0$ as $n \to \infty$. This shows that T is continuous.

Step 3. T maps bounded sets into equicontinuous sets on Q,

$$
\begin{aligned}
&\|Tx(t_2) - Tx(t_1)\| \\
&\leqslant + \frac{1-\beta}{M(\beta)}\|f(t_2, x_{t_2}) - f(t_1, x_{t_1}) \\
&\quad + \frac{\beta}{M(\beta)\Gamma(\beta)}\Big\| \int_{t_0}^{t_1} ((t_1 - p)^{\beta-1} - (t_2 - p)^{\beta-1}) f(p, x_p)\,dp + \int_{t_1}^{t_2} (t_2 - p)^{\beta-1} f(p, x_p)\Big\| dp \\
&\leqslant + \frac{1-\beta}{M(\beta)}\|f(t_2, x_{t_2}) - f(t_1, x_{t_1})\| + \frac{\beta}{M(\beta)\Gamma(\beta)} \int_{t_0}^{t_1} \Big\|((t_1 - p)^{\beta-1} - (t_2 - p)^{\beta-1}) f(p, x_p)\Big\| dp \\
&\quad + \frac{\beta}{M(\beta)\Gamma(\beta)} \int_{t_1}^{t_2} \Big\|(t_2 - s)^{\beta-1} f(p, x_p)\Big\| dp \\
&\leqslant + \frac{1-\beta}{M(\beta)}|f(t_2, x_{t_2}) - f(t_1, x_{t_1}) \\
&\quad + \frac{\beta}{M(\beta)\Gamma(\beta)} \int_{t_0}^{t_1} ((t_1 - p)^{\beta-1} - (t_2 - p)^{\beta-1})\mu\,dp + \int_{t_1}^{t_2} (t_2 - p)^{\beta-1}\mu\,dp \\
&\leqslant + \frac{1-\beta}{M(\beta)}\|f(t_2, x_{t_2}) - f(t_1, x_{t_1})\| + \frac{\beta}{M(\beta)\Gamma(\beta)} \int_{t_0}^{t_1} (t_1 - p)^{\beta-1} - (t_2 - p)^{\beta-1}\mu\,dp \\
&\quad + \frac{\beta}{M(\beta)\Gamma(\beta)} \int_{t_1}^{t_2} (t_2 - p)^{\beta-1}\mu\,dp \\
&\leqslant + \frac{1-\beta}{M(\beta)} + \|f(t_2, x_{t_2}) - f(t_1, x_{t_1})\| + \frac{\beta\mu}{M(\beta)\Gamma(\beta+1)}\big((t_1 - t_0)^{\beta} + (t_2 - t_1)^{\beta} - (t_2 - t_0)^{\beta}\big) \\
&\quad + \frac{\beta\mu}{M(\beta)\Gamma(\beta+1)}(t_2 - t_1)^{\beta} \leqslant + \frac{1-\beta}{M(\beta)} + \|f(t_2, x_{t_2}) - f(t_1, x_{t_1})\| + \frac{2\mu}{M(\beta)\Gamma(\beta)}(t_2 - t_1)^{\beta}.
\end{aligned}
$$

As $t_2 \to t_1$ the right-hand side of the above inequality tends to zero, then $Tx(t)$ is equicontinuous on $t \in [t_0, T]$. Thus, using the Arzelà-Ascoli theorem for equicontinuous functions, we conclude that $Tx(t)$ is relatively compact, and hence T is completely continuous on $C(Q, \mathbb{R})$. By Schauder fixed point theorem, T has a fixed point in $C(Q, \mathbb{R})$, which is a solution of the IVP (1.1) on Q. □

Next we show that there exists a unique solution of equation (1.1) on $t \in [t_0, T]$ using the Banach contraction mapping principle.

Theorem 3.2. *Assume the conditions of Theorem 3.3 are satisfied with the following additional condition.*

(D_4) There exist positive constants ν such that $\|f(t, x_t) - f(t, y_t)\| \leqslant \nu\|x - y\|$.

Then the IVP (1.1) has a unique solution on Q provided

$$0 < \Big[\frac{1-\beta}{M(\beta)}\nu + \frac{\nu}{M(\beta)\Gamma(\beta)}T^{\beta}\Big] < 1.$$

Proof.

$$
\begin{aligned}
\|Tx(t) - Ty(t)\| &\leqslant \frac{1-\beta}{M(\beta)}\|f(t, x_t) - f(t, y_t)\| + \frac{\beta}{M(\beta)\Gamma(\beta)}\Big\| \int_{t_0}^{t} (t - p)^{\beta-1}[f(p, x_p) - f(p, y_p)]\,dp\Big\| \\
&\leqslant \frac{1-\beta}{M(\beta)}\nu\|x - y\| + \frac{\beta}{M(\beta)\Gamma(\beta)} \int_{t_0}^{t} (t - p)^{\beta-1}\nu\|x_p - y_p\|\,dp \\
&\leqslant \frac{1-\beta}{M(\beta)}\nu\|x - y\| + \frac{\beta}{M(\beta)\Gamma(\beta)} \int_{t_0}^{t} (t - p)^{\beta-1}\nu\|x_p - y_p\|_{\infty}
\end{aligned}
$$

$$\leqslant \left(\frac{1-\beta}{M(\beta)}\nu + \frac{\nu}{M(\beta)\Gamma(\beta)}T^{\beta}\right)\|x-y\|_{\infty},$$

where

$$\frac{1-\beta}{M(\beta)}\nu + \frac{\nu}{M(\beta)\Gamma(\beta)}T^{\beta} < 1.$$

It follows that T is a strict contraction. Therefore there exists a unique solution of the IVP (1.1). □

3.2. Impulsive fractional delay differential equations

Theorem 3.3. *Assume the following conditions are satisfied.*

(F_1) *The function* $f : Q \times PC \to \mathbb{R}^n$ *is continuous.*

(F_2) *There exist positive constants* μ *such that* $\|f(t,x)\| \leqslant \mu$, *for each* $t \in Q$ *and* $x \in \mathbb{R}$.

(F_3) *There exists a positive constant* μ_1 *such that* $\|I_\tau(t_\tau, x)\| \leqslant \mu_1$ *for each* $x \in PC(Q,\mathbb{R})$ *and* $\tau = 1,2,\ldots,p$.

Then the IVP (1.2) *has a solution on* Q.

We transform the Atangana-Baleanu impulsive delay equation (1.2) into a fixed point problem.

$$T(x_{t_0}) = \theta_0, \quad t \in [-r, 0],$$

$$T(x(t)) = \begin{cases} \psi_0(0) + \frac{1-\beta}{M(\beta)}f(t,x_t) + \frac{\beta}{M(\beta)\Gamma(\beta)}\int_{t_0}^{t}(t-p)^{\beta-1}f(p,x_p)dp, & t \in [t_0 T], \\ \psi_0(0) + \frac{1-\beta}{M(\beta)}f(t,x_t) \\ \quad + \frac{\beta}{M(\beta)\Gamma(\beta)}\int_{t_0}^{t}(t-p)^{\beta-1}f(p,x_p)dp + \sum_{\tau=1}^{k} I_\tau(t_\tau, x(t_\tau)), & t \in (t_\tau, t_{\tau+1}],\ \tau = 1,\ldots,p. \end{cases}$$

Proof. Using Schauder fixed point theorem we prove that T defined by (3.1) has a fixed point. The proof will be given in steps.

Step 1: $T : PC(Q,\mathbb{R}) \to PC(Q,\mathbb{R})$. For $t \in [t_\tau, t_{\tau+1}]$,

$$\|T(x(t))\| \leqslant \|\psi_0(0)\| + \frac{1-\beta}{M(\beta)}\|f(t,x_t)\| + \frac{\beta}{M(\beta)\Gamma(\beta)}\int_{t_0}^{t}(t-p)^{\beta-1}\|f(p,x_p)\|dp + \sum_{\tau=1}^{k}\|I_\tau(t_\tau, x(t_\tau))\|,$$

$$\leqslant \|\psi_0(0)\| + \frac{1-\beta}{M(\beta)}\mu + \frac{\beta\mu}{M(\beta)\Gamma(\beta)}\int_{t_0}^{t}(t-p)^{\beta-1}dp + \sum_{\tau=1}^{k}\|I_\tau(t_\tau, x(t_\tau))\|,$$

$$\|T(x(t))\|_{\infty} \leqslant \|\psi(0)\| + \frac{1-\beta}{M(\beta)}\mu + \frac{\mu T^{\beta}}{M(\beta)\Gamma(\beta)} + p\mu_1 = \rho,$$

$$T : PC([t_0, T]) \to PC([t_0, T]).$$

Step 2: Let $\{\psi^n\}$ be a sequence such that $x_n(t) \to x(t)$ on $PC([t_0,T])$ and $f(t,x^n(t)) \to f(t,x(t))$ and $I_\tau(t_\tau, x^n(t)) \to I_\tau(t_\tau, x(t))$ on $PC(Q,\mathbb{R})$ as $n \to \infty$, $\tau = 1,\ldots,k$ for each $t \in [t_0,T]$, then for $t \in [t_\tau, t_{\tau+1}]$,

$$\|T(x^n(t)) - T(x(t))\|$$

$$\leqslant \frac{1-\beta}{M(\beta)}\|f(t,x_t^n) - f(t,x_t)\|$$

$$+ \frac{\beta}{M(\beta)\Gamma(\beta)}\int_{t_0}^{t}(t-p)^{\beta-1}\|f(p,x_p^n) - f(p,x_p)\|dp + \sum_{\tau=1}^{k}\|I_\tau(t_\tau, x_t^n) - I_\tau(t_\tau, x_t)\|,$$

$$\leqslant \frac{1-\beta}{M(\beta)}\|f(t,x_t^n) - f(t,x_t)\| + \frac{\beta}{M(\beta)\Gamma(\beta+1)}T^{\beta}\|f(.,x_.^n) \to f(t,x_.)\| + \sum_{0<t_\tau<t}\|I_\tau(.,x_.^n) - I_\tau(.,x_.^n)\|,$$

$$\leqslant \frac{1-\beta}{M(\beta)}\|f(.,x_.^n) \to f(.,x_.)\|_{\infty} + \frac{\beta}{M(\beta)\Gamma(\beta+1)}T^{\beta}\|f(.,x_.^n) - f(.,x_.)\|_{\infty} + k\|I_\tau(.,x_.^n) - I_\tau(.,x_.)\|_{\infty},$$

as $n \to \infty$, $f(.,x_.^n) \to f(.,x_.)$, and $I_\tau(t_\tau, x_t^n) \to I_\tau(t_\tau, x_t)$, therefore $\|T(x^n(t)) - T(x(t))\|_{\infty} \to 0$ as $n \to \infty$ This show that T is continuous.

Step 3: T maps bounded sets into equicontinuous sets on Q. Let $t_1, t_2 \in t \in [t_\tau, t_{\tau+1}]$ and $t_1 < t_2$, $PC(Q, \mathbb{R})$ be a bounded set as in Step 1 and Step 2 and $x \in PC(Q, \mathbb{R})$. We obtain

$$
\begin{aligned}
&\|Tx(t_2) - Tx(t_1)\| \\
&\leqslant + \frac{1-\beta}{M(\beta)}\|f(t_2, x_{t_2}) - f(t_1, x_{t_1})\| \\
&\quad + \frac{\beta}{M(\beta)\Gamma(\beta)}\Big\| \int_{t_0}^{t_1} ((t_1 - p)^{\beta-1} - (t_2 - p)^{\beta-1}) f(p, x_p)\,dp + \int_{t_1}^{t_2} (t_2 - p)^{\beta-1} f(p, x_p)\Big\| dp + \|I_\tau(t_\tau, x_{t_\tau})\| \\
&\leqslant + \frac{1-\beta}{M(\beta)}\|f(t_2, x_{t_2}) - f(t_1, x_{t_1})\| + \frac{\beta}{M(\beta)\Gamma(\beta)} \int_{t_0}^{t_1} \Big\|((t_1 - p)^{\beta-1} - (t_2 - p)^{\beta-1}) f(p, x_p)\Big\| dp \\
&\quad + \frac{\beta}{M(\beta)\Gamma(\beta)} \int_{t_1}^{t_2} \Big\|(t_2 - s)^{\beta-1} f(p, x_p)\Big\| dp + \|I_\tau(t_\tau, x_{t_\tau})\| \\
&\leqslant + \frac{1-\beta}{M(\beta)}|f(t_2, x_{t_2}) - f(t_1, x_{t_1})\| \\
&\quad + \frac{\beta}{M(\beta)\Gamma(\beta)} \int_{t_0}^{t_1} ((t_1 - p)^{\beta-1} - (t_2 - p)^{\beta-1})\mu\,dp + \int_{t_1}^{t_2} (t_2 - p)^{\beta-1}\mu\,dp + \|I_\tau(t_\tau, x_{t_\tau})\| \\
&\leqslant + \frac{1-\beta}{M(\beta)}\|f(t_2, x_{t_2}) - f(t_1, x_{t_1})\| + \frac{\beta}{M(\beta)\Gamma(\beta)} \int_{t_0}^{t_1} (t_1 - p)^{\beta-1} - (t_2 - p)^{\beta-1}\mu\,dp \\
&\quad + \frac{\beta}{M(\beta)\Gamma(\beta)} \int_{t_1}^{t_2} (t_2 - p)^{\beta-1}\mu\,dp + \|I_\tau(t_\tau, x_{t_\tau})\| \\
&\leqslant + \frac{1-\beta}{M(\beta)} + \|f(t_2, x_{t_2}) - f(t_1, x_{t_1})\| + \frac{\beta\mu}{M(\beta)\Gamma(\beta+1)}\big((t_1 - t_0)^\beta + (t_2 - t_1)^\beta - (t_2 - t_0)^\beta\big) \\
&\quad + \frac{\beta\mu}{M(\beta)\Gamma(\beta+1)}(t_2 - t_1)^\beta \leqslant + \frac{1-\beta}{M(\beta)} + \|f(t_2, x_{t_2}) - f(t_1, x_{t_1})\| + \frac{2\mu}{M(\beta)\Gamma(\beta)}(t_2 - t_1)^\beta.
\end{aligned}
$$

As $t_2 \to t_1$ the right-hand side of the above inequality tends to zero, then $Tx(t)$ is equicontinuous on $t \in [t_\tau, t_{\tau+1}]$, hence $Tx(t)$ is equicontinuous on $t \in [t_\tau, t_{\tau+1}]$, therefore $Tx(t)$ is piecewise equicontinuous on Q. Thus, using the Arzelà-Ascoli theorem for equicontinuous functions, we conclude that $Tx(t)$ is relatively compact, and hence T is completely continuous on $PC(Q, \mathbb{R})$. By Schauder Fixed point theorem, T has a fixed point in $PC(Q, \mathbb{R})$, which is a solution of the IVP (1.2) on Q. □

Next we prove uniqueness theorem using the Banach contraction mapping principle.

Theorem 3.4. *Assume the conditions of Theorem 3.3 are satisfied with following additional conditions.*

- (F_4) *There exist positive constants* ν *such that* $\|f(t, x_t) - f(t, y_t)\| \leqslant \nu\|x - y\|$.
- (F_5) *There exists a positive constant* ν_1 *such that* $\|I_\tau(t_\tau, x) - I_\tau(t_\tau, y)\| \leqslant \nu_1\|x - y\|$, *for each* $x, y \in PC(Q, \mathbb{R})$ *and* $\tau = 1, 2, \ldots, k$.

Then the IVP (1.2) has a unique solution on Q *provided*

$$0 < \Big[\frac{1-\beta}{M(\beta)}\nu + \frac{\nu}{M(\beta)\Gamma(\beta)}T^\beta + p\nu_1\Big] < 1.$$

Proof.

$$
\begin{aligned}
\|Tx(t) - Ty(t)\| &\leqslant + \frac{1-\beta}{M(\beta)}\|f(t, x_t) - f(t, y_t)\| \\
&\quad + \frac{\beta}{M(\beta)\Gamma(\beta)}\Big\| \int_{t_0}^{t} (t - p)^{\beta-1}[f(p, x_p) - f(p, y_p)]\,dp + \sum_{0<t_\tau<t} \|I_\tau(t_\tau, x_t) - I_\tau(t_\tau, y_t)\|
\end{aligned}
$$

$$\begin{aligned}
&\leqslant \frac{1-\beta}{M(\beta)}\nu\|x-y\| + \frac{\beta}{M(\beta)\Gamma(\beta)}\int_{t_\tau}^{t}(t-p)^{\beta-1}\nu\|x_p-y_p\|dp + \sum_{0<t_\tau<t}\nu_1 \max_{t\in Q}\|x-y\| \\
&\leqslant \frac{1-\beta}{M(\beta)}\nu\|x-y\| + \frac{\beta}{M(\beta)\Gamma(\beta)}\int_{t_0}^{t}(t-p)^{\beta-1}\nu\|x_p-y_p\|_\infty + k\nu_1\|x-y\|_\infty \\
&\leqslant \Big(\frac{1-\beta}{M(\beta)}\nu + \frac{\nu}{M(\beta)\Gamma(\beta)}T^\beta + k\nu_1\Big)\|x-y\|_\infty,
\end{aligned}$$

where

$$\frac{1-\beta}{M(\beta)}\nu + \frac{\nu}{M(\beta)\Gamma(\beta)}T^\beta + k\nu_1 < 1.$$

It follows that T is a strict contraction. Therefore there exists a unique fixed point which is a unique solution of the IVP (1.2).

□

3.3. Stability of fractional delay differential equation

In this section, the stability of the solution of equation (1.1) is examined.

Theorem 3.5. *For any two solutions $x(t)$ and $y(t)$ with initial conditions $x_{t_0} = \psi_0$, where $\psi_0 \in C$ and $y_{t_0} = \phi_0$, where $\phi_0 \in C$, respectively, moreover, for very small $\epsilon > 0$, there exists $\delta = (1-\lambda)\epsilon$. Then the solution of equation (1.1) is stable.*

Proof. Let $x(t)$ be a solution of (1.2), and $y(t)$ be a solution of (1.2) satisfying the initial value condition $y_{t_0} = \phi_0$, where $\phi_0 \in C(Q, \mathbb{R})$, then

$$(x(t)-y(t)) = \begin{cases} (\psi_0(0)-\phi_0(0)) + \frac{1-\beta}{M(\beta)}(f(t,x_t)-f(t,y_t)) \\ \quad + \frac{\beta}{M(\beta)\Gamma(\beta)}\int_{t_0}^{t}(t-p)^{\beta-1}(f(p,x_p)-f(p,y_p))dp, t\in[t_0,T]. \end{cases}$$

For $t \in [t_0, T]$,

$$\begin{aligned}
\|x(t)-y(t)\| &= \|\psi_0(0)-\phi_0(0)\| + \frac{1-\beta}{M(\beta)}\|(f(t,x_t)-f(t,y_t))\| \\
&\quad + \frac{\beta}{M(\beta)\Gamma(\beta)}\Big\|\int_{t_0}^{t}(t-p)^{\beta-1}[(f(p,x_p)-f(p,y_p))]dp \\
&\leqslant \|\psi_0(0)-\phi_0(0)\| + \frac{1-\beta}{M(\beta)}\nu\|\beta_0(0)-\phi_0(0)\| + \frac{\nu\beta}{M(\beta)\Gamma(\beta)}\int_{t_0}^{t}(t-p)^{\beta-1}dp\|\psi-\phi\| \\
&\leqslant \|\psi_0(0)-\phi_0(0)\| + \frac{1-\beta}{M(\beta)}\nu\|x-y\|_\infty + \frac{\beta\nu}{M(\beta)\Gamma(\beta)}T^\beta\|x-y\|_\infty \\
&\leqslant \|\psi_0(0)-\phi_0(0)\| + \frac{1-\beta}{M(\beta)}\nu\|x-y\|_\infty + \frac{\beta\nu}{M(\beta)\Gamma(\beta)}T^\beta\|x-y\|_\infty \\
&\leqslant \|\psi_0(0)-\phi_0(0)\| + \lambda\|x-y\|_\infty,
\end{aligned}$$

since $\lambda < 1$, we have

$$\|x-y\|_\infty \leqslant \|\psi_0-\phi_0\| + \lambda\|x-y\|_\infty, \quad \|x-y\|_\infty \leqslant \frac{1}{1-\lambda}\|\psi_0-\phi_0\|,$$

where

$$\lambda = \frac{1-\beta}{M(\beta)}\nu + \frac{\beta\nu}{M(\beta)\Gamma(\beta)}T^\beta$$

accounts for the fractional growth. There exist $\delta = (1-\lambda)\epsilon$, such that if $\|\psi-\phi\| \leqslant (1-\lambda)\epsilon$, then $\|x-y\| \leqslant \epsilon$ implies that the solution of (1.2) is stable. □

3.4. Stability of impulsive fractional delay differential equation

In this section, the stability of the solution of equation (1.2) is examined.

Theorem 3.6. *For any two solutions $x(t)$ and $y(t)$ with initial conditions $x_{t_0} = \psi_0$, where $\theta_0 \in PC$ and $y_{t_0} = \phi_0$, where $\phi_0 \in PC(Q, \mathbb{R})$, respectively, moreover, for very small $\epsilon > 0$, there exists $\delta = (1-c)\epsilon$. Then the solution of equation (1.2) is stable.*

Proof. Let $x(t)$ be a solution of (1.2), and $y(t)$ be a solution of (1.2) satisfying the initial value condition $y_{t_0} = \phi_0$, where $\phi_0 \in PC(Q, \mathbb{R})$, then

$$
\begin{aligned}
&(x(t) - y(t)) \\
&= \begin{cases} (\psi_0(0) - \phi_0(0)) + \frac{1-\beta}{M(\beta)}(f(t, x_t) - f(t, y_t)) + \frac{\beta}{M(\beta)\Gamma(\beta)} \int_{t_0}^{t} (t-p)^{\beta-1}(f(p, x_p) - f(p, y_p))dp, \ t \in [t_0, T], \\ (\psi_0(0) - \phi_0(0)) + \frac{1-\beta}{M(\beta)}(f(t, x_t) - f(t, y_t)) + \frac{\beta}{M(\beta)\Gamma(\beta)} \int_{t_0}^{t} (t-p)^{\beta-1}(f(p, x_p) - f(p, y_p))dp \\ \quad + \sum_{\tau=1}^{k} I_\tau((t_\tau, x(t_\tau)) - (t_\tau, y(t_\tau))), t \in (t_\tau, t_{\tau+1}], \ \tau = 1, \ldots, k. \end{cases}
\end{aligned}
$$

For $t \in (t_\tau, t_{\tau+1}]$,

$$
\begin{aligned}
\|x(t) - y(t)\| &= \|\psi_0(0) - \phi_0(0)\| + \frac{1-\beta}{M(\beta)}\|(f(t, x_t) - f(t, y_t))\| \\
&\quad + \frac{\beta}{M(\beta)\Gamma(\beta)}\Big\| \int_{t_0}^{t} (t-p)^{\beta-1}[f(p, x_p) - f(p, y_p)]\|dp + \sum_{\tau=1}^{k} \|I_\tau(x_{t_\tau}, \psi(0)) - I_\tau(y_{t_\tau}, \phi(0))\| \\
&\leqslant \|\psi_0(0) - \phi_0(0)\| + \frac{1-\beta}{M(\beta)}\nu\|\psi - \phi\| \\
&\quad + \frac{\beta}{M(\beta)\Gamma(\beta)} \int_{t_0}^{t} (t-p)^{\beta-1}dp\nu\big\|\psi - \phi\big\| + \sum_{\tau=1}^{k} \mu^* \max_{t \in Q} \|\psi(0) - \phi(0)\| \\
&\leqslant \|\psi_0(0) - \phi_0(0)\| + \frac{1-\beta}{M(\beta)}\nu\big\|x - y\big\|_\infty + \frac{\beta\nu}{M(\beta)\Gamma(\beta)}T^\beta\big\|x - y\big\|_\infty + k\mu^*\big\|x - y\big\|_\infty \\
&\leqslant \|\psi_0(0) - \phi_0(0)\| + \Big(\frac{1-\beta}{M(\beta)}\nu + \frac{\beta\nu}{M(\beta)\Gamma(\beta)}T^\beta + k\mu^*\Big)\big\|x - y\big\|_\infty \leqslant \|\psi_0(0) - \phi_0(0)\| + c\big\|x - y\big\|_\infty,
\end{aligned}
$$

since $c < 1$, we have

$$
\|x - y\|_\infty \leqslant \|\theta_0 - \phi_0\| + c\big\|x - y\big\|_\infty, \quad \|x - y\|_\infty \leqslant \frac{1}{1-c}\|\psi_0 - \phi_0\|,
$$

where

$$
c = \frac{1-\beta}{M(\beta)}\nu + \frac{\beta\nu}{M(\beta)\Gamma(\beta)}T^\beta + k\mu^*
$$

accounts for the fractional growth. $p\mu^*$ represents impulse contributions. There exist $\delta = (1-c)\epsilon$, such that if $\|\psi - \phi\| \leqslant (1-c)\epsilon$, then $\|x - y\| \leqslant \epsilon$ implies that the solution of (1.2) is stable. □

4. Multiline equations example

$$
\begin{aligned}
{}^{ABC}D^\beta x(t) &= \frac{x_t}{10(1+e^t)(1+x_t)}, \quad t \in Q = [0,1], \quad t \neq t_1 \\
\Delta x(t)|_{t=t_1} &= \frac{1}{2}, \quad x(t) = \psi(t), \quad t \in [-r, 0], \\
f(t, \psi) &= \frac{\psi}{10(1+e^t)(1+\psi)}.
\end{aligned}
$$

Let $\psi_1, \psi_2 \in PC([-r, 0], \mathbb{R}^n)$ and let $t \in [0, 1]$, then, we have

$$\begin{aligned}|f(t, \psi_1) - f(t, \psi_2)| &= \frac{1}{10(1+e^t)}\Big|\frac{\psi_1}{1+\psi_1} - \frac{\psi_2}{1+\psi_2}\Big| \\ &= \frac{\|\psi_1 - \psi_2\|}{10(1+e^t)(1+\psi_1)(1+\phi_2)} \leqslant \frac{\|\psi_1 - \psi_2\|}{10(1+e^t)} \leqslant \frac{1}{20}\|\psi_1 - \psi_2\|\end{aligned}$$

for all $\psi \in PC([-r, 0], \mathbb{R}^n)$ and $t \in [0, T]$,

$$|f(t, \psi)| = \frac{1}{10(1+e^t)}\Big\|\frac{\psi}{1+\psi}\Big\| \leqslant \frac{1}{10(1+e^t)} = \frac{1}{20}.$$

According to the inequality

$$\frac{1-\beta}{M(\beta)}\nu + \frac{\nu}{M(\beta)\Gamma(\beta)}T^\beta + k\nu_1 < 0.11.$$

5. Applications to Hutchinson's model

In this section, impulsive fractional biological systems involving derivative with Mittag-Leffler kernel will be proposed. Existence, uniqueness and stability of the solutions will be investigated. Considered the integer order delay differential equation of form

$$\dot{x}(t) = zx(t)\big(1 - \frac{x(t-r)}{C}\big), \quad t \geqslant 0. \tag{5.1}$$

Many writers have explored the biological system (5.1), which is a delayed generalisation of the basic logistic equation. Hutchinson's equation (5.1), is a model for population expansion in a single species [23, 33]. The impulsive generalisation of the model (5.1) is the situation in which impulsive causes suddenly affect the population at specific times, causing the population number $x(t)$ to fluctuate by jumps. In our work, we use the fixed point approach to examine the existence, uniqueness, and stability of this model by extending it to fractional derivative with the Mittag-Leffler kernel. This is the generalised model:

$$\begin{cases} {}^{ABC}D^\beta x(t) = zx(t)\big(1 - \frac{x(t-r)}{C}\big), \ t \geqslant 0, \ t \neq t_\tau, \\ \Delta x(t_\tau) = I_\tau(t_\tau, x(t_\tau)), \ t = t_\tau, \ \tau = 1, 2, \ldots, k, \\ x(t) = \theta(t), \end{cases} \tag{5.2}$$

where ${}^{ABC}D^\beta x(t)$ denote the fractional derivative with Mittag-Leffler kernel $z > 0$ is the intrinsic growth rate, $C > 0$ is the carrying capacity, $r > 0$ is the delay parameter, t_τ $(\tau = 1, 2, \ldots, k)$ are the impulsive times with $0 < t_1 < t_2 < \cdots < t_\tau < T$, and I_τ are the impulse functions. To demonstrate qualitative properties of this model, we state the following hypotheses.

1. For $\theta \in PC([-r, 0], \mathbb{R})$ we set the initial data to $x(0) = \theta(0)$.

2. For a fix $A > 0$ with $\max\{\|\theta\|_{[-r,0]}, 1\}$, we have solutions that satisfy the bound $|x(t)| \leqslant A$ on the interval $[0, T]$.

3. We compute the local Lipschitz constant for Hutchinson nonlinearity on the closed ball $\{(u, v) : |u| \leqslant A, |v| \leqslant A\}$, the map $(u, v) \to ru\big(1 - \frac{v}{C}\big)$ is Lipschitz with constant $r(1 + \frac{A}{C})$, for verification, $|u_i|, |v_i| \leqslant A$,

$$\begin{aligned}\big|ru_1\big(1 - \frac{v_1}{C}\big) - ru_2\big(1 - \frac{v_2}{C}\big)\big| &= r\big|(u_1 - u_2)\big(1 - \frac{v_1}{C}\big) + u_2\big(1 - \frac{v_2 - v_1}{C}\big)\big| \\ &\leqslant r\big(\big|1 - \frac{v_1}{C}\big||u_1 - u_2| + \frac{|u_2|}{C}|v_2 - v_1|\big).\end{aligned}$$

Using the coefficient $\left|1-\frac{\nu_1}{C}\right| \leqslant 1+\frac{|\nu_1|}{C}$ with $|\nu_1| \leqslant A$, this gives $\left|1-\frac{\nu_1}{C}\right| \leqslant 1+\frac{A}{C}$. We have the cases

$$|1-\frac{\nu_1}{C}| \leqslant 1+\frac{|\nu_1|}{C} \leqslant 1+\frac{A}{C}, \quad |\frac{u_2}{C}| \leqslant \frac{A}{C} \leqslant 1+\frac{A}{C}, \quad \frac{A}{C} \geqslant 0.$$

Therefore, we have

$$\begin{aligned}
&\left|ru_1\left(1-\frac{\nu_1}{C}\right)-ru_2\left(1-\frac{\nu_2}{C}\right)\right| = r\left(\left(1+\frac{A}{C}\right)|u_1-u_2|+\frac{A}{C}|\nu_2-\nu_1|\right) \leqslant r\left(1+\frac{A}{C}\right)\left(|u_1-u_2|+|\nu_2-\nu_1|\right),\\
&\left|ru_1\left(1-\frac{\nu_1}{C}\right)-ru_2\left(1-\frac{\nu_2}{C}\right)\right| \leqslant r\left(1+\frac{A}{C}\right)\left(|u_1-u_2|+|\nu_2-\nu_1|\right).
\end{aligned}$$

4. Impulse Lipschitz constant, each I_τ is Lipschitz on $(t_\tau, t_{\tau+1})$ with Lipschitz constant L_τ,

$$|I_\tau(u_1,\nu_1)-I_\tau(u_2,\nu_2)| \leqslant L_\tau(|u_1-u_2|+|\nu_1,\nu_2|)\forall u,\nu \in (t_\tau,t_{\tau+1}).$$

To prove the existence and uniqueness theorem, we transform the Atangana-Baleanu impulsive delay issue (5.2) into a fixed point problem. For the operator T, we defined a mapping as follows: $PC(Q,\mathbb{R}) \to PC(Q,\mathbb{R})$,

$$\begin{aligned}
&T(x_{t_0}) = \theta_0,\\
&T(x(t)) = \begin{cases}
\theta_0(0)+\frac{1-\beta}{M(\beta)}zx(t)\left(1-\frac{x(t-r)}{C}\right)+\frac{\beta}{M(\beta)\Gamma(\beta)}\int_{t_0}^{t}(t-p)^{\beta-1}zx(p)\left(1-\frac{x(p-r)}{C}\right)dp, \ t\in[t_0,T],\\
\theta_0(0)+\frac{1-\beta}{M(\beta)}zx(t)\left(1-\frac{x(t-p)}{C}\right)+\frac{\beta}{M(\beta)\Gamma(\beta)}\int_{t_0}^{t}(t-p)^{\beta-1}zx(p)\left(1-\frac{x(p-r)}{C}\right)dp\\
\quad +\sum_{\tau=1}^{k}I_\tau(t_\tau,x(t_\tau)), \ t\in(t_\tau,t_{\tau+1}], \ \tau=1,\dots,k,
\end{cases}
\end{aligned} \tag{5.3}$$

Next we prove using the Banach contraction mapping principle that T has a unique fixed point for $t \in [t_\tau, t_{\tau+1}]$,

$$\begin{aligned}
&\|Tx(t)-Ty(t)\|\\
&\leqslant +\frac{1-\beta}{M(\beta)}\|zx(t)\left(1-\frac{x(t-r)}{C}\right)-zy(t)\left(1-\frac{y(t-r)}{C}\right)\|\\
&\quad +\frac{\beta}{M(\beta)\Gamma(\beta)}\Big\|\int_{t_0}^{t}(t-p)^{\beta-1}\left[zx(p)\left(1-\frac{x(p-r)}{C}\right)-zy(p)\left(1-\frac{y(p-r)}{C}\right)\right]dp\\
&\quad +\sum_{0<t_\tau<t}\|I_\tau(t_\tau,x_t)-I_\tau(t_\tau,y_t)\|\\
&\leqslant \frac{1-\beta}{M(\beta)}\left\|zu_1\left(1-\frac{\nu_1}{C}\right)-zu_2\left(1-\frac{\nu_2}{C}\right)\right\|\\
&\quad +\frac{\beta}{M(\beta)\Gamma(\beta)}\Big\|\int_{t_0}^{t}(t-p)^{\beta-1}\left[\left|zu_1\left(1-\frac{\nu_1}{C}\right)-zu_2\left(1-\frac{\nu_2}{C}\right)\right|\right]ds+\sum_{0<t_\tau<t}\|I_\tau(t_\tau,x_t)-I_\tau(t_\tau,y_t)\|\\
&\leqslant \frac{1-\beta}{M(\beta)}\left\|z\left(\left(1+\frac{A}{C}\right)|u_1-u_2|+\frac{A}{C}|\nu_2-\nu_1|\right)\right\|\\
&\quad +\frac{\beta}{M(\beta)\Gamma(\beta)}\Big\|\int_{t_0}^{t}(t-p)^{\beta-1}\left[\left|z\left(\left(1+\frac{A}{C}\right)|u_1-u_2|+\frac{A}{C}|\nu_2-\nu_1|\right)\right|\right]dp+\sum_{0<t_\tau<t}\|I_\tau(t_\tau,x_t)-I_\tau(t_\tau,y_t)\|\\
&\leqslant \frac{1-\beta}{M(\beta)}\left\|z\left(\left(1+\frac{A}{C}\right)|u_1-u_2|+\frac{A}{C}|\nu_2-\nu_1|\right)\right\|\\
&\quad +\frac{\beta}{M(\beta)\Gamma(\beta)}\Big\|\int_{t_0}^{t}(t-p)^{\beta-1}\left[\left|z\left(\left(1+\frac{A}{C}\right)|u_1-u_2|+\frac{A}{C}|\nu_2-\nu_1|\right)\right|\right]dp+\sum_{0<t_\tau<t}\|I_\tau(t_\tau,x_t)-I_\tau(t_\tau,y_t)\|\\
&\leqslant \frac{1-\beta}{M(\beta)}z\left(1+\frac{A}{C}\right)\left(\|u_1-u_2\|+\|\nu_2-\nu_1\|\right)
\end{aligned}$$

$$
\begin{aligned}
&+\frac{\beta}{M(\beta)\Gamma(\beta)}\int_{t_\tau}^{t}(t-p)^{\beta-1}z\big(1+\frac{A}{C}\big)\big(\|u_1-u_2\|+\|v_2-v_1\|\big)\,dp+\sum_{0<t_\tau<t}\nu_1\max_{t\in Q}\|x_t-y_t\|\\
&\leqslant\frac{1-\beta}{M(\beta)}z\big(1+\frac{A}{C}\big)\big(\|u_1-u_2\|+\|v_2-v_1\|\big)\\
&\quad+\frac{\beta}{M(\beta)\Gamma(\beta)}\int_{t_\tau}^{t}(t-p)^{\beta-1}z\big(1+\frac{A}{C}\big)\big(\|u_1-u_2\|+\|v_2-v_1\|\big)\,dp+kL_\tau\big(\|u_1-u_2\|+\|v_2-v_1\|\big)_\infty\\
&\leqslant\big(\frac{1-\beta}{M(\beta)}z\big(1+\frac{A}{C}\big)+\frac{1}{M(\beta)\Gamma(\beta)}T^\beta z\big(1+\frac{A}{C}\big)+kL_\tau\big)\big(\|u_1-u_2\|+\|v_2-v_1\|\big),
\end{aligned}
$$

where

$$
\big(\frac{1-\beta}{M(\beta)}z\big(1+\frac{A}{C}\big)+\frac{1}{M(\beta)\Gamma(\beta)}T^\beta z\big(1+\frac{A}{C}\big)+kL_\tau\big)<1.
$$

It follows that T is a strict contraction. Therefore there exists a unique fixed point which is a unique solution of the IVP (5.3). For any two solutions $x(t)$ and $y(t)$ with initial conditions $x_{t_0}=\theta_0$, where $\theta_0\in PC$ and $y_{t_0}=\phi_0$, where $\phi_0\in PC$, respectively, moreover, for very small $\epsilon>0$, there exists $\delta=(1-\big(\frac{1-\beta}{M(\beta)}z\big(1+\frac{A}{C}\big)+\frac{1}{M(\beta)\Gamma(\beta)}T^\beta z\big(1+\frac{A}{C}\big)+kL_\tau\big))\epsilon$,

$$
\begin{aligned}
&\|x-y\|_\infty\leqslant\|\theta_0-\phi_0\|+\big(\frac{1-\beta}{M(\beta)}z\big(1+\frac{A}{C}\big)+\frac{1}{M(\beta)\Gamma(\beta)}T^\beta z\big(1+\frac{A}{C}\big)+kL_\tau\big)\big(\|u_1-u_2\|+\|v_2-v_1\|\big),\\
&\|x-y\|_\infty\leqslant\frac{1}{1-\big(\frac{1-\beta}{M(\beta)}z\big(1+\frac{A}{C}\big)+\frac{1}{M(\beta)\Gamma(\beta)}T^\beta z\big(1+\frac{A}{C}\big)+kL_\tau\big)}\|\theta_0-\phi_0\|.
\end{aligned}
$$

Then the solution of equation (5.2) is stable.

6. Conclusions

This study examines the existence, uniqueness, and stability of solutions for impulsive fractional differential equations (with delay) and fractional delay differential equations with Mittag-Leffler Kernel (without impulses) by applying the Banach contraction principle and the Schuader fixed point. Theoretical understanding of fractional differential equations involving delay factors, impulse perturbations, and memory effects is thus expanded through practical applications. Lastly, a practical example and application of the developed notion to impulsive fractional delay biological systems with Mittag-Leffler kernel involving a single species population growth model known as the Hutchinson's equations with fractional order derivative were presented.

Acknowledgment

The authors express their sincere gratitude to the Editor and anonymous reviewers for their valuable comments and constructive suggestions, which have greatly improved the quality and clarity of this paper.

Funding

The authors extend their appreciation to Prince Sattam bin Abdulaziz University, Alkharj, Saudi Arabia for funding this research work through the project number PSAU/2025/01/33466.

References

[1] M. I. Abbas, *On the initial value problems for the Caputo–Fabrizio impulsive fractional differential equations*, Asian-Eur. J. Math., **2021** (2021), 1–12. 1, 2.26

[2] S. Abbas, M. Benchohra, J. J. Nieto, *Caputo-Fabrizio fractional differential equations with instantaneous impulses*, AIMS Math., **6** (2021), 2932–2946. 1, 2.27

[3] S. Abbas, M. Banerjee, S. Momani, *Dynamic Analysis of Fractional-order modified logistic model*, Comput. Math. Appl., **62** (2011), 1098–1104. 1

[4] T. Abdeljawad, *Fractional operators with exponential kernels and a Lyapunov type inequality*, Adv. Differ. Equ., **2017** (2017), 11 pages. 2.4

[5] T. Abdeljawad, *A Lyapunov type inequality for fractional operators with non singular Mittag-Leffler kernel*, J. Inequal. Appl., **2017** (2017), 11 pages. 2.15, 2.16

[6] T. Abdeljawad, D. Baleanu, *Discrete fractional differences with nonsingular discrete Mittag-Leffler kernels*, Adv. Differ. Equ., **2016** (2016), 18 pages. 2.14

[7] T. Abdeljawad, D. Baleanu, *On fractional derivatives with exponential kernel and their discrete versions*, Rep. Math. Phys., **80** (2017), 11–27. 2.5

[8] T. Abdeljawad, D. Baleanu, *Integration by parts and its applications of a new nonlocal fractional derivative with Mittag-Leffler nonsingular kernel*, J. Nonlinear Sci. Appl., **10** (2017), 1098–1107. 2.8

[9] M. S. Abdo, T. Abdeljawad, K. Shah, F. Jarad, *Study of impulsive problems under Mittag-Leffler power law*, Heliyon, **6** (2020), 1–8. 1

[10] K. H. Alam, Y. Rohen, N. Saleem, M. Aphane, A. Rzzaque, *Convergence of Fibonacci-Ishikawa iteration procedure for monotone asymptotically nonexpansive mappings*, J. Inequal. Appl., **2024** (2024), 14 pages. 1

[11] M. Al Nuwairan, A. G. Ibrahim, *Nonlocal impulsive differential equations and inclusions involving Atangana-Baleanu fractional derivative in infinite dimensional spaces*, AIMS Math., **8** (2023), 11752–11780. 1

[12] J. E. Ante, E. E. Abraham, U. E. Ebere, W. K. Udogworen, C. S. Akpan, *On the Global Existence of solution and Lyapunov Asymptotic Practical Stability for Nonlinear Impulsive Caputo Fractional Derivative via Comparison Principle*, Sch. J. Phys. Math. Stat,., **11** (2024), 160–172.

[13] M. S. Aslam, M. F. Bota, M. S. R. Chowdhury, L. Guran, N. Saleem, *Common Fixed Points Technique for Existence of a Solution of Urisohn Type Integral Equations System in Compleax Valued b-metric Spcaes*, Mathematics, **9** (2021), 18 pages. 1

[14] A. Atangana, D. Baleanu, *New fractional derivatives with nonlocal and non-singular kernel: theory and application to heat transfer model*, Therm. Sci., **20** (2016), 763–769. 1, 2.6, 2.7, 2.9, 2.13, 2.17

[15] J. U. Atsu, J. E. Ante, C. M. Orazulume, S. O. Essang, R. E. Francis, S. I. Okeke, W. K. Udogworen, *A Lyapunov-based approach to uniform strict stability analysis for nonlinear impulsive Caputo fractional derivatives with weakly singular kernel*, Int. J. Math. Anal. Model., **8** (2025), 198–215. 1

[16] D. D. Bainov, I. M. Stamova, *Lipschitz stability of impulsive functional-differential equations*, Anziam J., **42** (2001), 504–514. 2.20

[17] R. I. Butt, T. Abdeljawad, M. ur Rehman, *Stability analysis by fixed point theorems for a class of non-linear Caputo nabla fractional difference equation*, Adv. Differ. Equ., **2020** (2020), 1–11. 2.18

[18] M. Caputo, M. Fabrizio, *A new Definition of Fractional Derivative without Singular Kernel*, Prog. Fract. Differ. Appl., **1** (2015), 73–85. 1

[19] F. Chen, *Fixed points and asymptotic stability of nonlinear fractional difference equations*, Electron. J. Qual. Theory Differ. Equ., **2011** (2011), 18 pages. 1

[20] F. Chen. A. Chen, X. Wang, *On the solutions for impulsive fractional functional differential equations*, Differ. Equ. Dyn. Syst., **17** (2009), 379–391. 1, 2.2

[21] S. Das, P. K. Gupta, *A mathematical model on fractional Lotka-Votterra equations*, J. Theor. Biol., **277** (2011), 1–6. 1

[22] A. M. A. El-Sayed, A. E. M. El-Mesiry, H. A. A. El-Saka, *On the fractional-order logistic equation*, Appl. Math. Lett., **20** (2007), 817–823 1

[23] G. F. Hutchinson, *Circular casual systems in ecology*, Ann. N.Y. Acad. Sci., **50** (1948), 221–246. 5

[24] D. K. Igobi, U. Abasiekwere, *Results on uniqueness of solution of nonhomogeneous impulsive retarded equation using the generalized ordinary differential equation*, Int. J. Differ. Equ., **2019** (2019), 9 pages. 1

[25] D. Igobi, W. Udogworen, *Results on Existence and Uniqueness of solutions of Fractional Differential Equations of Caputo-Fabrizio type in the sense of Riemann-Liouville*, IAENG Int. J. Appl. Math., **54** (2024), 1163–1171. 2.1, 2.3, 2.12

[26] M. Ineh, E. Akpan, *On Lyapunov stability of Caputo fractional dynamic equations on time scale using vector Lyapunov functions*, Khayyam J. Math., **11** (2025), 116–143. 1

[27] M. P. Ineh, E. P. Akpan, H. A. Nabwey, *A novel approach to Lyapunov stability of Caputo fractional dynamic equations on time scale using a new generalized derivative*, AIMS Math., **9** (2024), 34406–34434.

[28] M. P. Ineh, U. Ishtiaq, J. E. Ante, M. Garayev, I.-L. Popa, S. Arabia, *A robust uniform practical stability approach for Caputo fractional hybrid systems*, AIMS Math., **10** (2025), 7001–7021. 1

[29] F. Lael, N. Saleem, M. Abbas, *On the fixed points of multivalued mappings in* b*-metric spaces and their application to linear systems*, Politehn. Univ. Bucharest Sci. Bull. Ser. A Appl. Math. Phys., **82** (2020), 121–130. 1

[30] V. Lakshmikantham, *Theory of fractional functional differential equations*, Nonlinear Anal., **69** (2008), 3337–3343. 1

[31] L. Mahto, S. Abbas, A. Favini, *Analysis of Caputo impulsive fractional order differential equations with applications*, Int. J. Differ. Equ., **2013** (2013), 11 pages. 1
[32] G. A. Mboro Nchama, L. D. Lau Alfonso, A. M. León Mecías, M. Rodríguez Ricard, *Properties of the Caputo-Fabrizio Fractional Derivative*, Int. J. Appl. Eng. Res., **16** (2021), 13–21. 2.10
[33] N. Saleem, K. Ahmad, U. Ishtiaq, M. De la Sen, *Multivalued neutrosophic fractals and Hutchinson-Barnsley operator in neutrosophic metric space*, Chaos Solitons Fractals, **172** (2023), 12 pages. 5
[34] N. Saleem, M. Zhou, S. Bashir, S. M. Husnine, *Some new generalizations of* F*-contraction type mappings that weaken certain conditions on Caputo fractional type differential equations*, AIMS Math., **6** (2021), 12718–12742. 1
[35] I. M. Stamova, *Stability analysis of impulsive functional differential equations*, Walter de Gruyter GmbH & Co. KG, Berlin, (2009). 1
[36] I. M. Stamova, G. T. Stamov, *functional and Impulsive differential equations of fractional order: Qualitative Analysis and Applications*, CRC Press, Boca Raton, USA, (2017) 1
[37] M. I. Syam, M. Al-Refai, *Fractional differential equations with Atangana-Baleanu fractional derivative: Analysis analysis and applications*, Chaos Solitons Fractals: X, **2** (2009), 5 pages. 2.21, 2.22
[38] W. Udogworen, D. Igobi, U. Jim, *Analysis of Atangana-Baleanu-Caputo Impulsive Fractional Delay Differential Equations*, IAENG Int. J. Appl. Math., **55** (2025), 3883–3891. 1, 1, 2.19, 2.23, 2.24
[39] J. Wang, Y. Zhou, M. Fečkan, *Nonlinear impulsive problems for fractional differential equations and Ulam stability*, Comput. Math. Appl., **64** (2012), 3389–3405.
[40] Y. Zhou, *Basic theory of fractional differential equations*, World Scientific Publishing Co. Pte. Ltd., Hackensack, NJ, (2014). 1, 2.28