ENGINEERING MATHEMATICS 3

LINEAR PROGRAMMING

NOORAZLINA BINTI ABD.KARIM
MATHEMATICS, SCIENCE & COMPUTER DEPARTMENT
POLITEKNIK PORT DICKSON

ENGINEERING MATHEMATICS 3: LINEAR PROGRAMMING

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PREFACE

Alhamdulillah praise be to Allah S.W.T with His grace and mercy, the first edition of Engineering Mathematic 3 –Linear Programming has finally been completed.

This module contains parts of the Engineering Mathematic 3 course syllabus of polytechnic which covers the topic of Linear Programming. There are some useful learning material and examples that can help the student to learn Linear Programming.

Those learning material can also be used as the guidance for those who didn't take additional mathematics in secondary school and as a reference for newly trained lecturer in this field.

Hopefully, this module can help the students and lecturer to expand their knowledge about mathematics.

Author

LINEAR PROGRAMMING

Learning outcome:

- At the end of this chapter, student should be able to:
  - Understand Problem Formulation
  - Use Graphical Method
  - Understand Simplex Method

1.0 Introduction:

Linear programming is used to find the best or optimal solution to a problem that requires a decision or set of decisions about how best to use a set of limited resources to achieve a state goal of objectives. It is the basic for the development of solution algorithms of other (more complex) types of Operations research (OR) models, including integer, nonlinear, and stochastic programming.

1.1 Understanding Problem Formulation

A linear programming problem may be defined as the problem of maximizing or minimizing a linear function subject to linear constraints. A simple linear programming problem may look like this:

Maximize $P = 2x + 4y$ objective function

$$\text{Subject to} \quad \left.\begin{array}{l} 5x + 3y \leq 5 \\ x - y \leq 12 \\ x \geq 0;\, y \geq 0 \end{array}\right\} \text{Constraints}$$

1.1.1 Problem Interpretation of the Related Inequalities or Equations

Before solve the linear programming problem, the problem (constraints) can be represented by set of linear inequalities.

Symbol inequalities	Situation
$>$	Greater than More than Exceeds In excess of Bigger than Over
$\geq$	Greater than or equal to Greater and equal than No less than At least Minimum Is not under
$<$	Less than Fewer than Smaller than Must not be Below Is under Is fewer
$\leq$	Less than or equal to Less and equal than Not more than At most maximum

Table 1: Inequalities to describe situations.

Constraint	Inequality
a) y **is more than** x y **is greater than** x y **in excess of** x	$y > x$
b) y **is less than** x y **is fewer than** x y **smaller than** x	$y < x$
c) y **is not more than** x y **is at most** x y **is less than or equal to** x	$y \leq x$
d) y **is greater than or equal to** x y is at least x y is not under x	$y \geq x$
e) y **is at least k times of** x	$y \geq kx$
f) y **is at most k times of** x	$y \leq kx$

g) The total of x and y is not more than k	$x + y \leq k$
h) x exceeds two times of y by k or less	$x - 2y \leq k$
i) The ration of y to x is k or more	$\frac{y}{x} \geq k$

Table 2: Inequalities to interpret constraints

Example 1:

Write the inequality for the following statements:

a. The number of student (s) in competition is smaller than 35
b. The minimum value of x is 12
c. x is more than 2y less than 8

Solution:

a. The number of student (s) in competition is smaller than 35

$$s < 30$$

b. The minimum value of x is 10

$$x \geq 12$$

c. 2x is more than 3y less than 4

$$x - 2y < 8$$

Exercise 1:

Write the inequality for the following statements:

a. The sum of x and y must not be 23.
b. y is not more than three times of x.
c. The value of x is at most 11.
d. The number of cars must exceed the number of motorcycles by at least 15.
e. The ratio of the number of scarves (y) bought to the number of dresses (x) bought not more than 3:1

1.2 FORM AND OBJECTIVE FUNCTION FOR LINEAR PROGARMMING PROBLEMS.

Guideline for Linear Programming Model Formulation:

1. **Understand** the problem thoroughly
2. Identify the **decision variables** and assign symbols to them. **(eg x,y,z....or x_1,x_2)**
3. Identify the set if **constraints** and express them in terms of inequalities involving the decision variables **.(symblos of constraint is ≥ ,≤ ,=)**
4. Identify the **objective function** and express it is terms of the decision variables.
5. Add the **non-negativity condition ($x_1,x_2,....,x_n \geq 0$)**

Example 2:

Naufal enterprise wishes to produce two types of souvenirs: type-A will result in a profit of RM1.00, and type-B in a profit of RM1.20.To manufacture a type-A souvenir requires 2 minutes on machine I and 1 minute on machine II. A type-B souvenir requires 1 minute on machine I and 3 minutes on machine II. There are 3 hours available on machine I and 5 hours available on machine II. State the variables, objective function and list the inequalities.

Solution:

Let's first **tabulate** the given information:

	Type A	Type B	Time Available
Profit/Unit	RM1.00	RM1.20	
Machine I	2 min	1 min	180 min
Machine II	1 min	3 min	300 min

State the decision variables:

x-type A souvenir

y-type B souvenir

state the abvious

$x \geq 0$

$y \geq 0$

State the constraints

Constraints associated with the total time available in machine 1 and machine 2.

machine 1 type A requires 2 minutes and type B requires 1 minutes

$$2x + y \leq 180$$

machine 2 type A requires 1 minutes and type B requires 3 minutes

$$x + 3y \leq 300$$

State the objective function/profit

Each type-A gives a profit of RM1.00, and type-B in a profit of RM1.20, so

$$P = x + 1.2y$$

So, the non-linear equation for the problem are :

$$2x + y \leq 180$$

$$x + 3y \leq 300$$

$$x \geq 0, y \geq 0$$

Example 3:

A boutique wants to sell 250 blouses and 150 skirts . They have decided to make two packages A and B. Package A consist of two blouses and one skirts and B consists of 3 blouses and 2 skirts. The profit of package A is RM50 while package B is RM75. They want to sell at least 15 of package A and at least 10 of package B. State the variables, objective function and list the inequalities.

Solution:

Let's first **tabulate** the given information:

	Package A	Package B	Package want to sell
Profit/unit	RM50	RM75	
Blouse	2	3	250
Skirts	1	2	150
	≥15	≥10	

State the decision variables:

x-package A

y-package B

state the abvious

$$x \geq 15$$

$$y \geq 10$$

State the constraints

Constraints associated with the total sell in blouse and skirts.

blouse package A requires 2 blouse and package B requires 3 blouse

$$2x + 3y \leq 250$$

skirts package A requires 1 skirts and package B requires 2 skirts

$$x + 2y \leq 150$$

State the objective function/profit

Each package-A gives a profit of RM50, and package-B in a profit of RM75, so

$$P = 50x + 75y$$

So, the non-linear equation for the problem are :

$$2x + 3y \leq 250$$

$$x + 2y \leq 150$$

$$x \geq 15$$

$$y \geq 10$$

Example 4:

A private college accepts student for Biology and Chemistry courses. It is given that the total number for student for these two courses canoot exceed 600. The number of student taking Biology courses cannot be more than twice the number of student taking the Chemistry. Yearly fees costs for Biology and Chemistry courses are RM4200 and 2800 respectively. The total amount of the fees costs received in a year for both courses is at least RM 3 500 000. Write the inequalities that satisfy the conditions above.

Solution :

State the decision variables:

x-Biology courses

y- Chemistry courses

State the constraints

a. the total number for student for these two courses canoot exceed 600

$$x + y \leq 600$$

b. The number of student taking Biology courses cannot be more than twice the number of student taking the Chemistry.

$$x \leq 2y \text{ @ } y \geq \frac{x}{2}$$

c. The fees cost for Biology courses are RM4200
The fees cost for Chemistry courses are 2800
The total amount of the fees costs received in a year for both courses is at least RM 3 500 000.

$$4200x + 2800y \geq 3500000 \text{ @ } 4.2x + 2.8y \geq 3500$$

So, the non-linear equation for the problem are :

$$x + y \leq 600$$

$$x \leq 2y \text{ @ } y \geq \frac{x}{2}$$

$$4200x + 2800y \geq 3500000 \text{ @ } 4.2x + 2.8y \geq 3500$$

$$x \geq 0, y \geq 0$$

Example 5:

Hilman using x pieces of small tile and y pieces of big tile for his bathroom decoration. The expenses are based on the following constraints:

i. The total number of tiles is not more than 350 pieces

ii. The number of small tiles is not more than 3 times big tile

iii. The number of big tile is more than the number of small tile at most 75 piece

Solution:

State the decision variables:

x-small tile

y-big tile

State the constraints

i. The total number of tiles is not more than 300 pieces

$$\mathbf{x + y \leq 350}$$

ii. The number of small tiles is not more than 3 times big tile

$$\mathbf{x \leq 3y} \ @ \ y \geq \tfrac{1}{3}x$$

iii. The number of big tile is more than the number of small tile at most 75 piece

$$\mathbf{y \leq x + 75 \ @ \ y - x \leq 75}$$

Exercise 2:

1. A small business enterprise makes dresses and trousers. To make a dress requires $\frac{1}{2}$ hour of cutting and 20 minutes of stitching. To make trousers requires 15 minutes of cutting and $\frac{1}{2}$ hour of stitching. The profit on a dress is RM 40 and on a pair of trousers RM 50. The business operates for maximum of 8 hours per day. Write inequalities and objective function that satisfy all the above condition.

2. A farmer has 80 hectares of his farm available for planting maize and cabbages. He must grow at least 10 hectares of maize and 20 hectares of cabbages to meet demands. He prefers to plant more cabbages than maize but his work force and equipment will only allow him to cultivate a maximum of three times the quantity of cabbages to that of maize. If the profit on maize is RM800 per hectares and on cabbages RM500 per hectares, how should the farmer plant the two crops to make a maximum profit and what is this profit.

3. A small business makes 3-speed and 10-speed bicycles at two different factories. Factory **A** produces 16, 3-speed and 20, 10-speed bikes in one day while factory **B** produces 12, 3-speed and 20, 10-speed bikes daily. It costs RM1000/day to operate factory **A** and RM800/day to operate factory **B**. An order for 96, 3-speed bikes and 140 10-speed bikes has just arrived. State inequalities and based on the information above.

4. Pak Amin can rear not more than 4500 ducks. The number of local duck is more than 450 but the number of export ducks is not more than four times the number of local ducks reared. The ration of the number or export ducks to local ducks is at least 2:3. State four inequalities based on the information above.

5. A developer is planning to build single and double story house. At most 800 houses could be built in that area. The number of single houses is not more than three times the number of double-story houses. At least 200 single houses need to be built. List down three inequalities other than $x \geq 0$ and $y \geq 0$ that fulfilled the condition above. **(Final Exam Dec 2015)**

6. A store sells two types of fan, A and B. The store's owner pays RM100 for fan A and RM125 for fan B. For each unit of fan A yields a profit of RM10 while each unit of fan B yields a profit of RM20. The store's owner estimates that not more than 200 fans should be sell every month and he does not plan to invest more than RM50,000 in the fan's inventory. List the inequalities that fulfilled the condition above. **(Final Exam June 17)**

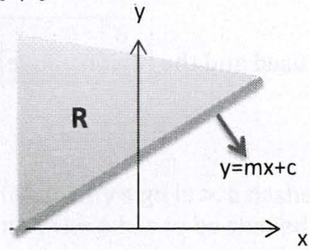
1.3 GRAPHICAL SOLUTION TO A LINEAR PROGRAMMING PROBLEM

1.3.1 Graphing Systems of Linear Inequalities

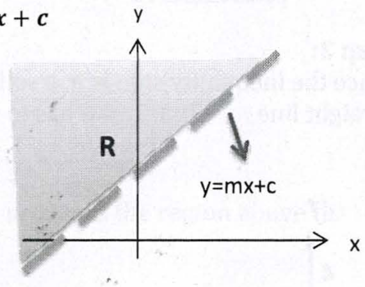
After writing the constraints of linear programming problems, it can be solved by using a graph. The straight line representing the equation for each inequality is then drawn graphically. Each ordered pair (x,y) satisfying all the linear inequalities is known as the feasible solution. The region containing all the feasible solution is known as the region of feasible solution.

If a given inequality is **greater than or equal to, i.e $y \geq mx + c$** or **greater than,** i.e $y > mx + c$ the region above the straight line

$y \geq mx + c$

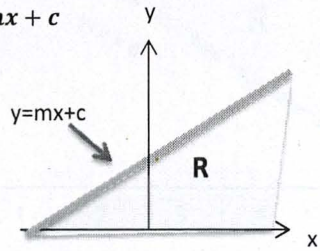


$y > mx + c$

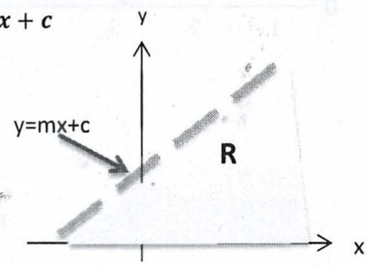


If a given inequality is **less than or equal to, i.e $y \geq mx + c$** or **less than,** i.e $y > mx + c$ the region above the straight line

$y \leq mx + c$



$y < mx + c$



Example 6:

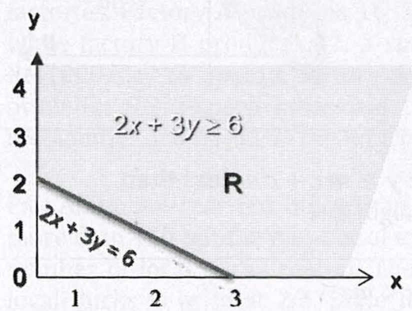
Determine the solution set for the inequality $2x + 3y \geq 6$.

Solution:

Step 1:
Replacing the inequality $\geq$ with equality =, we obtain the equation $2x + 3y = 6$. Sketch the straight line $2x + 3y = 6$. By determining the points where the straight line cuts the x-axis and the y-axis.

x	0	3
y	2	0

Step 2:
Since the inequality sign is $\geq$, a solid line should be used and the region above the straight line $2x + 3y = 6$ has to be shaded.



Example 7:

Determine the solution set for the inequality $x > 6-y$.

Solution:

Step 1:

Shift the terms in y to left-hand side of the inequality and at the same time; make sure that its coefficient is positive.

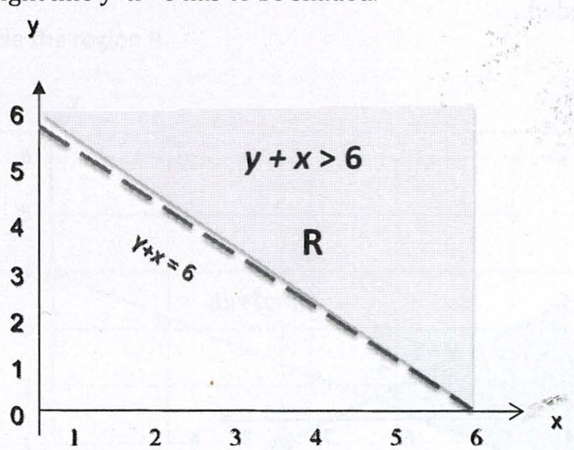
$$x > 6-y \longrightarrow y + x > 6$$

Replacing the inequality > with equality =, we obtain the equation y+x= 6. Sketch the straight line y+x= 6.By determining the points where the straight line cuts the x-axis and the y-axis.

x	0	6
y	6	0

Step 2:

Since the inequality sign is **>,** a dashed line should be used and the region above the straight line y+x= 6 has to be shaded.



Example 8:

Determine the solution set for the inequality $x \geq 4y$.

Solution:

Step 1:

Shift the terms in y to left-hand side of the inequality and at the same time; make sure that its coefficient is positive.

$$x \geq 4y \longrightarrow y \leq \frac{x}{4}$$

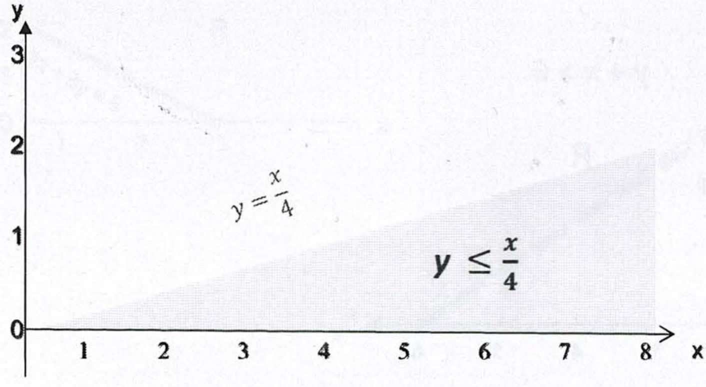
Replacing the inequality > with equality =, we obtain the equation $y = \frac{x}{4}$

Sketch the straight line $y = \frac{x}{4}$. By determining the points where the straight line cuts the x-axis and the y-axis.

X	0	8
y	0	2

Step 2:

Since the inequality sign is ≤, a solid line should be used and the region above the straight line has $y = \frac{x}{4}$ to be shaded.



Example 9:

Construct and shaded the region R that satisfies the inequalities x<5, y≤ x-1 and 4x+5y≥20

Solution:

Step 1

Sketch the line

i. X=5 (dashed line)

ii. y=x-1 (solid line)

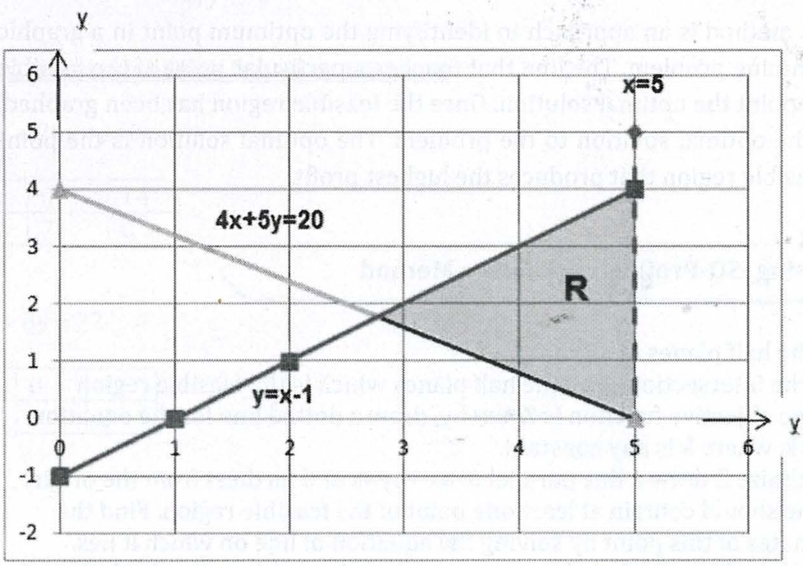
x	0	1
y	-1	0

iii. 4x+5y=20 (solid line)

x	0	5
y	4	0

Step 2

Shade the region R.



Exercise 3:

1. sketch and shade the region that satisfies each of the following inequalities
   a. $3x + y \geq 3$
   b. $2x - y < 2$
   c. $3y \geq 18 - 2x$

2. Construct and shade the region R that satisfies the following inequalities.
   a. $x + y \geq 5,\ y \leq x, x < 5$
   b. $x + 2y \geq 4, x - y \leq 1,\ y < 2$
   c.
3. Sketch and shade the feasible regions which satisfy the given condition
   a. $x \geq -2$
   b. $y \leq 6$
   c. $2y \geq 6x + 12$
   d. $6x - 6y \geq 12$

   (Final exam Jun 18)

1.3.2 ISO-Profit Line Solution Method

ISO-profit line method is an approach to identifying the optimum point in a graphic linear programming problem. The line that touches a particular point of the feasible region will pinpoint the optimal solution. Once the feasible region has been graphed, one can find the optimal solution to the problem. The optimal solution is the point lying in the feasible region that produces the highest profit

Guideline using ISO-Profit Line Solution Method

1. Draw the half planes of all constraints
2. Shade the intersection of all the half planes which is the feasible region
3. Since the objective function is Z=ax+by, draw a dotted line for the equation ax+by=k, where k is any constant.
4. To maximize Z draw a line parallel to ax +by=k and farthest from the origin. This line should contain at least one point of the feasible region. Find the coordinates of this point by solving the equation of line on which it lies.
5. if (x1,y1) is the point found in step 4, then x=x1, y=y1,is optimal solution of the LPP and ;

$$Z = ax_1 + by_1$$

Example 10:

You need to buy some filing cabinets. You know that Cabinet X costs RM10 per unit, requires six square feet of floor space, and holds eight cubic feet of files. Cabinet Y costs RM20 per unit, requires eight square feet of floor space, and holds twelve cubic feet of files. You have been given RM140 for this purchase, though you don't have to spend that much. The office has room for no more than 72 square feet of cabinets. How many of which model should you buy, in order to maximize storage volume? **Use ISO-line to solve the problem.**

Solution :

State the decision variables:

x: number of model X cabinets purchased
y: number of model Y cabinets purchased

State the constraints

Cost: $10x + 20y \leq 140$,
space: $6x + 8y \leq 72$
volume: $V = 8x + 12y$

$$x, y \geq 0$$

Find the straight line for all constraints

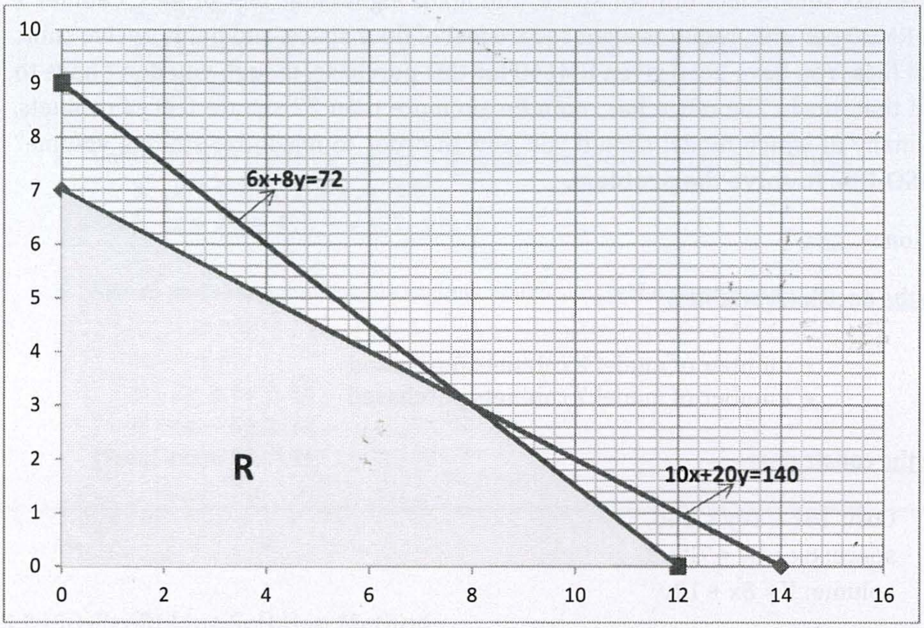
$10x + 20y = 140$

X	0	14
y	7	0

$6x + 8y = 72$

x	0	12
y	9	0

Draw the half planes of all constraints and shade the intersection of all the half planes which is the feasible region

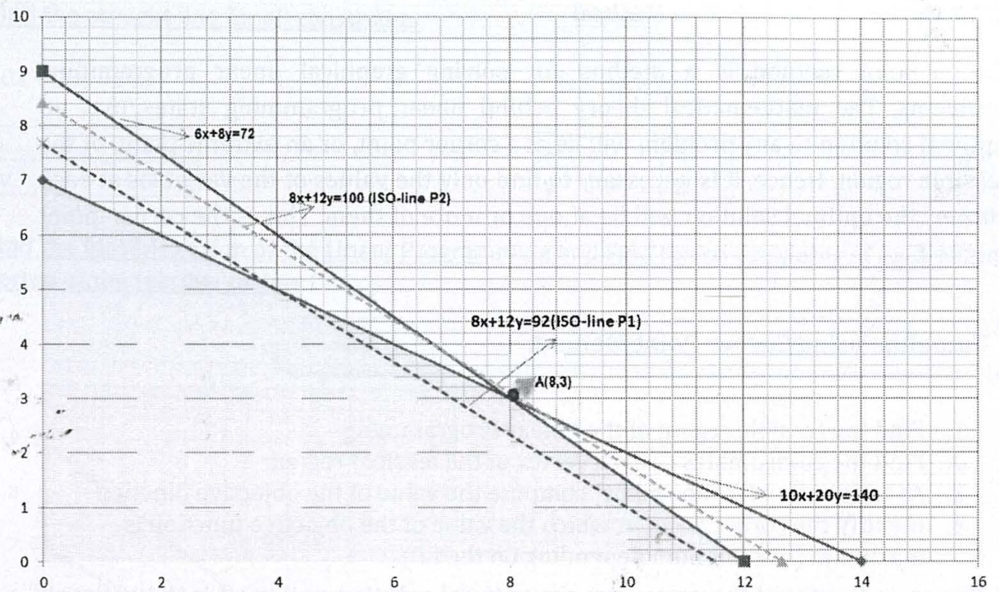


Since the objective function is Z=ax+by, draw a dotted line for the equation ax+by=k, where k is any constant. To maximize Z draw a line parallel to ax +by=k and farthest from the origin. **This line should contain at least one point of the feasible region.** Find the coordinates of this point by solving the equation of line on which it lies

Given $V = 8x + 12y$

8x+12y=k

Any point in the feasible region	Value of constant	ISO-profit equation.	ISO-profit line coordinate
P1(4,5)	K=92	92=8x+12y	(07.67;11.5,0)
P2(4.4,5.4)	K=100	100=8x+12y	(0,8.33;12.5,0)



If (x1,y1) is the point found ,then x=x1, y=y1,is optimal solution of the LPP and ;

$$Z = ax_1 + by_1$$

Coordinate Point A (8,3)

V=8x+12y

V=8(8)+12(3)

V=100

A maximal volume of 100 cubic feet by buying eight of model X and three of model Y.

1.3.3 Corner-Point Solution Method

Corner-point method is a method for solving graphical linear programming problems. The mathematical theory behind linear programming states that an optimal solution to any problem will lie at a corner point, or an extreme point, of the feasible region. Hence, it is necessary to find only the values of the variables at each corner; the optimal solution will lie at one or more of them. This is the corner-point method.

Guideline using Corner Point Method:

1. Find the feasible region of the Linear Programming
2. Find the co-ordinates of each vertex of the feasible region
3. At each vertex (corner point) compute the value of the objective function
4. Identify the corner point at which the value of the objective function is maximum (or minimum depending on the LP)

**** The co-ordinates of the vertex are the optimal solution and the Z is the optimal value.***

Example:

You need to buy some filing cabinets. You know that Cabinet X costs RM10 per unit, requires six square feet of floor space, and holds eight cubic feet of files. Cabinet Y costs RM20 per unit, requires eight square feet of floor space, and holds twelve cubic feet of files. You have been given RM140 for this purchase, though you don't have to spend that much. The office has room for no more than 72 square feet of cabinets. How many of which model should you buy, in order to maximize storage volume? Use **corner point method** to solve the problem.

Solution :

State the decision variables:

x: number of model X cabinets purchased
y: number of model Y cabinets purchased

State the constraints

Cost: $10x + 20y \leq 140$,
space: $6x + 8y \leq 72$

$x, y \geq 0$

State the objective function

Volume: $V = 8x + 12y$

Find the straight line for all constraints

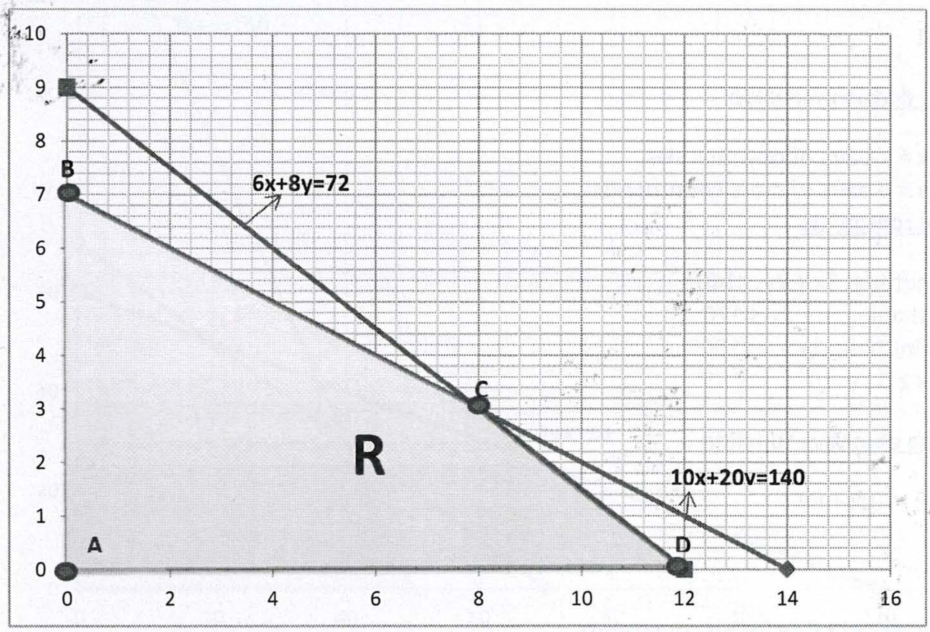
$10x + 20y = 140$

X	0	14
y	7	0

$6x + 8y = 72$

x	0	12
y	9	0

Find the feasible region of the Linear Programming and Find the co-ordinates of each vertex of the feasible region



Coordinate point	V=8x+12y
A (0,0)	0
B(0,7)	84
C (8,3)	100
D(12,0)	96

When you test the corner points at (8, 3), (0, 7), and (12, 0), you should obtain a maximal volume of 100 cubic feet by buying eight of model X and three of model Y.

Example 11:

Ahmad Products makes downhill and cross-country skis. A pair of downhill skis requires 2 man-hours for cutting, 1 man-hour for shaping and 3 man-hours for finishing while a pair of cross-country skis requires 2 man-hours for cutting, 2 man-hours for shaping and 1 man-hour for finishing. Each day the company has available 140 man-hours for cutting, 120 man-hours for shaping and 150 man-hours for finishing. How many pairs of each type of ski should the company manufacture each day in order to maximize profit if a pair of downhill skis yields a profit of RM10 and a pair of cross-country skis yields a profit of RM8? Use **ISO-line and corner point method** to solve the problem.

Solution

State the decision variables:

x = # pairs of downhill skis
y = # pairs of cross country skis

State the constraints

cutting: $2x + 2y \leq 140$
shaping: $x + 2y \leq 120$
finishing: $3x + y \leq 150$
$x \geq 0$, $y \geq 0$

State the objective function

$P = 10x + 8y$

Find the straight line for all constraints

$2x + 2y = 140$

X	0	70
y	70	0

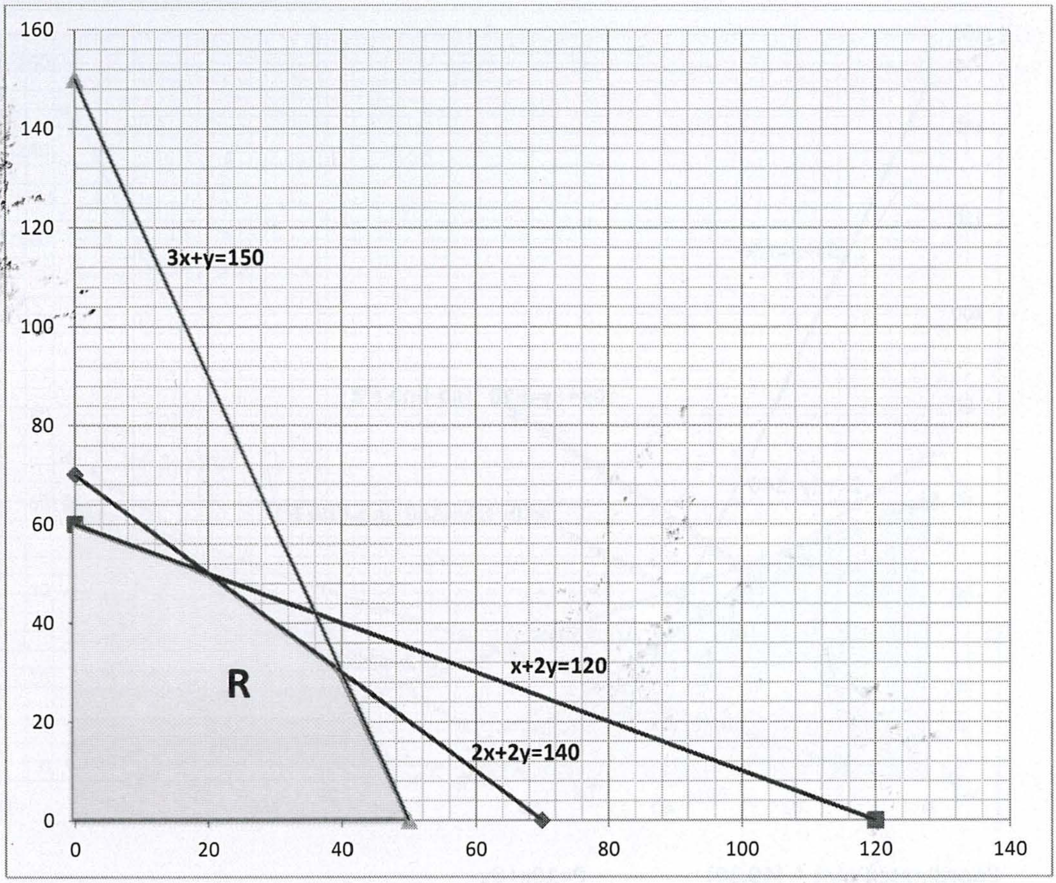
$x + 2y = 120$

x	0	60
y	120	0

$3x + y = 150$

X	0	50
y	150	0

Draw the half planes of all constraints and shade the intersection of all the half planes which is the feasible region

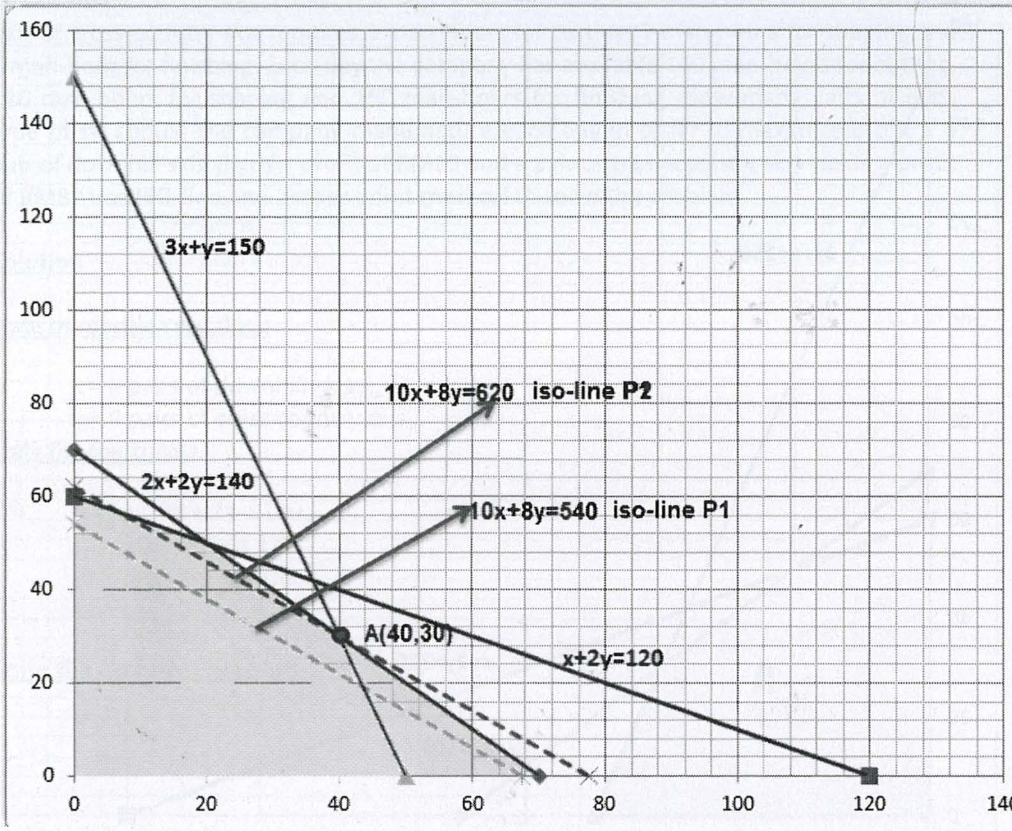


Use ISO-line to solve the problem

P = 10x + 8y, so

K=10x+8y

Any point in the feasible region	Value of constant	ISO-profit equation.	ISO-profit line coordinate
P1(30,30)	K=540	540=10x+8y	(0,54;67.5,0)
P2(34,35)	K=620	620=10x+8y	(0,62;77.5,0)



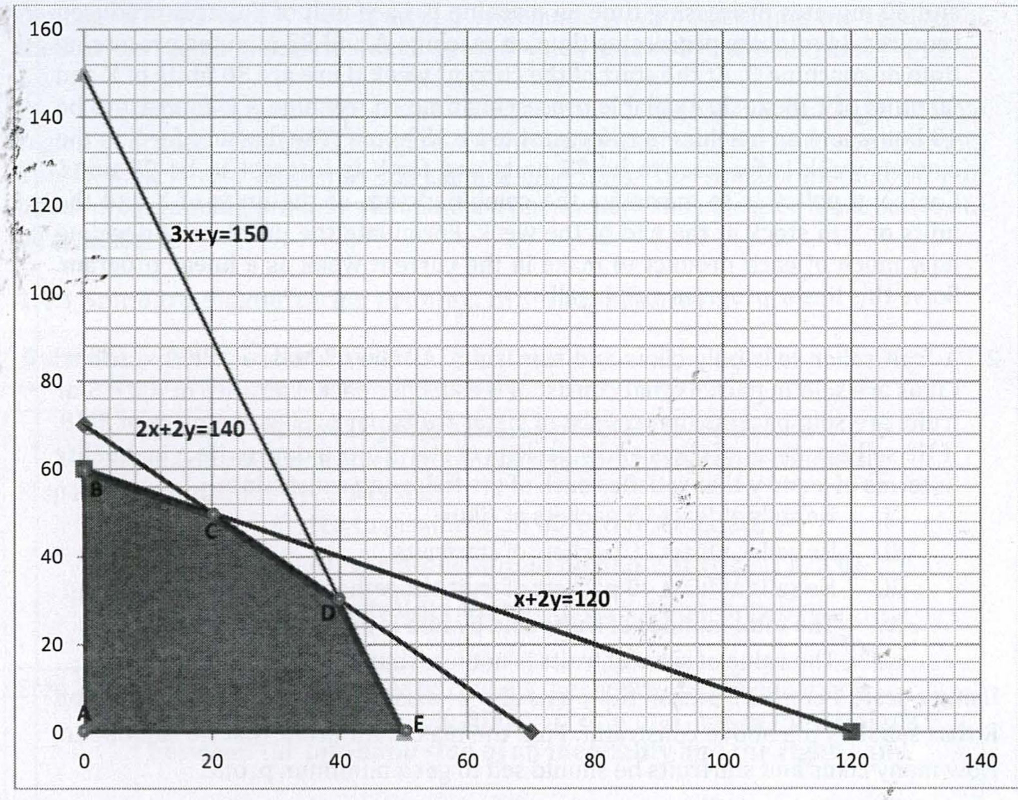
Coordinate Point A (40,30) **P=10x+8y**

P=10(40)+8(30)=RM640

Make 40 pairs of downhill skis and 30 pairs of cross country skis for a profit of RM640

Use corner point -line to solve the problem

Find the feasible region of the Linear Programming and Find the co-ordinates of each vertex of the feasible region



Coordinate point	P=10x+8y
A (0,0)	RM0
B(0,60)	RM480
C (20,50)	RM600
D(40,30)	RM640
E(50,0)	RM500

Make 40 pairs of downhill skis and 30 pairs of cross country skis for a profit of RM640

Exercise 4:

1. A company makes two products (X and Y) using two machines (A and B). Each unit of X that is produced requires 50 minutes processing time on machine A and 30 minutes processing time on machine B. Each unit of Y that is produced requires 24 minutes processing time on machine A and 33 minutes processing time on machine B. At the start of the current week there are 30 units of X and 90 units of Y in stock. Available processing time on machine A is forecast to be 40 hours and on machine B is forecast to be 35 hours. The demand for X in the current week is forecast to be 75 units and for Y is forecast to be 95 units. Company policy is to maximize the combined sum of the units of X and the units of Y in stock at the end of the week. Formulate the problem of deciding how much of each product to make in the current week as a linear program. Solve this linear program graphically.

2. A fruit seller sells only cikus and star fruits. All the fruits are sold in packets. Cikus are sold in packets that consist of 6 cikus per packet at price of RM7. Star fruits are sold packets that consist of 4 star fruits per packet at a price of RM8. If He sell x packets of cikus and y packets of star fruits a day, write a inequality in terms of x and y that satisfies each of the following constraints.
   i. He sells at least 15 packets of cikus
   ii. He sells at least 20 packets of starfruits
   iii. He sells at least 40 packets of fruit altogether
   iv. The total number of fruits sold by him is not more than 340
   v. The value of sales of fruits is not less than RM150

Using a scale 1 cm: 10 fruits on the x-axis and y –axis and shade the feasible region R that satisfies the above constraint. Find the maximum profit that he can obtain. How many cikus and starfruits he should sell to get a minimum profit.

3. A company produces two types of chairs, C1 and C2. It takes 2 hours to produce the parts of one unit of C1, 1 hour to assemble and 2 hours to polish. It takes 3 hours to produce the parts of one unit of C2, 2.5 hours to assemble and 1.5 hours to polish. Per month, 7,000 hours are available for producing the parts, 4,000 hours for assembling the parts and 5,500 hours for polishing the chairs. The profit per unit of C1 is RM90 and per unit of C2 is RM110.
   i. List THREE (3) inequalities other than and that satisfy all of the above constraints.
   ii. Using a scale of 2 cm to 500 units on axes, draw and shade the feasible region that satisfied all the given constraints.
   iii. Based on your graph, how many of each type of chairs should be produced in order to maximize the total of monthly profit? **(Final Exam Jun 18)**

1.4 SIMPLEX METHODS

For problems involving more than two variables or problems involving a large number of constraints, it is better to use solution methods that are adaptable to computers. One such method is called the **simplex method,** developed by George Dantzig in 1946. It provides us with a systematic way of examining the vertices of the feasible region to determine the optimal value of the objective function.

The simplex method presents an organized strategy for evaluating a feasible region's vertices. This helps to figure out the optimal value of the objective function. The method is also known as *the simplex algorithm.* The simplex method only to linear programming problems in standard form where the objective function was to be *maximized.*

1.4.1 Standard Maximization Problem in Standard Form

Guideline to Maximization Problem

Step1.	**Check if the linear programming problem is a** standard maximization problem in standard form**, i.e., if all the following conditions are satisfied:** ✓ **It's to** maximize **an objective function;** ✓ **All variables should be non-negative (i.e. ≥ 0).** ✓ **Constraints should all be ≤ a non-negative.**
Step 2.	Create **slack variables** to convert the inequalities to equations **Slack variables-A nonnegative variable that "takes up the slack" between the left-hand side of an inequality and the right-hand** $x + 4y \leq 8$ $\quad\quad$ $x + 4y + s = 8$
Step 3	Write the *objective* function as an equation in the form "left hand side"= 0 where terms involving variables are negative. ***Example***: ***Z = 3x + 4y becomes – 3x – 4y + Z = 0.***
Step 4.	Place the system of equations with slack variables, into a matrix (form the initial tableau). Place the revised objective equation in the bottom row.
Step 5	Select **pivot column** by finding the **most negative indicator.** (***Indicators*** *are those elements in bottom row except last two elements in that row)*
Step 6	Select **pivot row**. (*Divide the last column by pivot column for each corresponding entry except bottom entry and negative entries. Choose the smallest positive result. The corresponding row is the pivot row. In case there is no positive entry in pivot column above dashed line, there is no optimal solution)*

Step 7	Find **pivot**: Circle the pivot entry at the intersection of the pivot column and the pivot row, and identify entering variable and exit variable at mean time. Divide pivot by itself in that row to obtain 1. **(NEVER SWAP TWO ROWS in Simplex Method!)** Also obtain zeros for all rest entries in pivot column by row operations.
Step 8	Do we get all **nonnegative** indicators? If yes, we may stop. Otherwise repeat step 5 to step 7.
Step 9	Read the results: Correspond the last column entries to the variables in front of the first column. The variables not showing are automatically equal to 0.

Example 12:

Given Linear Programming problem where maximum $P = x + 2y$

Subject to the constraints of

$$x + 4y \leq 8$$
$$x + y \leq 12$$
$$x, y \geq 0$$

Solve the linear programming problem using Simplex Method.

Solution:

Step1-Check if the linear programming problem is a **standard maximization problem in standard form (√)**

Step 2- convert to standard form and add slack variables into equation

$$x + 4y + s = 8$$
$$x + y + t = 12$$

Step 3- rewrite the objective function

$$-x - 2y + p = 0$$

Step 4- Place the system of equations with slack variables, into a matrix (*form the initial tableau*). Place the revised objective equation in the bottom row.

	x	y	s	t	P	Constant	
R1	1	4	1	0	0	8	
R2	1	1	0	1	0	12	
	−1	−2	0	0	1	0	→ Objective function

Step 5- Select **pivot column** by finding the most negative indicator. (***Indicators** are those elements in bottom row except last two elements in that row)*

x	y	s	t	P	Constant
1	4	1	0	0	8
1	1	0	1	0	12
−1	−2	0	0	1	0

Pivot column

*Since the entry -2 is the most negative entry to the left of the vertical line in the last row of the tableau, the second column in the tableau is the pivot column

Step 6 - Select **pivot row**. (*Divide the last column by pivot column for each corresponding entry except bottom entry and negative entries. Choose the smallest positive result. The corresponding row is the pivot row. In case there is no positive entry in pivot column above dashed line, there is no optimal solution)*

	x	y	s	t	P	Constant	
Pivot row →	1	$\boxed{4}$	1	0	0	8	$R_1 = \frac{8}{4}=2$
	1	1	0	1	0	12	$R_2 = \frac{12}{1}=12$
	−1	−2	0	0	1	0	

Pivot column Pivot element

* We see that the ratio $\frac{8}{4}= 2$ is less than the ratio $\frac{12}{1} = 12$, so row 1 is the pivot row. The entry 4 lying in the pivot column and the pivot row is the pivot element.

Step 7- Find **pivot**: Circle the pivot entry at the intersection of the pivot column and the pivot row, and identify entering variable and exit variable at mean time. **Divide pivot by itself in that row to obtain 1.**

	x	y	s	t	P	Constant
$\frac{1}{4}R_1 \longrightarrow$	1/4	1	1/4	0	0	2
	1	1	0	1	0	12
	−1	-2	0	0	1	0

Use elementary row operations to convert the pivot column into a unit column.

	x	y	s	t	P	Constant
	1/4	1	1/4	0	0	2
$R_2 - R_1 \longrightarrow R_2$	3/4	0	-1/4	1	0	10
$R_3 + 2R_1 \longrightarrow R_3$	−1/2	0	1/2	0	1	4

Step 8- Do we get all **nonnegative** indicators? If yes, we may stop. Otherwise repeat step 5 to step 7.

*For this example, the last row of the tableau contains a negative number, so an optimal solution *has not* been reached. Therefore, we repeat step 5 to step 7.

x	y	s	t	P	Constant
1/4	1	1/4	0	0	2
3/4	0	-1/4	1	0	10
-1/2	0	1/2	0	1	4

Step 5- Select **pivot column** by finding the most negative indicator. (**Indicators** are those elements in bottom row except last two elements in that row)

x	y	s	t	P	Constant
1/4	1	1/4	0	0	2
3/4	0	-1/4	1	0	10
−1/2	0	1/2	0	1	4

Pivot column

*Since the entry $-\frac{1}{2}$ is the most negative entry to the left of the vertical line in the last row of the tableau, the first column in the tableau is now the pivot column.

Step 6 - Select **pivot row.**

x	y	s	t	P	Constant
1/4	1	1/4	0	0	2
3/4	0	-1/4	1	0	10
−1/2	0	1/2	0	1	4

Pivot row

Pivot column

Pivot element

$R_1 = \frac{2}{\frac{1}{4}} = 8$

$R_2 = \frac{10}{\frac{3}{4}} = 13.3$

*We see that the ratio $\frac{2}{\frac{1}{4}} = 8$ is less than the ratio $\frac{10}{\frac{3}{4}} = 13.3$, so row 1 is the pivot row and pivot element

Step 7- Divide pivot by itself in that row to obtain 1.

	x	y	s	t	P	Constant
$4R_1 \longrightarrow$	$\boxed{1}$	4	1	0	0	8
	3/4	0	-1/4	1	0	10
	-1/2	0	1/2	0	1	4

Use elementary row operations to convert the pivot column into a unit column.

	x	y	s	t	P	Constant
	1	4	1	0	0	8
$R_2 - \frac{3}{4}R_1 \longrightarrow R_2$	0	-3	-1	1	0	4
$R_3 + \frac{1}{2}R_1 \longrightarrow R_3$	0	2	1	0	1	8

Step 8- Do we get all **nonnegative** indicators? If yes, we may stop. Otherwise repeat step 5 to step 7. (Yes)

Locate the basic variables in the final tableau.
In this case, the basic variables are x, y, and P.

The optimal value for x is 8.
The optimal value for y is 0.
The optimal value for P is 8

1.4.2 Standard Minimization Problem in Standard Form

Previously, we learned the simplex method to solve linear programming problems that were labeled as standard maximization problems. However, many problems are not maximization problems. Often we will be asked to minimize the objective function. There are two types of minimization problems.

The first type of standard minimization problem is one in which

1. The objective function is to be minimized,
2. All variables involved in the problem are nonnegative, and
3. All other linear constraints may be written so that the expression involving the variables is less than or equal to a nonnegative constant.

The process for solving the first type is very similar to the process for solving the standard maximization problem. Given a type one standard minimization problem with some objective function C, we solve the problem by maximizing -C. After changing our objective, the problem becomes exactly the same as before.

Example 13:

Minimize C= -3x+2y subject to 2x + 3y ≤ 20;6x-2y≤ 12 and x,y ≥ 0

Solution:

Let P = -C. Then P=3x-2y. We seek to maximize P subject to the above constraints.

(Step 1) $2x + 3y + s = 20$
$6x - 2y + t = 12$

(Step 2) $-3x + 2y + P = 0$

(Step 3)

x	y	s	t	P	**Constant**
2	3	1	0	0	20
6	-2	0	1	0	12
$\boxed{-3}$	2	0	0	1	0

(Step 4)

x	y	s	t	P	Constant	
2	3	1	0	0	20	$R_1 = \frac{20}{2} = 10$
6	-2	0	1	0	12	$R_2 = \frac{12}{6} = 2$
-3	2	0	0	1	0	

(Step 5)

	x	y	s	t	P	Constant
	2	3	1	0	0	20
$\frac{1}{6}R_2 \longrightarrow$	1	-1/3	0	1/6	0	2
	-3	2	0	0	1	0

(Step 6)

	x	y	s	t	P	Constant
$R_1 - 2R_2 \longrightarrow R_1$	0	11/3	1	-1/3	0	16
	1	-1/3	0	1/6	0	2
$R_3 + 3R_2 \longrightarrow R_3$	0	1	3	1/2	1	6

(Step 7)- The optimal solution is reached

Locate the basic variables in the final tableau.
In this case, the basic variables are *x*, *y*, and *P*.

The optimal value for *x* is 2.
The optimal value for *y* is 0.
The optimal value for *P* is 6
P=-C
6=-C
C=-6

The second type of standard minimization problem is one in which

1. The objective function is to be minimized,
2. All variables involved in the problem are nonnegative, and
3. All other linear constraints may be written so that the expression involving the variables is greater than or equal to a constant.

Notice that the direction of the inequality in the constraints has changed, and notice that we no longer require the constant involved in the constraint to be nonnegative. To solve minimization problems of type two, use the following steps

Example 14:

Minimize C= x+2y subject to x + 4y ≥ 8; x+y≥ 12 and x,y ≥ 0

Solution:

Step 1: converting this problem to a maximization problem is to form the augmented matrix for this system of inequalities

$$\begin{matrix} x + 4y \geq 8 \\ x + y \geq 12 \\ c = x + 2y \end{matrix} \longrightarrow \left|\begin{array}{cc|c} 1 & 4 & 8 \\ 1 & 1 & 12 \\ \hline 1 & 2 & 0 \end{array}\right|$$

Step 2: form the **transpose** of this matrix by interchanging its rows and columns.

$$\left|\begin{array}{cc|c} 1 & 4 & 8 \\ 1 & 1 & 12 \\ \hline 1 & 2 & 0 \end{array}\right| \longrightarrow \left|\begin{array}{cc|c} 1 & 1 & 1 \\ 4 & 1 & 2 \\ \hline 8 & 12 & 0 \end{array}\right| \quad \textbf{Transpose}$$

Step 3 :

Interpret the new matrix as a *maximization* problem as follows.
New variables, *m* and *n*. We call this corresponding maximization problem the **dual** of the original minimization problem.

$$\left|\begin{array}{cc|c} 1 & 1 & 1 \\ 4 & 1 & 2 \\ \hline 8 & 12 & 0 \end{array}\right| \longrightarrow \begin{matrix} m + n \leq 1 \\ 4m + n \leq 2 \\ 8m + 12n = p \end{matrix}$$

Step 4:

The solution of the original minimization problem can be found by applying the simplex method to the new dual problem, as follows.

$$m + n \leq 1$$

$$4m + n \leq 2$$

$$8m + 12n = p$$

Step 5: convert to standard form and add slack variables into equation

$$m + n + s = 1$$
$$4m + n + t = 2$$

Step 6:
Rewrite the objective function

$$-8m - 12n + p = 0$$

Step 7: form the initial tableau

m	n	s	t	P	**Constant**
1	1	1	0	0	1
4	1	0	1	0	2
-8	-12	0	0	1	0

Step 8: find pivot column

m	n	s	t	P	**Constant**
1	1	1	0	0	1
4	1	0	1	0	2
-8	$\boxed{-12}$	0	0	1	0

Step 9: find pivot row/ pivot element.

m	n	s	t	P	Constant
1	1	1	0	0	1
4	1	0	1	0	2
-8	-12	0	0	1	0

$R_1 = \frac{1}{1}=1$

$R_2 = \frac{2}{1}=2$

Step11 : Perform the pivot operation

$R_2 - R_1 \longrightarrow R_2$

$R_3 + 12R_1 \longrightarrow R_3$

m	n	s	t	P	Constant
1	1	1	0	0	1
3	0	-1	1	0	1
4	0	12	0	1	12

*The last row of the tableau contains *no* negative numbers, so an optimal solution *has* been reached

m	n	s	t	P	Constant	
1	1	1	0	0	1	
3	0	-1	1	0	1	
4	0	12	0	1	12	← P/C
		x	y			

Remember we interpret the final matrix in special way. C is minimized at 12 and it occurs when x=12 and y=0.

C=x+2y

C=12+2(0)

C=12

Exercise 4:

1. Given the following Liner Programming problem. Write the problem in Standard Simplex Method Form. **(Final Exam Dis 16)**
   i. Maximize $z = 20x_1 + 50x_2$ subject to constrains

   $$-0.2x_1 + 0.8x_2 \le 0$$
   $$2x_1 + 4x_2 \le 240$$
   $$x_1 \le 100$$
   $$x_{1,}x_2 \ge 0$$

   ii. Minimize $C = 2x_1 - 3x_2$ subject to constrains

   $$x_1 + x_2 \le 5$$
   $$x_1 + 3x_2 \ge 9$$
   $$-2x_1 + x_2 \le 2$$
   $$x_{1,}x_2 \ge 0$$

2. Given Linear Programming problem, where Maximize $P = 5a + 2b + 8c$ subject to constrains

   $$2a - 4b + c \le 42$$
   $$2a + 3b - c \le 27$$
   $$6a - b + 3c \le 12$$
   $$a \ge 0, b \ge 0, c \ge 0$$

   i. Rewrite the objective function above in standard form
   ii. Rewrite the corresponding system of constraints equation in standard form
   iii. Convert the standard form above into First Initial Tableau Table. **(Final Exam June 17)**

3. Solve the linear Programming Problem using Simplex method.**(Final Exam June 18)**

   Maximize $P = 7x + 8y + 10z$ subject to constrains

   $$2x + 3y + 2z \le 1000$$
   $$x + y + 2z \le 800$$
   $$x_1 \le 100$$
   $$x \ge 0, y \ge 0, z \ge 0$$

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