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A cone-theoretic comparison principle for Caputo fractional differential equations

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Abstract

This paper presents a comprehensive cone-theoretic comparison principle for Caputo fractional differential equations, thereby addressing a key gap in the qualitative theory of fractional systems. Although classical comparison techniques based on scalar and vector Lyapunov functions have been extended to fractional settings, the more general and powerful framework of cone-valued Lyapunov functions has received little attention. In this work, we develop a complete theoretical foundation for this extension. We establish core results on cone-preserving fractional differential inequalities, prove the existence and characterization of maximal solutions with respect to arbitrary closed convex cones, and derive a general comparison principle for Caputo fractional systems. The central result is a unified comparison theorem that enables stability analysis of fractional systems using cone-valued Lyapunov functions. This approach incorporates both scalar and vector Lyapunov methods as special cases, and can be naturally extended to systems whose dynamics follow non-standard partial orderings. In all, our results provide a robust theoretical basis for studying complex fractional-order systems that lie beyond the reach of traditional comparison techniques.

Keywords: Cone, Lyapunov function, Caputo, fractional differential equations.

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1. Introduction

The theory of differential inequalities together with comparison principles have long been recognized as one of the most powerful tools in the qualitative analysis of dynamical systems [11]. The comparison principle, in particular, enables one to infer the qualitative behavior/properties of a complex system from that of a simpler comparison system, hence providing significant analytical and interpretive advantages.

A major advancement in this methodology came with the introduction of vector Lyapunov functions, which offered a multidimensional generalization of the classical scalar approach. Instead of compressing system behavior into a single Lyapunov function, this framework employs a vector of functions

$$\mathcal{L}(t,x) = (\mathcal{L}_1(t,x), \mathcal{L}_2(t,x), \dots, \mathcal{L}_n(t,x))^T,$$

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with each component potentially capturing a different aspect of the system's energy or deviation. The key idea is to show that the evolution of $\mathcal{L}(t, x(t))$ is dominated by the solution $\rho(t)$ of a corresponding comparison system,

$$\rho' = g(t, \rho), \quad \rho(t_0) = \rho_0,$$

so that if $\mathcal{L}(t, x(t)) \leqslant \rho(t)$, one can deduce stability properties of the original system from those of the comparison system.

However, the vector approach imposes a significant structural constraint. For the core comparison condition $\mathcal{L}(t, x(t)) \leqslant \rho(t)$ to hold, the function $g(t, \rho)$ must satisfy a quasimonotone nondecreasing condition relative to the standard cone $\mathbb{R}^n_+$ (the nonnegative orthant), which induces the familiar componentwise ordering $\rho \leqslant \nu$ if and only if $\rho_i \leqslant \nu_i$ for all $i = 1, \ldots, n$.

Formally, this means that for any vectors ρ, ν with $\rho \leqslant \nu$ and $\rho_i = \nu_i$ for some index i, we must have $g_i(t, \rho) \leqslant g_i(t, \nu)$. In essence, this requirement rules out inhibitory cross-couplings; an increase in one component cannot cause a decrease in the growth rate of another. As a result, the known vector Lyapunov method applies mainly to cooperative or competitive systems whose interaction structures are compatible with this coordinatewise monotonicity [11].

The draw back of this approach becomes evident when analyzing systems whose stability mechanisms arise from more complex or unbalanced interaction patterns. As Lakshmikantham insightfully observed in [11], many physically meaningful systems remain stable even though they fail to satisfy quasimonotonicity with respect to $\mathbb{R}^n_+$. Examples of such systems include those with asymmetric coupling or nonreciprocal interactions, features common in ecological models, neural networks, and multi-agent systems with directed information flow. In such cases, the componentwise ordering induced by $\mathbb{R}^n_+$ fails to reflects the true organizational structure of the dynamics, thus rendering the traditional vector Lyapunov method inapplicable despite the system's inherent stability.

The conceptual breakthrough by Lakshmikantham and Leela was the realization that this shortcoming did not stem from the vector Lyapunov approach itself, but from the restrictive notion of ordering it employed. The solution lies in generalizing the very concept of order in the state space by replacing the standard cone $\mathbb{R}^n_+$ with a suitably chosen cone $\mathcal{V} \subset \mathbb{R}^n$ that reflects the specific structural properties of the system.

This cone-theoretic framework introduces a more flexible partial ordering, denoted as $\leqslant_{\mathcal{V}}$ (see Definition 2.3), where $\mathcal{V}$ is a closed, convex, pointed cone with a nonempty interior. The pioneering work of Lakshmikantham and Leela in [11] established the foundations of this theory, developing:

1. A general theory of differential inequalities relative to arbitrary cones in $\mathbb{R}^n$
2. A generalized notion of quasimonotonicity with respect to a cone K
3. Comparison principles for cone-valued Lyapunov functions
4. Illustrative examples demonstrating how an appropriate choice of cone overcomes the limitations of standard vector Lyapunov methods

This conceptual shift provides several key advantages:

1. Geometric flexibility: The cone $\mathcal{V}$ can be tailored to match the system's inherent geometry, such as the dominant directions of its linearized dynamics or the coupling structure among its subsystems.
2. Broader applicability: Systems that are not quasimonotone with respect to $\mathbb{R}^n_+$ may still satisfy a generalized quasimonotonicity condition relative to a suitable cone $\mathcal{V}$, allowing the comparison principle to be applied.
3. Unified framework: The cone-theoretic formulation encompasses both scalar and vector Lyapunov methods as special cases, corresponding to $\mathcal{V} = \mathbb{R}_+$ and $\mathcal{V} = \mathbb{R}^n_+$, respectively.

Building on this foundation, Akpan and Akinyele in [3, 4] extended the theory to functional differential equations, where they introduced the concept of ϕ_0-stability. They were able to show through concrete examples, that the cone-valued Lyapunov method can establish stability results where both scalar and vector Lyapunov approaches fail. Further contributions by Akinyele in [5] applied this framework to impulsive control systems, where he derived new comparison theorems and stability criteria for systems experiencing abrupt state changes.

Parallel to these developments, fractional differential equations have emerged as powerful tools for modeling complex phenomena characterized by memory and nonlocal effects. Among the several fractional derivatives, the Caputo derivative has gained particular prominence due to its compatibility with initial conditions of classical derivatives (see [7, 9, 10, 12]) . The nonlocality inherent in fractional derivatives, expressed by [6]

$$ {}^{C}D^{\tau}_{t_0} f(t) = \frac{1}{\Gamma(1-\tau)} \int_{t_0}^{t} \frac{f'(\xi)}{(t-\xi)^{\tau}}\, d\xi, \quad 0 < \tau < 1, $$

introduces distinctive mathematical challenges that demand careful treatment when extending classical theories.

Recent advances have adapted comparison principles to the fractional setting. In [2], Agarwal et al. introduced a Caputo fractional Dini derivative for Lyapunov-like functions,

$$ {}^{C}_{+}D^{\tau}_{(3.1)}\mathcal{L}(t,x) = \limsup_{h\to 0^{+}} \frac{1}{h^{\tau}} \left\{ \mathcal{L}(t,x) - \mathcal{L}(t_0,x_0) - \sum_{r=1}^{[(t-t_0)/h]} (-1)^{r+1} ({}_{\tau}C_{r}) \times \left[\mathcal{L}(t-rh, x-h^{\tau} f(t,x)) - \mathcal{L}(t_0,x_0) \right] \right\}, $$

and established foundational comparison results for fractional systems. More recently, Achuobi et al. in [1] extended this framework to vector Lyapunov functions for time-dependent fractional systems with delay, while Ineh et al. in [8] developed a unified framework for Caputo fractional derivatives on arbitrary time domains.

In [13], Wu established a general comparison principle for Caputo fractional-order ordinary differential equations. His work addressed key gaps in earlier research by providing:

1. A complete theory of maximal solutions for Caputo fractional systems via continuation techniques, and
2. The use of Vainikko's representation of the Caputo derivative to derive comparison results under minimal regularity assumptions.

While Wu's results marked a major advancement for scalar fractional systems, the extension to vector or cone-valued Lyapunov functions remained unexplored.

Despite these substantial developments so outlined, a crucial gap persists; and that is the absence of a cone-theoretic comparison framework for fractional differential equations. While scalar and vector Lyapunov methods have been extended to fractional systems, the cone-valued approach which offers even greater flexibility and applicability to non-quasimonotone systems has not yet been systematically developed in the fractional context.

This paper therefore addresses this fundamental gap by establishing a cone-theoretic comparison principle for Caputo fractional differential equations. Our results extend the classical cone-theoretic methods of Lakshmikantham, Leela, Akpan, and others to the fractional domain, overcoming the additional mathematical difficulties introduced by nonlocal fractional operators. The proposed framework offers several key advantages:

1. A unified structure that generalizes both scalar and vector Lyapunov methods.
2. The ability to analyze systems whose natural ordering is not captured by coordinate-wise comparisons.
3. The flexibility to design comparison systems that more accurately reflect the structure of the original dynamics.
4. Greater geometric adaptability for constructing problem-specific comparison principles.

It is worth emphasizing that the traditional vector Lyapunov method is obtained as a particular case of the cone-theoretic framework developed in this work. Specifically, when the cone is chosen as $\mathcal{V} = \mathbb{R}^{n}_{+}$, the induced order becomes the standard componentwise ordering. Under this choice, the comparison function $g(t, \rho)$ must satisfy the familiar requirement of being quasimonotone nondecreasing with respect to $\mathbb{R}^n_+$, a structural condition that naturally fits cooperative or competitive systems but significantly restricts general applicability.

The present framework avoids this limitation by allowing $\mathcal{V}$ to be any closed convex cone. The corresponding generalized quasimonotonicity condition (Definition 2.4) becomes strictly weaker whenever $\mathcal{V} \neq \mathbb{R}^n_+$. This added flexibility makes it possible to treat systems whose coupling is asymmetric, non–coordinate-aligned, or otherwise incompatible with the classical structure, thereby removing one of the principal constraints of the traditional vector Lyapunov approach.

In this paper, we focus on establishing the foundational comparison theory within this cone-theoretic framework for fractional systems. The applications of these results to stability analysis will be developed in a subsequent paper, allowing a clear separation between the theoretical development and its practical implications.

The remainder of the paper is organized as follows. Section 2 introduces notation and the key definitions used throughout the paper. Section 3 presents the necessary preliminaries on fractional calculus, cone theory, and fractional differential inequalities. Section 4 develops the main results, Section 5 describes the verification procedure for candidate cones and comparison systems. Section 6 concludes the paper and outlines potential directions for future research.

2. Notations and definitions

Let $\mathbb{R}^N$ denote the N-dimensional Euclidean space equipped with the Euclidean norm $\|\cdot\|$, and let $\langle \cdot, \cdot \rangle$ represent the standard inner product.

Definition 2.1. A subset $\mathcal{V} \subset \mathbb{R}^N$ is called a closed convex pointed cone if it satisfies the following conditions:

1. Closure under addition: If $v_1, v_2 \in \mathcal{V}$, then their sum $v_1 + v_2$ also belongs to $\mathcal{V}$.
2. Closure under non-negative scaling: If $v \in \mathcal{V}$, then for any non-negative scalar α, $\alpha v \in \mathcal{V}$. If $\alpha = 0$, then $0.v = 0 \in \mathcal{V}$
3. Topological closure: $\mathcal{V}$ contains all its limit points, i.e., $\mathcal{V} = \bar{\mathcal{V}}$.
4. Non-Empty Interior: The cone has a non-empty interior, i.e., $\mathcal{V}^0 \neq \emptyset$.

Definition 2.2 ([3, Dual Cone]). The dual cone (or adjoint cone) of a cone $\mathcal{V}$ is the set

$$\mathcal{V}^* = \{\psi \in \mathbb{R}^N : \langle \psi, x \rangle \geqslant 0, \forall x \in \mathcal{V}\},$$

where, $\langle \psi, x \rangle$ denotes the standard dot product of ψ and x. $x \in \partial\mathcal{V}$ (boundary of $\mathcal{V}$) if and only if $\langle \psi, x \rangle = 0$ for some $\psi \in \mathcal{V}^* \setminus \{0\}$.

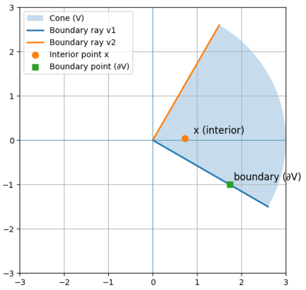

Figure 1: Cone $\mathcal{V}$ in $\mathbb{R}^N$.

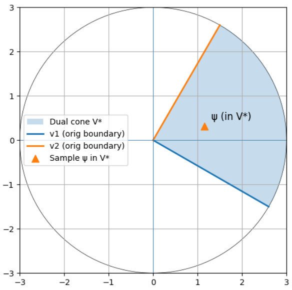

Figure 2: Dual cone $\mathcal{V}^*$ in $\mathbb{R}^n$.

Definition 2.3 ([3, Partial Order relation induced by a cone])**.** Let $\mathcal{V} \subset \mathbb{R}^N$ be a pointed cone (i.e, $\mathcal{V} \cap (-\mathcal{V}) = \{0\}$) with a non empty interior. The cone $\mathcal{V}$ induces a partial order relation $\leqslant_{\mathcal{V}}$ on $\mathbb{R}^N$ defined by

$$u \leqslant_{\mathcal{V}} v, \iff v - u \in \mathcal{V}$$

and

$$u <_{\mathcal{V}^0} v, \iff v - u \in \mathcal{V}^0, \forall u, v \in \mathbb{R}^N.$$

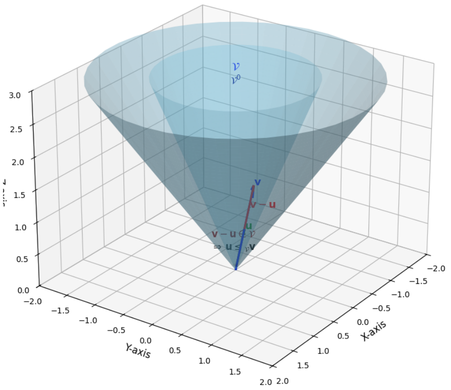

Figure 3: Illustration of the partial order $u \leqslant_{\mathcal{V}} v$ induced by the cone $\mathcal{V} \subset \mathbb{R}^N$.

This relation, denoted by $\leqslant_{\mathcal{V}}$, satisfies the following properties for all $u, v, w \in \mathbb{R}^N$:

1. Reflexivity: $u \leqslant_{\mathcal{V}} u$, since $u - u = 0 \in \mathcal{V}$.
2. Antisymmetry: If $u \leqslant_{\mathcal{V}} v$ and $v \leqslant_{\mathcal{V}} u$, then $(v - u) + (u - v) = 0 \in \mathcal{V}$.
3. Transitivity: If $u \leqslant_{\mathcal{V}} v$ and $v \leqslant_{\mathcal{V}} w$, then $u \leqslant_{\mathcal{V}} w$, because $(w - v) + (v - u) = w - u \in \mathcal{V}$.

Definition 2.4 (**Quasimonotonicity Relative to a Cone** [4])**.** Let $g \in C[E, \mathbb{R}^N]$, where $E \subset \mathbb{R}_+ \times \mathbb{R}^N$. The function $g(t, \rho)$ is said to be quasimonotone in ρ relative to the cone $\mathcal{V}$ if, for every $t \in \mathbb{R}_+$ and for all $x, y \in \mathbb{R}^N$ such that $(t, x), (t, y) \in E$ and $y - x \in \partial\mathcal{V}$, there exists $\psi \in \mathcal{V}^* \setminus \{0\}$ satisfying:

1. $\langle \psi, y - x \rangle = 0$,
2. $\langle \psi, g(t, y) - g(t, x) \rangle \geqslant 0$.

Remark 2.5 (Linear Case and Cooperative Systems)*.* When $g(\rho) = A\rho$ is linear and $\mathcal{V} = \mathbb{R}^N_+$, quasimonotonicity is equivalent to A having non-negative off-diagonal elements. Such matrices are called Metzler matrices and arise naturally in cooperative dynamical systems where each variable has a non-negative effect on the others.

Remark 2.6 (Comparison Principle)*.* This condition is the minimal requirement for the comparison principle to hold in vector Lyapunov methods. It ensures that solutions of the comparison system bound the Lyapunov function components appropriately.

Definition 2.7. [13][Caputo Fractional Non-Autonomous System]

Consider the Caputo fractional differential equation of order $\tau \in (0, 1)$:

$$\begin{cases} {}^{C}D_t^{\tau} x(t) = f(t, x(t)), \\ x(t_0) = x_0, \quad t > t_0, \end{cases} \tag{2.1}$$

where${}^{C}D_t^{\tau}$ denotes the Caputo fractional derivative of order τ with respect to time t. Here, $x(t) \in \mathbb{R}^N$ represents the state vector at time t; $f : \mathcal{D} \to \mathbb{R}^N$ is a continuous vector field defined on an open domain

$\mathcal{D} \subseteq \mathbb{R}_+ \times \mathbb{R}^N$; $t_0 \in \mathbb{R}_+$ is the initial time; and $x_0 \in \mathbb{R}^N$ denotes the initial condition vector. For any $M > 0$, the open ball of radius M centered at the origin is defined as:

$$B_M = \{x \in \mathbb{R}^N : \|x\| < M\}.$$

3. Preliminaries

Lemma 3.1 ([13]). *Assume that* f *is continuous on the closed set*

$$\tilde{S} = \{(t, x) : t \in [t_0, t_0 + a],\ \|x - x_0\| \leqslant b\},$$

for some $a > 0$ *and* $b > 0$, *with* $\tilde{S} \subset D$. *Then, the Caputo fractional differential equation* (2.1) *admits a solution* $x(t) \in C^{\tau}[t_0, t_0 + h]$, *where*

$$h = \min\left\{a, \left(\frac{b\,\Gamma(\tau+1)}{M}\right)^{1/\tau}\right\}, \quad \text{and} \quad M = \max_{(t,x)\in\tilde{S}} \|f(t, x)\|.$$

Lemma 3.2 ([13]). *Under the assumptions of Lemma 3.1, a function* $x(t) \in C[t_0, t_0 + h]$ *is a solution of the Caputo fractional differential equation* (2.1) *if and only if it satisfies the corresponding Volterra integral equation of the second kind:*

$$x(t) = x_0 + \frac{1}{\Gamma(\tau)} \int_{t_0}^{t} (t - \xi)^{\tau-1} f(\xi, x(\xi))\, d\xi.$$

Definition 3.3 (Uniform Boundedness and Equicontinuity). Let $[a, b] \subset \mathbb{R}$ and $\{g_k\}$ be a sequence of functions $g_k : [a, b] \to \mathbb{R}^n$. The sequence is called:

1. uniformly bounded if there exists $M > 0$ such that for all k and all $t \in [a, b]$, $\|g_k(t)\| \leqslant M$;
2. equicontinuous if for any $\epsilon > 0$, there exists $\delta > 0$ such that for all k and all $t_1, t_2 \in [a, b]$, $|t_1 - t_2| < \delta$ implies $\|g_k(t_1) - g_k(t_2)\| < \epsilon$.

Definition 3.4. [2] We say that $m \in C^{\tau}([t_0, T], \mathbb{R}^n)$ if $m(t)$ is differentiable (i.e., $m'(t)$ exists), and the Caputo derivative ${}^{C}D^{\tau}m(t)$ exists and satisfies (2.1) for all $t \in [t_0, T]$.

Remark 3.5. If $m \in C^{\tau}([t_0, T], \mathbb{R}^n)$, then the upper and lower Caputo fractional Dini derivatives coincide with the standard Caputo derivative. However, for composite functions of the form $\mathcal{L}(t, \rho(t))$, where $\rho(t)$ is a solution of the fractional system (2.1) and $\mathcal{L}$ is locally Lipschitz, the relationship requires careful consideration. The Grünwald-Letnikov approximation employed in our Dini derivative definition ensures that the local Lipschitz condition is sufficient to guarantee the existence and appropriate behavior of the derivative, despite the non-local nature of fractional operators. This follows from the fact that the approximation preserves the linear structure and accommodates the memory effects through the summation terms, allowing the chain rule in Lemma 4.7 to hold under the stated regularity conditions.

Lemma 3.6. *[13][Arzela–Ascoli Theorem] If* $\{g_k\}$ *is a uniformly bounded and equicontinuous sequence of functions defined on* $[a, b]$, *then there exists a subsequence that converges uniformly on* $[a, b]$.

Lemma 3.7. *[13][Arzela–Ascoli Corollary] If* $\{g_m\}$ *is a uniformly bounded and equicontinuous sequence of functions defined on* $[a, b]$ *and every uniformly convergent subsequence converges to the same limit* g, *then*

$$\lim_{m\to\infty} g_m = g$$

uniformly on $[a, b]$.

Lemma 3.8. *(Linearity of Caputo Fractional Dini Derivative). The left and right Caputo fractional Dini derivatives, as defined in Equation (4.5), are linear operators. That is, for any functions* $f, g \in C([t_0, T], \mathbb{R}^n)$ *for which the Dini derivatives exist, and for any scalars* $\alpha, \beta \in \mathbb{R}$, *we have:*

$$^{C}D^{\tau}_{\pm}[\alpha f(t) + \beta g(t)] = \alpha\, ^{C}D^{\tau}_{\pm} f(t) + \beta\, ^{C}D^{\tau}_{\pm} g(t)$$

Proof. *The linearity follows directly from the Grünwald-Letnikov approximation structure inherent in the definition. For the left derivative:*

$$\begin{aligned}
^{C}D^{\tau}_{-}[\alpha f + \beta g](t) &= \liminf_{h \to 0^+} \frac{1}{h^{\tau}} \Bigg[(\alpha f(t) + \beta g(t)) - (\alpha f(t_0) + \beta g(t_0)) \\
&\quad - \sum_{l=1}^{\left\lfloor \frac{t-t_0}{h} \right\rfloor} (-1)^l \binom{\tau}{l} \left((\alpha f(t-lh) + \beta g(t-lh)) - (\alpha f(t_0) + \beta g(t_0))\right)\Bigg] \\
&= \alpha \liminf_{h \to 0^+} \frac{1}{h^{\tau}} \left[f(t) - f(t_0) - \sum_{l=1}^{\left\lfloor \frac{t-t_0}{h} \right\rfloor} (-1)^l \binom{\tau}{l} (f(t-lh) - f(t_0)) \right] \\
&\quad + \beta \liminf_{h \to 0^+} \frac{1}{h^{\tau}} \left[g(t) - g(t_0) - \sum_{l=1}^{\left\lfloor \frac{t-t_0}{h} \right\rfloor} (-1)^l \binom{\tau}{l} (g(t-lh) - g(t_0)) \right] \\
&= \alpha\, ^{C}D^{\tau}_{-} f(t) + \beta\, ^{C}D^{\tau}_{-} g(t)
\end{aligned}$$

Note that although we are considering the left derivative (looking backward in time), the limit is still taken as $h \to 0^+$ *because we are using a positive increment* h *and examining points* $t - lh$ *that lie to the left of* t *when* $h > 0$. *The notation* $h \to 0^+$ *indicates that we approach zero through positive values, which is standard in difference quotient definitions of derivatives.*

The proof for the right derivative follows similarly, with the appropriate adjustment in the summation indices to look forward in time. This linearity property is essential for the analysis in Theorems 4.1 and 4.8.

4. Main Results

4.1. Theory of Cone-Based Differential Inequalities and Comparison Principles

This section establishes the mathematical foundation for analyzing fractional systems through cone-valued Lyapunov functions. We develop the theory of differential inequalities in cone-ordered spaces and derive comparison theorems that enable stability analysis of Caputo fractional differential equations. The principal object of investigation in this work is the Caputo fractional comparison system of order $\tau \in (0, 1)$:

$$^{C}D^{\tau}_{t} \rho(t) = g(t, \rho(t)), \quad \rho(t_0) = \rho_0, \tag{4.1}$$

where: $^{C}D^{\tau}_{t}$ is the Caputo fractional derivative operator, $\rho(t) \in \mathbb{R}^n$ is the state vector of the comparison system, $g : \mathcal{D}' \to \mathbb{R}^n$ is a continuous vector field defined on $\mathcal{D}' \subseteq \mathbb{R}_+ \times \mathcal{V}$, $t_0 \in \mathbb{R}_+$ is the initial time instant, and $\rho_0 \in \mathbb{R}^n$ is the initial condition vector with $B(M) = \{\rho \in \mathcal{V} : \|\rho\| < M\}$.

Theorem 4.1 (Cone–Preserving Fractional Differential Inequality)**.** *Let* $\mathcal{V} \subseteq \mathbb{R}^n$ *be a closed convex cone with nonempty interior, and let* $g : \mathbb{R}_+ \times \mathbb{R}^n \to \mathbb{R}^n$ *be continuous and quasimonotone relative to* $\mathcal{V}$. *Suppose* $\rho, \nu \in C[\mathbb{R}_+, \mathbb{R}^n]$, *and the Caputo derivatives* $^{C}_{t_0}D^{\tau}_{t} \rho(t)$ *and* $^{C}_{t_0}D^{\tau}_{t} \nu(t)$ *exist for all* $t \geqslant t_0$. *Then, the following inequality holds for* $t \geqslant t_0$:

$$^{C}D^{\tau}_{-} \rho(t) \leqslant_{\mathcal{V}} g(t, \rho(t)), \tag{4.2}$$

$$g(t,\nu(t)) <_{\mathcal{V}^0} {}^{C}D_{-}^{\tau}\nu(t), \tag{4.3}$$

where ${}^{C}D_{-}^{\tau}$ *is the left Caputo fractional Dini derivative of order* $\tau \in (0,1)$*. If* $\rho(t_0) <_{\mathcal{V}^0} \nu(t_0)$*, then:*

$$\rho(t) <_{\mathcal{V}^0} \nu(t), \quad \forall t \geqslant t_0.$$

Proof. Assume by contradiction that there exists a time $t_s > t_0$ such that:

$$\begin{aligned} \nu(t_s) - \rho(t_s) &\in \partial\mathcal{V}, \\ \nu(t) - \rho(t) &\in \mathcal{V}^0 \quad \text{for } t \in [t_0, t_s). \end{aligned}$$

By quasimonotonicity of g, there exists $\psi \in \mathcal{V}^* \setminus \{0\}$ such that:

$$\begin{aligned} \langle \psi, \nu(t_s) - \rho(t_s) \rangle &= 0, \\ \langle \psi, g(t_s, \nu(t_s)) - g(t_s, \rho(t_s)) \rangle &\geqslant 0. \end{aligned}$$

Define

$$H(t) = \langle \psi, \nu(t) - \rho(t) \rangle. \tag{4.4}$$

Then $H(t) > 0$ for $t < t_s$ and $H(t_s) = 0$. Additionally, $H(t_s) = 0$ suggests that $H(t)$ is decreasing as $t \to t_s^-$. In particular, as $t \to t_s$, $H(t) \to 0$, indicating that the function $H(t)$ is decreasing near t_s. Thus, the left Caputo fractional Dini derivative ${}^{C}D_{-}^{\tau}H(t)$ will reflect this decreasing behavior. Since $H(t) \to 0$ but remains positive just before t_s, it follows that the derivative must be non-positive. That is,

$${}^{C}D_{-}^{\tau}H(t_s) \leqslant 0, \tag{4.5}$$

where

$${}^{C}D_{-}^{\tau}H(t_s) = \liminf_{h \to 0^+} \frac{1}{h^{\tau}} \left[H(t_s) - H(t_0) - \sum_{l=1}^{\left\lfloor \frac{t_s - t_0}{h} \right\rfloor} (-1)^{l} \binom{\tau}{l} \left(H(t_s - lh) - H(t_0) \right) \right]. \tag{4.6}$$

At $t = t_s$, Equation (4.4) becomes

$$H(t_s) = \langle \psi, \nu(t_s) - \rho(t_s) \rangle. \tag{4.7}$$

From (4.2) and Definition 2.3, we deduce that

$$g(t,\rho(t)) - {}^{C}D_{-}^{\tau}\rho(t) \in \mathcal{V} \quad \text{and} \quad {}^{C}D_{-}^{\tau}\nu(t) - g(t,\nu(t)) \in \mathcal{V}^0.$$

So that

$${}^{C}D_{-}^{\tau}\nu(t) - g(t,\nu(t)) + [g(t,\rho(t)) - {}^{C}D_{-}^{\tau}\rho(t)] \in \mathcal{V}^0 + \mathcal{V}.$$

Since $\mathcal{V}^0$ is the interior of $\mathcal{V}$ and $\mathcal{V}$ is a closed and convex cone, it follows that $\mathcal{V}^0 + \mathcal{V} = \mathcal{V}^0$. Thus, we have

$${}^{C}D_{-}^{\tau}\nu(t) - {}^{C}D_{-}^{\tau}\rho(t) - [g(t,\nu(t)) - g(t,\rho(t))] \in \mathcal{V}^0.$$

However, recall that $y - x \in \mathcal{V}^0 \implies y - x >_{\mathcal{V}^0} 0$. Therefore,

$${}^{C}D_{-}^{\tau}\nu(t) - {}^{C}D_{-}^{\tau}\rho(t) - [g(t,\nu(t)) - g(t,\rho(t))] \in \mathcal{V}^0 >_{\mathcal{V}^0} 0.$$

Rearranging terms, we have for $t = t_s$ that

$${}^{C}D_{-}^{\tau}\nu(t_s) - {}^{C}D_{-}^{\tau}\rho(t_s) > g(t_s,\nu(t_s)) - g(t_s,\rho(t_s)). \tag{4.8}$$

Now, from Equations (4.7) and (4.8), and using the fact that $g(t_s, \nu(t_s)) - g(t_s, \rho(t_s))$ is non-negative by the assumption of quasimonotonicity, alongside the linearity of the Caputo fractional Dini derivative, we have

$$\begin{aligned} {}^{C}D_{-}^{\tau}H(t_s) &= \langle \psi, {}^{C}D_{-}^{\tau}\nu(t_s) - {}^{C}D_{-}^{\tau}\rho(t_s)\rangle, && (4.9)\\ &> \langle \psi, g(t_s, \nu(t_s)) - g(t_s, \rho(t_s))\rangle \geqslant 0. && (4.10)\end{aligned}$$

Thus, we conclude that

$$^{C}D_{-}^{\tau}H(t_s) > 0.$$

But $H(t) \to 0$ implies that ${}^{C}D_{-}^{\tau}H(t_s) \leqslant 0$. Hence, no such t_s exists. Therefore, our assumption that $\rho(t)$ and $\nu(t)$ do not maintain their order throughout the interval $[t_0, t_s)$ is incorrect. Hence, $\rho(t) <_{\mathcal{V}^0} \nu(t)$, for all $t \geqslant t_0$. □

Definition 4.2. Let $\mathcal{V} \subset \mathbb{R}^n$ be a closed convex cone with non-empty interior, which induces a partial order $\leqslant_{\mathcal{V}}$. A solution $r(t)$ of Equation (4.1), defined on an interval I is called the maximal solution with respect to the cone $\mathcal{V}$ if for any other solution $u(t)$ with same initial data defined on I, we have

$$u(t) \leqslant_{\mathcal{V}} r(t), \text{ for all } t \in I.$$

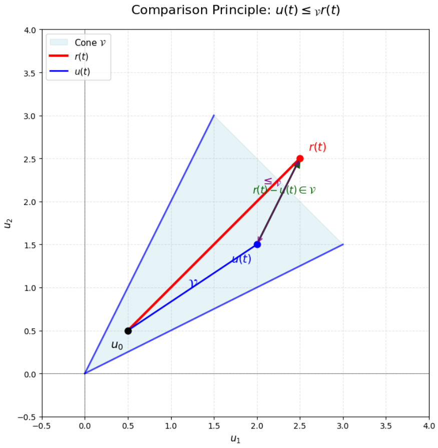

Figure 4: Illustration of the maximal solution $r(t)$ of the comparison system, satisfying $u(t) \leqslant_{\mathcal{V}} r(t)$ for all $t \geqslant t_0$.

Theorem 4.3. *Let* $g : R_j \to \mathbb{R}^n$ *be continuous on the closed domain*

$$R_j = \{(t, \rho) : t \in [t_0, t_0 + a],\ \|\rho - \rho_0\| \leqslant b\},$$

where $a > 0$, $b > 0$, *and* $\rho_0 \in \mathbb{R}^n$. *Assume that:*

1. $\|g(t, \rho)\| \leqslant N$ *for all* $(t, \rho) \in R_j$, *where*

$$N = \sup_{(t,\rho)\in R_j} \|g(t, \rho)\|.$$

2. g *is quasimonotone in* ρ *relative to the cone* $\mathcal{V}$.

Then there exists a maximal solution $r(t)$ *of*

$$^{C}D_{t}^{\tau}\rho(t) = g(t, \rho(t)), \quad \rho(t_0) = \rho_0, \quad \tau \in (0, 1), \tag{4.11}$$

on the interval $[t_0, t_0 + \beta]$, *where*

$$\beta = \min\left(a, \left(\frac{b\,\Gamma(\tau+1)}{N + \frac{b}{2}}\right)^{1/\tau}\right).$$

Proof. Let $\xi \in \mathcal{V}^0$ be a unit vector ($\|\xi\| = 1$) and $0 < \epsilon \leqslant b/2$. Consider the perturbed system:

$$ {}^{C}D_t^{\tau}\rho(t,\epsilon) = g(t,\rho) + \epsilon\xi, \quad \rho(t_0) = \rho_0 + \epsilon\xi. \tag{4.12}$$

The corresponding Volterra integral equation, according to Lemma 3.2, is

$$\rho(t,\epsilon) = \rho_0 + \epsilon\xi + \frac{1}{\Gamma(\tau)}\int_{t_0}^{t}(t-s)^{\tau-1}\left(g(s,\rho(s,\epsilon)) + \epsilon\xi\right)ds. \tag{4.13}$$

Define $g_\epsilon(t,\rho) = g(t,\rho) + \epsilon\xi$. For $(t,\rho) \in R_\epsilon = \{(t,\rho) : t \in [t_0, t_0 + a], \|\rho - (\rho_0 + \epsilon\xi)\| \leqslant b\}$, we have

$$\|g_\epsilon(t,\rho)\| \leqslant \|g(t,\rho)\| + \epsilon\|\xi\| \leqslant N + \frac{b}{2}.$$

By Lemma 3.2, the perturbed system (4.12) has a solution $\rho(t,\epsilon)$ on $[t_0, t_0+\beta]$, where

$$\beta = \min\left(a, \left(\frac{b\,\Gamma(\tau+1)}{2N+b}\right)^{1/\tau}\right).$$

For $t \in [t_0, t_0+\beta]$,

$$\|\rho(t,\epsilon)\| \leqslant \|\rho_0\| + \epsilon\|\xi\| + \frac{1}{\Gamma(\tau)}\int_{t_0}^{t}(t-s)^{\tau-1}\left(N + \frac{b}{2}\right)ds.$$

Computing the integral gives

$$\|\rho(t,\epsilon)\| \leqslant \|\rho_0\| + \frac{b}{2} + \frac{(2N+b)}{2\Gamma(\tau+1)}\beta^{\tau} \leqslant \|\rho_0\| + b.$$

Thus, the family $\{\rho(t,\epsilon)\}$ is uniformly bounded. For $t_1, t_2 \in [t_0, t_0+\beta]$ with $t_1 < t_2$,

$$\begin{aligned}\|\rho(t_2,\epsilon) - \rho(t_1,\epsilon)\| &\leqslant \frac{2N+b}{\Gamma(\tau)}\left|\int_{t_0}^{t_2}(t_2-s)^{\tau-1}ds - \int_{t_0}^{t_1}(t_1-s)^{\tau-1}ds\right| \\ &\leqslant \frac{2N+b}{\Gamma(\tau)}\frac{(t_2-t_0)^{\tau} - (t_1-t_0)^{\tau}}{\tau}.\end{aligned}$$

Using subadditivity of x^{τ} for $\tau \in (0,1)$, i.e. $(a+b)^{\tau} \leqslant a^{\tau} + b^{\tau}$, we have

$$(t_2-t_0)^{\tau} - (t_1-t_0)^{\tau} \leqslant (t_2-t_1)^{\tau}.$$

Hence,

$$\|\rho(t_2,\epsilon) - \rho(t_1,\epsilon)\| \leqslant \frac{(2N+b)}{\Gamma(\tau+1)}|t_2-t_1|^{\tau}.$$

This estimate,

$$\|\rho(t_2,\epsilon) - \rho(t_1,\epsilon)\| \leqslant M|t_2-t_1|^{\tau}, \quad \text{where } M = \frac{(2N+b)}{\Gamma(\tau+1)},$$

shows that the family of solutions $\{\rho(t,\epsilon)\}$ is Holder continuous with exponent τ. This implies equicontinuity.

By the Arzela-Ascoli theorem, there exists a subsequence $\{\rho(t,\epsilon_k)\}$ with $\epsilon_k \to 0$ that converges uniformly to $r(t)$. From (4.13), we have

$$\rho(t,\epsilon_k) = \rho_0 + \epsilon_k\xi + \frac{1}{\Gamma(\tau)}\int_{t_0}^{t}(t-s)^{\tau-1}\left(g(s,\rho(s,\epsilon_k)) + \epsilon_k\xi\right)ds. \tag{4.14}$$

Taking the limit as $k \to \infty$ in (4.14), we obtain

$$r(t) = \rho_0 + \frac{1}{\Gamma(\tau)} \int_{t_0}^{t} (t-s)^{\tau-1} g(s, r(s))ds.$$

Thus, $r(t)$ solves Equation (4.11).

Let $x(t)$ be any other solution of (2.1) with $x(t_0) = x_0 \leqslant_{\mathcal{V}} \rho_0$. For each $\epsilon > 0$, consider the perturbed solution $\rho(t, \epsilon)$ satisfying

$${}^{C}D_t^{\tau} \rho(t, \epsilon) = g(t, \rho(t, \epsilon)) + \epsilon\xi, \quad \rho(t_0, \epsilon) = \rho_0 + \epsilon\xi.$$

Since

$$x(t_0) = x_0 \leqslant_{\mathcal{V}} \rho_0 <_{\mathcal{V}^\circ} \rho_0 + \epsilon\xi = \rho(t_0, \epsilon),$$

and

$${}^{C}D_t^{\tau} x(t) = g(t, x(t)) \leqslant_{\mathcal{V}} g(t, x(t)) + \epsilon\xi, \quad {}^{C}D_t^{\tau} \rho(t, \epsilon) = g(t, \rho(t, \epsilon)) + \epsilon\xi,$$

by Theorem 4.1 (applied to $x(t)$ and $\rho(t, \epsilon)$), we obtain

$$x(t) <_{\mathcal{V}^\circ} \rho(t, \epsilon), \quad \forall t \in [t_0, t_0 + \beta].$$

Taking the limit as $\epsilon \to 0$ through the subsequence converging to $r(t)$ yields

$$x(t) \leqslant_{\mathcal{V}} r(t), \quad \forall t \in [t_0, t_0 + \beta].$$

Hence, $r(t)$ is the maximal solution with respect to the cone order $\leqslant_{\mathcal{V}}$.

The maximality of $r(t)$ follows from two key properties preserved under our construction: First, for each $\epsilon > 0$, the perturbed solution $\rho(t, \epsilon)$ dominates all other solutions due to the strict inequality $x(t) <_{\mathcal{V}} \rho(t, \epsilon)$ established through Theorem 4.1. Second, since $\mathcal{V}$ is closed and convex, the cone order $\leqslant_{\mathcal{V}}$ is preserved under uniform convergence. Specifically, if a sequence $\{y_n\}$ converges uniformly to y and $x \leqslant_{\mathcal{V}} y_n$ for all n, then $x \leqslant_{\mathcal{V}} y$. This preservation property, combined with the dominance of each $\rho(t, \epsilon)$ over all solutions, ensures that the limit $r(t)$ inherits this maximal dominating property. The choice of perturbation direction $\xi \in \mathcal{V}^0$ guarantees that the dominance is strict in the interior of the cone, which is essential for establishing maximality rather than mere solution existence. □

Remark 4.4. This result extends the classical maximal solution theorem to Caputo fractional systems with cone-induced ordering. The case $\mathcal{V} = \mathbb{R}_+^n$ recovers componentwise comparison.

Lemma 4.5. *Assume the following:*

1. $g \in C[\mathbb{R}_+ \times \mathcal{V}, \mathbb{R}^n]$ *and* $g(t, \rho)$ *is quasimonotone in* ρ *relative to the cone* $\mathcal{V}$*, for each* $(t, \rho) \in \mathbb{R}_+ \times \mathbb{R}^n$.
2. $l \in C[\mathbb{R}_+, \mathcal{V}]$ *such that*
$$ {}^{C}D_{-}^{\tau} l(t) \leqslant_{\mathcal{V}} g(t, l(t)), \quad t \geqslant t_0.$$

Let $r(t)$ *be the maximal solution of the comparison system* (4.1)*, then*

$$l(t_0) \leqslant_{\mathcal{V}} \rho(t_0) \implies l(t) <_{\mathcal{V}} r(t), \quad t \geqslant t_0.$$

Proof. Let $\xi \in \mathcal{V}^0$ with $\|\xi\| = 1$ and let $0 < \epsilon < b/2$. Consider the perturbed system:

$${}^{C}D^{\tau} \rho(t, \epsilon) = g_{\epsilon}(t, \rho(t)), \quad \rho(t_0) = \rho_0 + \epsilon\xi, \tag{4.15}$$

where $g_{\epsilon}(t, \rho(t)) = g(t, \rho(t)) + \epsilon\xi$. By Theorem 4.3, for each $\epsilon > 0$, there exists a solution $\rho(t, \epsilon)$ of (4.15) on $[t_0, t_0 + \beta]$, such that $\lim_{k \to \infty} \rho(t, \epsilon_k) = r(t)$, where $r(t)$ is the maximal solution of (4.1). Define $\rho(t) = l(t)$ and $\nu(t) = \rho(t, \epsilon)$. From hypothesis (2), we have

$${}^{C}D_{-}^{\tau} l(t) \leqslant_{\mathcal{V}} g(t, l(t)).$$

Also, for the perturbed system, we have

$$\begin{aligned} {}^{C}D_{-}^{\tau}\rho(t,\epsilon) = {}^{C}D^{\tau}\rho(t,\epsilon) &= g_{\epsilon}(t,\rho) \\ &= g(t,\rho) + \epsilon\xi, \\ &>_{\mathcal{V}^0} g(t,\rho), \end{aligned}$$

since $\epsilon\xi \in \mathcal{V}^0$.

Again, from the hypothesis $l(t_0) \leqslant_{\mathcal{V}} \rho(t_0)$ and the construction $\rho(t_0) = \rho_0 + \epsilon\xi$, we have

$$l(t_0) \leqslant_{\mathcal{V}} \rho_0 < \rho_0 + \epsilon\xi = \rho(t_0).$$

By Theorem 4.1, we conclude that

$$l(t) \leqslant_{\mathcal{V}} \rho(t,\epsilon), \quad \text{for } t \geqslant t_0.$$

Since $\rho(t,\epsilon) \to r(t)$ uniformly as $\epsilon \to 0$, we have

$$l(t) <_{\mathcal{V}} r(t), \quad \text{for } t \geqslant t_0.$$

□

Lemma 4.6. *Let $\mathcal{V} \subset \mathbb{R}^n$ be a closed, convex, pointed cone with nonempty interior. Assume the following:*

1. $\ell(t), \rho(t) \in C([t_0,T],\mathbb{R}^n)$
2. *There exists $t_s \in (t_0,T]$ such that $\ell(t_s) = \rho(t_s)$ and $\ell(t) <_{\mathcal{V}^0} \rho(t)$ for all $t \in [t_0,t_s)$*
3. *The Caputo fractional Dini derivatives ${}^{C}D_{+}^{\tau}\ell(t_s)$ and ${}^{C}D_{+}^{\tau}\rho(t_s)$ exist for some $\tau \in (0,1)$*

Then there exists a nonzero vector $\psi \in \mathcal{V}^*$ that satisfies the strict scalar inequality

$$\langle \psi, {}^{C}D_{+}^{\tau}\ell(t_s)\rangle > \langle \psi, {}^{C}D_{+}^{\tau}\rho(t_s)\rangle$$

Proof. Since $\rho(t) - \ell(t) \in \mathcal{V}^0$ for $t \in [t_0,t_s)$ and $\rho(t_s) - \ell(t_s) = 0 \in \partial\mathcal{V}$, the supporting hyperplane theorem for cones guarantees the existence of a nonzero vector $\psi \in \mathcal{V}^*$ such that

$$\langle \psi, \rho(t) - \ell(t)\rangle > 0 \quad (t \in [t_0,t_s)), \qquad \langle \psi, \rho(t_s) - \ell(t_s)\rangle = 0.$$

Define the scalar function

$$m(t) := \langle \psi, \ell(t) - \rho(t)\rangle.$$

By construction, $m(t) < 0$ for $t \in [t_0,t_s)$ and $m(t_s) = 0$. Since the Caputo fractional Dini derivatives of ℓ and ρ exist at t_s, the derivative of $\ell - \rho$ exists at t_s and equals ${}^{C}D_{+}^{\tau}\ell(t_s) - {}^{C}D_{+}^{\tau}\rho(t_s)$. Consider the scalar derivative of m at t_s:

$${}^{C}D_{+}^{\tau}m(t_s) = \limsup_{h\to 0^+} \frac{1}{h^{\tau}} \left[m(t_s) - m(t_0) - \sum_{r=1}^{\left\lfloor \frac{t_s - t_0}{h} \right\rfloor} (-1)^r \binom{\tau}{r} (m(t_s - rh) - m(t_0)) \right].$$

Substituting $m(t) = \langle \psi, \ell(t) - \rho(t)\rangle$ gives

$$\begin{aligned} {}^{C}D_{+}^{\tau}m(t_s) = \limsup_{h\to 0^+} \frac{1}{h^{\tau}} \Big\langle \psi, &(\ell(t_s) - \rho(t_s)) - (\ell(t_0) - \rho(t_0)) \\ &- \sum_{r=1}^{N_h} (-1)^r \binom{\tau}{r} \left[(\ell(t_s - rh) - \rho(t_s - rh)) - (\ell(t_0) - \rho(t_0)) \right] \Big\rangle. \end{aligned}$$

Since the vector Caputo fractional Dini derivative of $\ell - \rho$ exists at t_s, we can interchange the inner product and the limit, yielding

$$ {}^{C}D_{+}^{\tau} m(t_s) = \left\langle \psi, {}^{C}D_{+}^{\tau}(\ell-\rho)(t_s) \right\rangle = \left\langle \psi, {}^{C}D_{+}^{\tau}\ell(t_s) - {}^{C}D_{+}^{\tau}\rho(t_s) \right\rangle . $$

Verification of Scalar Comparison Lemma Prerequisites:

We now explicitly verify that $m(t)$ satisfies all prerequisites of the scalar comparison lemma (Lemma 1 in [2]):

1. Continuity: Since $\ell(t)$ and $\rho(t)$ are continuous on $[t_0, T]$ and the inner product is continuous, $m(t)$ is continuous on $[t_0, T]$.
2. Behavior on $[t_0, t_s)$: By construction, $m(t) < 0$ for all $t \in [t_0, t_s)$ since $\ell(t) <_{\mathcal{V}^0} \rho(t)$ implies $\rho(t) - \ell(t) \in \mathcal{V}^0$, and for $\psi \in \mathcal{V}^* \setminus \{0\}$, this gives $\langle \psi, \rho(t) - \ell(t) \rangle > 0$.
3. Behavior at t_s: $m(t_s) = 0$ by the assumption $\ell(t_s) = \rho(t_s)$.
4. Existence of Dini derivative: The existence of ${}^{C}D_{+}^{\tau} m(t_s)$ follows from the linearity property (Lemma 3.8) and the existence of ${}^{C}D_{+}^{\tau}\ell(t_s)$ and ${}^{C}D_{+}^{\tau}\rho(t_s)$.
5. Monotonicity condition: The function $m(t)$ increases to zero as $t \to t_s^-$, which implies the required monotonicity behavior for the scalar comparison lemma.

Since all prerequisites are satisfied, we can apply the scalar comparison lemma 1 in [2] to conclude that ${}^{C}D_{+}^{\tau} m(t_s) > 0$. Hence,

$$ \left\langle \psi, {}^{C}D_{+}^{\tau}\ell(t_s) - {}^{C}D_{+}^{\tau}\rho(t_s) \right\rangle > 0, $$

or equivalently,

$$ \left\langle \psi, {}^{C}D_{+}^{\tau}\ell(t_s) \right\rangle > \left\langle \psi, {}^{C}D_{+}^{\tau}\rho(t_s) \right\rangle . \quad \square $$

□

Lemma 4.7. *Let the following conditions be satisfied:*

1. $\rho(t) \in C^{\tau}([t_0, T], \mathbb{R}^N)$ *is a solution of the Caputo fractional differential equation* (2.1).
2. $\mathcal{L} \in C[\mathbb{R}_+ \times B_M, \mathcal{V}]$ *is locally Lipschitz in* ρ, *that is,*

$$ \|\mathcal{L}(t,x) - \mathcal{L}(t,y)\| \leqslant L\|x-y\|, \quad \forall (t,x),(t,y) \in \mathbb{R}_+ \times B_M . $$

3. *The Caputo fractional Dini derivative satisfies the inequality*

$$ {}^{C}D_{+}^{\tau}\mathcal{L}(t,\rho) \leqslant_{\mathcal{V}} g(t, \mathcal{L}(t,\rho)), \quad \forall (t,\rho) \in \mathbb{R}_+ \times B_M . $$

Then for the composite function $\mathcal{L}(t, \rho(t))$, *we have*

$$ {}^{C}D_{+}^{\tau}\mathcal{L}(t,\rho(t)) \leqslant_{\mathcal{V}} g(t, \mathcal{L}(t,\rho(t))), \quad \forall t \geqslant t_0 . $$

Proof. Using the Grunwald–Letnikov approximation of the Caputo fractional Dini derivative,

$$ {}^{C}D_{+}^{\tau}\mathcal{L}(t,\rho(t)) = \limsup_{h\to 0^+} \frac{1}{h^{\tau}} \left[\mathcal{L}(t,\rho(t)) - \mathcal{L}(t_0,\rho_0) - \sum_{r=1}^{N} (-1)^{r+1} \binom{\tau}{r} \left(\mathcal{L}(t-rh, \rho(t-rh)) - \mathcal{L}(t_0,\rho_0) \right) \right], $$

where $N = \left\lfloor \frac{t-t_0}{h} \right\rfloor$.

Substituting $\mathcal{L}(t, \rho(t))$ gives

$$ \begin{aligned} & \mathcal{L}(t,\rho(t)) - \mathcal{L}(t_0,\rho_0) - \sum_{r=1}^{N} (-1)^{r+1} \binom{\tau}{r} \left(\mathcal{L}(t-rh, \rho(t-rh)) - \mathcal{L}(t_0,\rho_0) \right) \\ & = \left[\mathcal{L}(t,\rho(t)) - \mathcal{L}(t_0,\rho_0) - \sum_{r=1}^{N} (-1)^{r+1} \binom{\tau}{r} \left(\mathcal{L}(t-rh, \rho(t) - h^{\tau} f(t,\rho(t))) - \mathcal{L}(t_0,\rho_0) \right) \right] \\ & \quad + \sum_{r=1}^{N} (-1)^{r+1} \binom{\tau}{r} \left[\mathcal{L}(t-rh, \rho(t) - h^{\tau} f(t,\rho(t))) - \mathcal{L}(t-rh, \rho(t-rh)) \right]. \end{aligned} $$

Let us denote:

$$A = \mathcal{L}(t,\rho(t)) - \mathcal{L}(t_0,\rho_0) - \sum_{r=1}^{N}(-1)^{r+1}\binom{\tau}{r}\big(\mathcal{L}(t-rh,\rho(t)-h^{\tau}f(t,\rho(t))) - \mathcal{L}(t_0,\rho_0)\big),$$

and

$$B = \sum_{r=1}^{N}(-1)^{r+1}\binom{\tau}{r}\Big[\mathcal{L}(t-rh,\rho(t)-h^{\tau}f(t,\rho(t))) - \mathcal{L}(t-rh,\rho(t-rh))\Big].$$

By hypothesis (3) and the definition of the Caputo fractional Dini derivative,

$$\limsup_{h\to 0^+}\frac{1}{h^{\tau}}A \leqslant_{\mathcal{V}} g(t,\mathcal{L}(t,\rho(t))).$$

Fix a vector $\xi \in \mathcal{V}^0$ with $\|\xi\| = 1$. Then for any $\varepsilon > 0$, there exists $h_\varepsilon > 0$ such that for all $0 < h < h_\varepsilon$,

$$\frac{1}{h^{\tau}}A \leqslant_{\mathcal{V}} g(t,\mathcal{L}(t,\rho(t))) + \varepsilon\xi.$$

Multiplying both sides by h^{τ} yields

$$A \leqslant_{\mathcal{V}} h^{\tau}g(t,\mathcal{L}(t,\rho(t))) + \varepsilon h^{\tau}\xi = h^{\tau}g(t,\mathcal{L}(t,\rho(t))) + o(h^{\tau}).$$

Using the Lipschitz condition (hypothesis (2)),

$$\|B\| \leqslant L\sum_{r=1}^{N}\left|\binom{\tau}{r}\right|\big\|\rho(t) - h^{\tau}f(t,\rho(t)) - \rho(t-rh)\big\| = O(h^{\tau}).$$

Hence, $B \leqslant_{\mathcal{V}} o(h^{\tau})\xi$.

$$\mathcal{L}(t,\rho(t)) - \mathcal{L}(t_0,\rho_0) - \sum_{r=1}^{N}(-1)^{r+1}\binom{\tau}{r}(\mathcal{L}(t-rh,\rho(t-rh)) - \mathcal{L}(t_0,\rho_0)) \leqslant_{\mathcal{V}} h^{\tau}g(t,\mathcal{L}(t,\rho(t))) + o(h^{\tau}).$$

Dividing by h^{τ} and taking $\limsup_{h\to 0^+}$ gives

$${}^{C}D_{+}^{\tau}\mathcal{L}(t,\rho(t)) \leqslant_{\mathcal{V}} g(t,\mathcal{L}(t,\rho(t))).$$

□

Theorem 4.8 (Comparison Principle for Caputo Fractional Systems)**.** *Assume the following:*

1. $g \in C[\mathbb{R}_+ \times \mathcal{V}, \mathbb{R}^n]$ *is quasimonotone in* ρ *relative to the cone* $\mathcal{V}$*: for* $\rho, \nu \in \mathcal{V}$ *with* $\nu - \rho \in \partial\mathcal{V}$*, there exists* $\psi \in \mathcal{V}^* \setminus \{0\}$ *such that*
$$\langle\psi, \nu-\rho\rangle = 0 \implies \langle\psi, g(t,\nu) - g(t,\rho)\rangle \geqslant 0.$$
2. $\mathcal{L} \in C[\mathbb{R}_+ \times B_M, \mathcal{V}]$ *is locally Lipschitz in* x *relative to* $\mathcal{V}$*: for all* $(t,x),(t,y) \in \mathbb{R}_+ \times B_\rho$,
$$\|\mathcal{L}(t,x) - \mathcal{L}(t,y)\| \leqslant L\|x-y\|.$$
3. *The Caputo fractional Dini derivative of* $\mathcal{L}(t,x)$ *satisfies*
$${}^{C}D_{+}^{\tau}\mathcal{L}(t,x) \leqslant_{\mathcal{V}} g(t,\mathcal{L}(t,x)), \quad \forall (t,x) \in \mathbb{R}_+ \times B_\rho.$$

Let $r(t)$ *be the maximal solution of the comparison system* (4.1), *and* $x(t)$ *be any solution of the CFDE* (2.1) *with* $\mathcal{L}(t_0, x_0) \leqslant_{\mathcal{V}} \rho_0$. *Then,*

$$\mathcal{L}(t, x(t)) \leqslant_{\mathcal{V}} r(t), \quad \forall t \geqslant t_0.$$

Proof. Let $x(t)$ be any solution of the CFDE (2.1) with $\mathcal{L}(t_0, x_0) \leqslant_{\mathcal{V}} \rho_0$. Define the composite function

$$\ell(t) = \mathcal{L}(t, x(t)) \in C[\mathbb{R}_+, \mathcal{V}],$$

representing the Lyapunov function evaluated along the trajectory of $x(t)$. By Lemma 4.7, we have

$$^{C}D_{+}^{\tau}\ell(t) \leqslant_{\mathcal{V}} g(t, \ell(t)).$$

Fix $\xi \in \mathcal{V}^0$ with $\|\xi\| = 1$ and $\varepsilon > 0$. Consider the perturbed system:

$$^{C}D^{\tau}\rho(t, \varepsilon) = g(t, \rho(t, \varepsilon)) + \varepsilon\xi, \quad \rho(t_0, \varepsilon) = \rho_0 + \varepsilon\xi.$$

By Theorem 4.3, this system admits a solution $\rho(t, \varepsilon)$ on $[t_0, t_0 + \beta]$. We claim that

$$\ell(t) <_{\mathcal{V}} \rho(t, \varepsilon), \quad \forall t \geqslant t_0. \tag{1}$$

At $t = t_0$, it holds that

$$\ell(t_0) = \mathcal{L}(t_0, x_0) \leqslant_{\mathcal{V}} \rho_0 <_{\mathcal{V}} \rho_0 + \varepsilon\xi = \rho(t_0, \varepsilon),$$

so (1) is true initially.

Suppose, for contradiction, that (1) fails. Let

$$t_s = \inf\{\, t > t_0 : \ell(t) \not<_{\mathcal{V}} \rho(t, \varepsilon) \,\}.$$

Then $\ell(t_s) = \rho(t_s, \varepsilon)$, and $\ell(t) <_{\mathcal{V}} \rho(t, \varepsilon)$ for all $t \in [t_0, t_s)$. Since $\rho(t_s, \varepsilon) - \ell(t_s) \in \partial\mathcal{V}$, by quasimonotonicity (assumption 1), there exists $\psi \in \mathcal{V}^* \setminus \{0\}$ such that

$$\langle \psi, \rho(t_s, \varepsilon) - \ell(t_s) \rangle = 0, \qquad \langle \psi, g(t_s, \rho(t_s, \varepsilon)) - g(t_s, \ell(t_s)) \rangle \geqslant 0.$$

Applying Lemma 4.6 with $\ell(t) = \mathcal{L}(t, x(t))$ and $\rho(t) = \rho(t, \varepsilon)$ yields

$$\langle \psi, {}^{C}D_{+}^{\tau}\ell(t_s) \rangle > \langle \psi, {}^{C}D_{+}^{\tau}\rho(t_s, \varepsilon) \rangle. \tag{2a}$$

From the perturbed system,

$$^{C}D_{+}^{\tau}\rho(t_s, \varepsilon) = g(t_s, \rho(t_s, \varepsilon)) + \varepsilon\xi = g(t_s, \ell(t_s)) + \varepsilon\xi.$$

Substituting into (2a) gives

$$\langle \psi, {}^{C}D_{+}^{\tau}\ell(t_s) \rangle > \langle \psi, g(t_s, \ell(t_s)) + \varepsilon\xi \rangle = \langle \psi, g(t_s, \ell(t_s)) \rangle + \varepsilon\langle \psi, \xi \rangle. \tag{2b}$$

Since $\xi \in \mathcal{V}^0$ and $\psi \in \mathcal{V}^* \setminus \{0\}$, we have $\langle \psi, \xi \rangle > 0$, implying

$$\langle \psi, {}^{C}D_{+}^{\tau}\ell(t_s) \rangle > \langle \psi, g(t_s, \ell(t_s)) \rangle. \tag{2c}$$

However, from Lemma 4.7,

$$^{C}D_{+}^{\tau}\ell(t_s) \leqslant_{\mathcal{V}} g(t_s, \ell(t_s)).$$

Since $\psi \in \mathcal{V}^*$, this implies

$$\langle \psi, {}^{C}D_{+}^{\tau}\ell(t_s) \rangle \leqslant \langle \psi, g(t_s, \ell(t_s)) \rangle. \tag{3}$$

Inequalities (2c) and (3) contradict each other. Hence, (1) must hold for all $t \geqslant t_0$:

$$\ell(t) <_{\mathcal{V}} \rho(t,\varepsilon), \quad \forall t \geqslant t_0.$$

Letting $\varepsilon \to 0$, the solutions $\rho(t,\varepsilon)$ converge uniformly on compact intervals to the maximal solution $r(t)$ of the comparison system. By continuity of the cone order, we obtain

$$\ell(t) \leqslant_{\mathcal{V}} r(t), \quad \forall t \geqslant t_0.$$

Recalling that $\ell(t) = \mathcal{L}(t,x(t))$, we conclude

$$\mathcal{L}(t,x(t)) \leqslant_{\mathcal{V}} r(t), \quad \forall t \geqslant t_0.$$

□

The cone-theoretic framework also clarifies how standard Lyapunov comparison methods fit within a single unified structure. Consider the Lyapunov-like function $\mathcal{L}(t,x)$. When $\mathcal{V} = \mathbb{R}_+$, the partial order $\leqslant_{\mathcal{V}}$ collapses to an ordinary scalar inequality, reproducing the classical scalar Lyapunov method. When $\mathcal{V} = \mathbb{R}^n_+$, the framework yields the vector Lyapunov method, where the generalized quasimonotonicity condition reduces to the familiar componentwise requirement.

The principal advantage of the generalized cone approach is its ability to tailor the very definition of "order" to the intrinsic geometry of a system's dynamics. This power becomes evident when the cone $\mathcal{V}$ is selected to encode a nonstandard order relation.

Consider the cone

$$\mathcal{V} = \{(u,v) \in \mathbb{R}^2 : u \geqslant v\}.$$

Unlike the standard cone $\mathbb{R}^2_+$ (which induces a componentwise order aligned with the coordinate axes), this cone defines a vector (u_1,v_1) to be "less than or equal to" (u_2,v_2) if and only if the inequality $u_2 - u_1 \geqslant v_2 - v_1$ holds. This ordering is not based on the individual magnitudes of u and v, but on their relative difference. It is fundamentally misaligned with the coordinate axes, making it inaccessible to traditional vector Lyapunov methods, which are confined to the componentwise order of $\mathbb{R}^2_+$.

This flexibility is critical for analyzing systems where stability is governed by relationships between state variables rather than their individual values. For instance, in a model of two competing species, stability might be determined by the balance between their populations $(u - v)$, not by each population in isolation. If the system's dynamics are quasimonotone with respect to this custom cone $\mathcal{V}$ but not with respect to $\mathbb{R}^2_+$, the cone-theoretic comparison principle (Theorem 4.8) can successfully establish stability, whereas both scalar and traditional vector Lyapunov methods would fail.

4.2. *Guidelines for Cone Selection in Practical Applications*

The effectiveness of our cone-theoretic framework hinges on selecting an appropriate cone $\mathcal{V}$ that aligns with the system's inherent structure. Below we provide practical guidance for cone selection across different application scenarios:

4.2.1. *Systems with Linearized Dynamics*

For systems that can be linearized around equilibrium points, analyze the system matrix A. Construct $\mathcal{V}$ as a cone that is invariant under A (i.e., $A\mathcal{V} \subseteq \mathcal{V}$). This can be determined by:

- Identifying the dominant eigenvectors of A and constructing $\mathcal{V}$ as their nonnegative span
- For Metzler matrices, the standard cone $\mathbb{R}^n_+$ is often appropriate
- For matrices with complex eigenstructure, consider rotated or elliptical cones aligned with the principal directions

4.2.2. *Networked Systems with Directed Interactions*

For systems with explicit graph structure or asymmetric coupling:

- For strongly connected directed graphs, base $\mathcal{V}$ on the left Perron-Frobenius eigenvector of the adjacency matrix, which captures the influence structure
- For hierarchical systems, use cones that encode the dominance ordering (e.g., $\{x \in \mathbb{R}^n : x_1 \geqslant x_2 \geqslant \cdots \geqslant x_n\}$)
- For systems with community structure, employ product cones that separate different clusters

4.2.3. *Physically Motivated Systems*

When the system has clear physical interpretations:

- Ecological models: Use cones based on trophic levels or competitive relationships (e.g., $\{(u, v) : u \geqslant v\}$ for predator-prey systems)
- Neural networks: Align cones with excitatory/inhibitory pathways or functional connectivity patterns
- Mechanical systems: Construct cones reflecting conservation laws or constraint manifolds
- Chemical reaction networks: Base cones on stoichiometric compatibility classes

4.2.4. *Computational and Geometric Considerations*

Practical implementation suggestions:

- Start with polyhedral cones (finite intersection of half-spaces) for computational simplicity
- Use Lorentz cones $\{x : x_1 \geqslant \sqrt{x_2^2 + \cdots + x_n^2}\}$ for systems with quadratic-type Lyapunov functions
- For high-dimensional systems, employ product cones $\mathcal{V}_1 \times \mathcal{V}_2$ to decompose the analysis
- Consider spectral cones when the system has natural coordinate transformations

5. Verification Procedure

After selecting a candidate cone $\mathcal{V}$, systematically verify:

1. Check that $\mathcal{V}$ is indeed closed, convex, pointed, and has nonempty interior (Definition 2.1)
2. Verify the generalized quasimonotonicity condition (Definition 2.4) for your specific vector field $g(t, \rho)$
3. Ensure the cone aligns with the physical intuition of your system's stability mechanisms
4. Test the framework on simplified versions of your system before full application

6. Conclusions

In this paper, we developed a cone-theoretic comparison principle for Caputo fractional differential equations, thereby extending cone-valued Lyapunov theory to the fractional-order setting. The main stability implication of this development is summarized in Theorem 4.8, which shows that the evolution of the cone-valued Lyapunov function $\mathcal{L}(t, x(t))$ for system (2.1) is dominated by the maximal solution $r(t)$ of the comparison system (4.1). More precisely, the bound

$$\mathcal{L}(t, x(t)) \leqslant_V r(t), \qquad t \geqslant t_0,$$

allows the stability properties of the comparison system to be transferred directly to the original one. This yields a broadly applicable and technically flexible method for establishing stability in fractional-order systems whose structures fall beyond the reach of conventional comparison techniques.

The usefulness of this approach is particularly evident in models whose interaction geometry is not captured by coordinate-wise monotonicity. Examples include neural networks with asymmetric synaptic coupling, ecological systems characterized by directional or non-reciprocal interactions, and multi-agent systems with directed information exchange. In such settings, choosing a cone $\mathcal{V}$ that reflects the underlying interaction structure enables stability results that classical scalar and vector methods cannot provide. The contributions of this work including the cone-preserving fractional inequality, the construction of maximal solutions, the fractional chain rule, and the general comparison principle, together form a unified foundation for modern stability analysis in fractional-order dynamical systems.

Going forward, several directions for future research emerge. These include applying the theory to derive explicit stability criteria for fractional systems, extending the results to other fractional derivatives such as those of Riemann-Liouville and Hadamard types, and generalizing the framework to time-delay and systems with impulsive effects. Concrete applications in physics, biology, and engineering such as neural networks with asymmetric couplings, ecological models with complex interactions, and quantum systems-stand to benefit from this cone-theoretic perspective.

In conclusion, the cone-theoretic comparison framework developed in this paper opens up new avenues for analyzing fractional-order systems with non-standard order structures. By overcoming the limitations of coordinate-wise monotonicity, it provides a powerful and flexible tool for studying the stability and dynamics of complex fractional systems.

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