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STEP BY STEP APPROACH

NUMERICAL METHOD

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Step by Step Approach Numerical Method

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PREFACE

This eBook is designed to guide learners through the subject in a simple, structured, and practical manner. Each chapter introduces the basic concept, explains the method clearly, and demonstrates the solution process with detailed step-by-step examples.

The eBook emphasizes two main aspects:

- Clarity of Concepts – Fundamental ideas are explained in simple and direct language, avoiding unnecessary technical jargon.

- Step-by-Step Approach – Every method is carefully broken down into manageable steps, supported by worked examples that demonstrate how theoretical knowledge is applied in practice.

It is our sincere hope that this resource not only supports learners in achieving academic success but also inspires curiosity, critical thinking, and problem-solving skills—qualities that are essential in today's fast-evolving technological world.

Finally, we would like to express our gratitude to colleagues, students, and coordinators whose feedback and encouragement have been invaluable in shaping this work. May this eBook serve as a reliable companion in your learning journey.

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Siti Nurul Huda binti Romli
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2025

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NUMERICAL METHOD

Numerical Methods play a vital role in solving mathematical problems that are too complex for exact analytical solutions. In particular, engineering and scientific problems often require systematic techniques to approximate solutions with accuracy and efficiency. Three different approaches are presented and compared:

- **Gaussian Elimination Method**
- **Doolittle Method**
- **Crout Method**

These methods not only provide efficient ways to handle systems of equations but also highlight the differences between elimination and factorization techniques. In addition, this eBook discusses iterative methods for solving nonlinear equations, focusing on two widely used techniques:

- **Fixed Point Iteration Method**
- **Newton Raphson Method**

GAUSSIAN ELIMINATION METHOD

INTRODUCTION

- Gaussian elimination is a fundamental method in linear algebra for solving sets of linear equations. This process, named after Carl Friedrich Gauss, transforms the system's matrix into a simpler version by performing a sequence of basic operations.

- Gaussian elimination is an algorithm that solves systems of linear equations by transforming the system's augmented matrix into row echelon form (REF) using elementary row operations (swapping, scaling, and addition). This simplified form preserves the original solution while making it straightforward to solve for the variables.

- Gaussian elimination is more than just a math concept. It is widely used in real-life problems in science, engineering, and data analysis. Some common applications include:
  - Linear regression: In data science, Gaussian elimination helps solve equations to find the best line that fits data, allowing predictions based on past information.
  - Network analysis: In electrical circuits, engineers use it to find unknown voltages or currents by solving equations that come from Kirchhoff's laws.
  - Matrix inversion: This method is also used to calculate matrix inverses, which are important in machine learning, statistics, and many other areas.

GAUSSIAN ELIMINATION METHOD STEP BY STEP

- Gaussian Elimination transforms a matrix into upper triangular form by creating zeros below the main diagonal using row operations. After obtaining the triangular matrix, the solution is found through back substitution.

$$\begin{bmatrix} 1 & -2 & 1 \\ 2 & 1 & -3 \\ 4 & -7 & 1 \end{bmatrix}$$

Original matrix is converted to upper triangular matrix

$$\begin{bmatrix} 1 & -2 & 1 \\ 0 & 5 & -5 \\ 0 & 0 & -2 \end{bmatrix}$$

- Gaussian Elimination Method is summarized by the following steps :

STEP 1: Write the system in matrix form AX =B

STEP 2: Convert to augmented matrix A | B

STEP 3: 1st Elimination – 1st order transformation (2nd row and 3rd row)

STEP 4: 2nd Elimination – 2nd order transformation (3rd row)

STEP 5: Solve and find the answer

EXAMPLE 1

Solve the linear equation below by using Gaussian Elimination Method :

$$x - 2y + z = 0$$

$$2x + y - 3z = 5$$

$$4x - 7y + z = -1$$

Step 1 : Write the system in matrix form AX = B

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

$$A \quad x = B$$

>>>

$$\begin{bmatrix} 1 & -2 & 1 \\ 2 & 1 & -3 \\ 4 & -7 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 5 \\ -1 \end{bmatrix}$$

Step 2 : Convert to augmented matrix A | B

$$\begin{bmatrix} 1 & -2 & 1 \\ 2 & 1 & -3 \\ 4 & -7 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 5 \\ -1 \end{bmatrix}$$

>>>

$$\left[\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 2 & 1 & -3 & 5 \\ 4 & -7 & 1 & -1 \end{array}\right]$$

Step 3 : 1st elimination - 1st order transformation (2nd row)

$$m_{21} = \frac{a_{21}}{a_{11}} \left[\begin{array}{ccc|c} a_{11} & a_{12} & a_{13} & b_1 \\ a_{21} & a_{22} & a_{23} & b_2 \\ a_{31} & a_{32} & a_{33} & b_3 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 2 & 1 & -3 & 5 \\ 4 & -7 & 1 & -1 \end{array}\right]$$

Produce **zero** below the first entry in the first column make $a_{21} = 0$, 1st order transformation for 2^{nd} **row** which translates into eliminating the **first variable, x** from the **second equations.**

$a'_{21} = a_{21} - a_{11}\left(\frac{a_{21}}{a_{11}}\right) = 0$ - - - - - - - → $a'_{21} = 2 - 1(2) = 0$

$a'_{22} = a_{22} - a_{12}\left(\frac{a_{21}}{a_{11}}\right)$ - - - - - - - → $a'_{22} = 1 - (-2)(2) = 5$

$a'_{23} = a_{23} - a_{13}\left(\frac{a_{21}}{a_{11}}\right)$ - - - - - - - → $a'_{23} = -3 - 1(2) = -5$

$b'_2 = b_2 - b_1\left(\frac{a_{21}}{a_{11}}\right)$ - - - - - - - → $b'_2 = 5 - 0(2) = 5$

Step 3 : 1st elimination - 1st order transformation (3rd row)

$$m_{31} = \frac{a_{31}}{a_{11}} \quad \left[\begin{array}{ccc|c} a_{11} & a_{12} & a_{13} & b_1 \\ a_{21} & a_{22} & a_{23} & b_2 \\ a_{31} & a_{32} & a_{33} & b_3 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 2 & 1 & -3 & 5 \\ 4 & -7 & 1 & -1 \end{array}\right]$$

- Produce **zero** below the first entry in the first column make a_{31} =0, 1st order transformation for 3rd **row**) which translates into eliminating the **first variable, x** from the **third equations.**

$a'_{31} = a_{31} - a_{11}\left(\frac{a_{31}}{a_{11}}\right) = 0$ - - - - - - - → $a'_{31} = 4 - 1(4) = 0$

$a'_{32} = a_{32} - a_{12}\left(\frac{a_{31}}{a_{11}}\right)$ - - - - - - - → $a'_{32} = -7 - (-2)(4) = 1$

$a'_{33} = a_{33} - a_{13}\left(\frac{a_{31}}{a_{11}}\right)$ - - - - - - - → $a'_{33} = 1 - (1)(4) = -3$

$b'_3 = b_3 - b_1\left(\frac{a_{31}}{a_{11}}\right)$ - - - - - - - → $b'_3 = -1 - 0(4) = -1$

The row operations which accomplish this are as follows :

$$\left[\begin{array}{ccc|c} a_{11} & a_{12} & a_{13} & b_1 \\ 0 & a'_{22} & a'_{23} & b'_2 \\ 0 & a'_{32} & a'_{33} & b'_3 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 0 & 5 & -5 & 5 \\ 0 & 1 & -3 & -1 \end{array}\right]$$

Step 4 : 2nd elimination - 1st order transformation (3rd row)

$$m_{32} = \frac{a'_{32}}{a'_{22}} \qquad \left[\begin{array}{ccc|c} a_{11} & a_{12} & a_{13} & b_1 \\ 0 & a'_{22} & a'_{23} & b'_2 \\ 0 & a'_{32} & a'_{33} & b'_3 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 0 & 5 & -5 & 5 \\ 0 & 1 & -3 & -1 \end{array}\right]$$

Produce a **zero** below the second entry in the second column make a_{32}= 0, 2nd order transformation for 3rd **row**) which translates into eliminating the **second variable, x** from the **third equations.**

$$a''_{32} = a'_{32} - a'_{22}\left(\frac{a'_{32}}{a'_{22}}\right) = 0 \quad \dashrightarrow \quad a''_{32} = 1 - 5\left(\frac{1}{5}\right) = 0$$

$$a''_{33} = a'_{33} - a'_{23}\left(\frac{a'_{32}}{a'_{22}}\right) \quad \dashrightarrow \quad a''_{33} = -3 - (-5)\left(\frac{1}{5}\right) = -2$$

$$b''_3 = b'_3 - b'_2\left(\frac{a'_{32}}{a'_{22}}\right) \quad \dashrightarrow \quad b''_3 = -1 - 5\left(\frac{1}{5}\right) = -2$$

The row operations which accomplish this are as follows :

$$\left[\begin{array}{ccc|c} a_{11} & a_{12} & a_{13} & b_1 \\ a'_{21} & a'_{22} & a'_{23} & b'_2 \\ a'_{31} & a''_{32} & a''_{33} & b''_3 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 0 & 5 & -5 & 5 \\ 0 & 0 & -2 & -2 \end{array}\right]$$

Step 5 : Solve and find the answer

Find the value x , y and z using back substitution

$$\left[\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 0 & 5 & -5 & 5 \\ 0 & 0 & -2 & -2 \end{array}\right]$$

$$-2z = -2$$
$$z = 1$$

$$x - 2y + z = 0$$
$$x - 2(2) + 1 = 0$$
$$x = 3$$

$$5y - 5z = 5$$
$$5y - 5(1) = 5$$
$$y = 2$$

Therefore, x = 3, y = 2 and z = 1

CLICK TO WATCH!

SCAN TO WATCH!

EXAMPLE 2

Solve the linear equation below by using Gaussian Elimination Method :

$$5x - 2y - 3z = 0$$
$$4y + 3z + 2 = 0$$
$$x - 4y + 9z - 60 = 0$$

Step 1 : Write the system in matrix form AX = B

Rearrange the equation into general form of linear equation :

$$5x - 2y - 3z = 0$$
$$4y + 3z = -2$$
$$x - 4y + 9z = 60$$

Step 2 : Convert to augmented matrix A | B

$$\begin{bmatrix} 5 & -2 & -3 \\ 0 & 4 & 3 \\ 1 & -4 & 9 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ -2 \\ 60 \end{bmatrix} \quad \ggg \quad \left[\begin{array}{ccc|c} 5 & -2 & -3 & 0 \\ 0 & 4 & 3 & -2 \\ 1 & -4 & 9 & 60 \end{array}\right]$$

Step 3 : 1st elimination - 1st order transformation (2nd row)

In this case, it is **unnecessary to perform the first-order transformation on the second row since a21 is already zero**; therefore, the process can proceed directly to the **third row.**

$$m_{31} = \frac{a_{31}}{a_{11}} \quad \left[\begin{array}{ccc|c} 5 & -2 & -3 & 0 \\ 0 & 4 & 3 & -2 \\ 1 & -4 & 9 & 60 \end{array}\right]$$

$$a'_{31} = 1 - 5\left(\frac{1}{5}\right) = 0$$

$$a'_{32} = -4 - (-2)\left(\frac{1}{5}\right) = -\frac{18}{5}$$

$$a'_{33} = 9 - (-3)\left(\frac{1}{5}\right) = \frac{48}{5}$$

$$b'_{3} = 60 - 0\left(\frac{1}{5}\right) = 60$$

The row operations which accomplish this are as follows :

$$\left[\begin{array}{ccc|c} 5 & -2 & -3 & 0 \\ 0 & 4 & 3 & -2 \\ 0 & -\frac{18}{5} & \frac{48}{5} & 60 \end{array}\right]$$

Step 4 : 2nd elimination - 1st order transformation (3rd row)

$$m_{32} = \frac{a'_{32}}{a'_{22}} \quad \left[\begin{array}{ccc|c} 5 & -2 & -3 & 0 \\ 0 & 4 & 3 & -2 \\ 0 & -\frac{18}{5} & \frac{48}{5} & 60 \end{array}\right]$$

$$a''_{32} = -\frac{18}{5} - 4\left(\frac{-\frac{18}{5}}{4}\right) = 0$$

$$a''_{33} = \frac{48}{5} - (3)\left(\frac{-\frac{18}{5}}{4}\right) = \frac{123}{10}$$

$$b''_{3} = 60 - (-2)\left(\frac{-\frac{18}{5}}{4}\right) = \frac{291}{5}$$

The row operations which accomplish this are as follows :

$$\left[\begin{array}{ccc|c} 5 & -2 & -3 & 0 \\ 0 & 4 & 3 & -2 \\ 0 & 0 & \frac{123}{10} & \frac{291}{5} \end{array}\right]$$

Step 5 : Solve and find the answer

Find the value x , y and z using back substitution

$$\left[\begin{array}{ccc|c} 5 & -2 & -3 & 0 \\ 0 & 4 & 3 & -2 \\ 0 & 0 & \frac{123}{10} & \frac{291}{5} \end{array}\right]$$

$$\frac{123}{10}z = \frac{291}{5}$$

$$z = \frac{194}{41} @ 4.73$$

$$5x - 2y - 3z = 0$$

$$5x - 2\left(-\frac{166}{41}\right) - 3\left(\frac{194}{41}\right) = 0$$

$$x = -\frac{50}{41} @ -1.22$$

$$4y + 3z = -2$$

$$4y + 3\left(\frac{194}{41}\right) = -2$$

$$y = -\frac{166}{41} @ -4.05$$

Therefore $x = -\frac{50}{41}$ $y = -\frac{166}{41}$ $z = \frac{194}{41}$

QUESTIONS

1

Solve the linear equation below by using Gaussian Elimination Method :

$$-3x_1 + 2x_2 - x_3 = -1$$
$$6x_1 - 6x_2 + 7x_3 = -7$$
$$3x_1 - 4x_2 + 4x_3 = -6$$

Answer : $x_1 = 2 \quad x_2 = 2 \quad x_3 = -1$

2

Solve the following equations by using Gaussian Elimination Method :

$$2x + y + 4z = 12$$
$$4x + 11y - z = 33$$
$$8x - 3y + 2z = 20$$

Answer : $x = 3 \quad y = 2 \quad z = 1$

3

Solve the linear equation below by using Gaussian Elimination Method :

$$x + 2y + z = 5$$
$$3x + 10y + 6z = 17$$
$$8y + 14z = 20$$

Answer : $x = 5 \quad y = -1 \quad z = 2$

DOOLITTLE METHOD

LU DECOMPOSITION

- LU decomposition is a method of breaking down a square matrix into the product of two triangular matrices: a lower triangular matrix L and an upper triangular matrix U. Multiplying these two matrices reconstructs the original matrix

- This technique has many applications, particularly in solving systems of linear equations. It also plays an important role in tasks such as analyzing electrical circuits, computing the inverse of a matrix, and determining a matrix's determinant.

DOOLITTLE ALGORITHM

- Doolittle's Algorithm provides a way to factor a matrix A into an LU decomposition.

- Any square matrix can be decomposed into the product of a lower triangular matrix and an upper triangular matrix, expressed as:

$$A = LU$$

- We begin by assuming that an LU decomposition exists for a general $n \times n$ matrix A. The forms of L and U are written directly, and then the equations arising from the multiplication $A = LU$ are systematically applied to determine the entries of both matrices.

- In the Doolittle Method, the lower triangular matrix L has unit diagonal elements (1s), while U contains the general upper triangular entries

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix} \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix}$$

$$A \quad = \quad L \quad U$$

DOOLITTLE METHOD STEP BY STEP

- Doolittle Method is summarized by the following steps :

STEP 1 Write the system in matrix form AX =B

STEP 2 Construct the matrices A=LU and find matrices L and U

STEP 3 Let LY=B to solve for y using back substitution

STEP 4 Let UX=Y to solve for x using back substitution

EXAMPLE 1

Solve the linear equation below by using Doolittle Method

$$3x + 2y - z = 10$$
$$7x - y + 6z = 8$$
$$3x + 2z = 5$$

Step 1 : Write the system in matrix form AX = B

$$\underset{A}{\begin{bmatrix} 3 & 2 & -1 \\ 7 & -1 & 6 \\ 3 & 0 & 2 \end{bmatrix}} \underset{x}{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}} \underset{=}{=} \underset{B}{\begin{bmatrix} 10 \\ 8 \\ 5 \end{bmatrix}}$$

Step 2 : Construct the matrices A=LU and find matrices L and U

Let ***A = LU*** where ***L*** is the **lower triangular matrix** and ***U*** is the **upper triangular matrix** assume that the diagonal entries L is equal to 1

$$\underset{A}{\begin{bmatrix} 3 & 2 & -1 \\ 7 & -1 & 6 \\ 3 & 0 & 2 \end{bmatrix}} = \underset{L}{\begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix}} \underset{U}{\begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix}}$$

$$\underset{A}{\begin{bmatrix} 3 & 2 & -1 \\ 7 & -1 & 6 \\ 3 & 0 & 2 \end{bmatrix}} = \underset{L}{\begin{bmatrix} 1 & 0 & 0 \\ a & 1 & 0 \\ b & c & 1 \end{bmatrix}} \underset{U}{\begin{bmatrix} d & e & f \\ 0 & g & h \\ 0 & 0 & i \end{bmatrix}}$$

You may replace the symbols u_{11}, u_{12}, u_{13} with other alphabets like d, e, f...

Multiply L and U

			U		
L			d	e	f
			0	g	h
			0	0	i
1	0	0	d	e	f
a	1	0	ad	$ae+g$	$af+h$
b	c	1	bd	$be+cg$	$bf+ch+i$

$d = 3$	$e = 2$	$f = -1$
$ad = 7$ $a(3) = 7$ $a = \frac{7}{3}$	$ae + g = -1$ $\frac{7}{3}(2) + g = -1$ $g = -\frac{17}{3}$	$af + h = 6$ $\frac{7}{3}(-1) + h = 6$ $h = \frac{25}{3}$
$bd = 3$ $b(3) = 3$ $b = 1$	$be + cg = 0$ $(1)(2) + c\left(-\frac{17}{3}\right) = 0$ $c = \frac{6}{17}$	$bf + ch + i = 2$ $(1)(-1) + \left(\frac{6}{17}\right)\left(\frac{25}{3}\right) + i = 2$ $i = \frac{1}{17}$

Matrix *L* and matrix *U* are written as follows :

$$\underset{L}{\begin{bmatrix} 1 & 0 & 0 \\ a & 1 & 0 \\ b & c & 1 \end{bmatrix}} \underset{U}{\begin{bmatrix} d & e & f \\ 0 & g & h \\ 0 & 0 & i \end{bmatrix}}$$

$$L = \begin{bmatrix} 1 & 0 & 0 \\ \frac{7}{3} & 1 & 0 \\ 1 & \frac{6}{17} & 1 \end{bmatrix} \qquad U = \begin{bmatrix} 3 & 2 & -1 \\ 0 & -\frac{17}{3} & \frac{25}{3} \\ 0 & 0 & \frac{1}{17} \end{bmatrix}$$

Step 3 : Let LY=B to solve for y using back substitution

Let Ly = B, solve for y's

$$\begin{bmatrix} 1 & 0 & 0 \\ \frac{7}{3} & 1 & 0 \\ 1 & \frac{6}{17} & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 10 \\ 8 \\ 5 \end{bmatrix}$$

$$y_1 = 10$$

$$\frac{7}{3}y_1 + y_2 = 8$$

$$\left(\frac{7}{3}\right)(10) + y_2 = 8$$

$$y_2 = -\frac{46}{3}$$

$$y_1 + \frac{6}{17}y_2 + y_3 = 5$$

$$10 + \left(\frac{6}{17}\right)\left(-\frac{46}{3}\right) + y_3 = 5$$

$$y_3 = \frac{7}{17}$$

Step 4 : Let Ux=Y to solve for x using back substitution

Let Ux = Y, solve for the variable vectors x, y and z

$$\begin{bmatrix} 3 & 2 & -1 \\ 0 & -\frac{17}{3} & \frac{25}{3} \\ 0 & 0 & \frac{1}{17} \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 10 \\ -\frac{46}{3} \\ \frac{7}{17} \end{bmatrix}$$

$$\frac{1}{17}z = \frac{7}{17}$$

$$z = 7$$

$$-\frac{17}{3}y + \frac{25}{3}z = -\frac{46}{3}$$

$$-\frac{17}{3}y + \left(\frac{25}{3}\right)(7) = -\frac{46}{3}$$

$$-\frac{17}{3}y = -\frac{221}{3}$$

$$y = 13$$

$$3x + 2y - z = 10$$

$$3x + (2)(13) - 7 = 10$$

$$3x = -9$$

$$x = -3$$

CLICK TO WATCH!

SCAN TO WATCH!

EXAMPLE 2

Solve the linear equation below by using Doolittle Method :

$$x + 4y - z = -1$$

$$3x - 2y + z = 6$$

$$4x + 2y - 2z = -3$$

Step 1 : Write the system in matrix form AX = B

$$\begin{bmatrix} 1 & 4 & -1 \\ 3 & -2 & 1 \\ 4 & 2 & -2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -1 \\ 6 \\ -3 \end{bmatrix}$$

$$A \qquad x \; = \; B$$

Step 2 : Construct the matrices A=LU and find matrices L and U

Let ***A = LU*** where ***L*** is the **lower triangular matrix** and ***U*** is the **upper triangular matrix** assume that the diagonal entries L is equal to 1

$$\begin{bmatrix} 1 & 4 & -1 \\ 3 & -2 & 1 \\ 4 & 2 & -2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix} \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix}$$

$$A \; = \; L \qquad U$$

$$\begin{bmatrix} 1 & 4 & -1 \\ 3 & -2 & 1 \\ 4 & 2 & -2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ a & 1 & 0 \\ b & c & 1 \end{bmatrix} \begin{bmatrix} d & e & f \\ 0 & g & h \\ 0 & 0 & i \end{bmatrix}$$

$$A \; = \; L \qquad U$$

You may replace the symbols u_{11}, u_{12}, u_{13} with other alphabets like d, e, f...

Multiply L and U

			U		
			d	e	f
	L		0	g	h
			0	0	i
1	0	0	d	e	f
a	1	0	ad	$ae+g$	$af+h$
b	c	1	bd	$be+cg$	$bf+ch+i$

$d=1$	$e=4$	$f=-1$
$ad=3$ $a(1)=3$ $a=3$	$ae+g=-2$ $(3)(4)+g=-2$ $g=-14$	$af+h=1$ $3(-1)+h=1$ $h=4$
$bd=4$ $b(1)=4$ $b=4$	$be+cg=2$ $(4)(4)+c(-14)=2$ $c=1$	$bf+ch+i=-2$ $(4)(-1)+(1)(4)+i=-2$ $i=-2$

Matrix *L* and matrix *U* are written as follows :

$$\underbrace{\begin{bmatrix} 1 & 0 & 0 \\ a & 1 & 0 \\ b & c & 1 \end{bmatrix}}_{L}\underbrace{\begin{bmatrix} d & e & f \\ 0 & g & h \\ 0 & 0 & i \end{bmatrix}}_{U}$$

$$L=\begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 4 & 1 & 1 \end{bmatrix} \qquad U=\begin{bmatrix} 1 & 4 & -1 \\ 0 & -14 & 4 \\ 0 & 0 & -2 \end{bmatrix}$$

Step 3 : Let Ly=B to solve for y using back substitution

Let Ly = B, solve for y's

$$\begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 4 & 1 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} -1 \\ 6 \\ -3 \end{bmatrix}$$

$$y_1 = -1$$

$$3y_1 + y_2 = 6$$
$$(3)(-1) + y_2 = 6$$
$$y_2 = 9$$

$$4y_1 + y_2 + y_3 = -3$$
$$4(-1) + 9 + y_3 = -3$$
$$y_3 = -8$$

Step 4 : Let UX=Y to solve for x using back substitution

Let Ux = Y, solve for the variable vectors x

$$\begin{bmatrix} 1 & 4 & -1 \\ 0 & -14 & 4 \\ 0 & 0 & -2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -1 \\ 9 \\ -8 \end{bmatrix}$$

$$-2z = -8$$
$$z = 4$$

$$-14y + 4z = 9$$
$$-14y + 4(4) = 9$$
$$-14y = -7$$
$$y = \frac{1}{2}$$

$$x + 4y - z = -1$$
$$x + 4\left(\frac{1}{2}\right) - 4 = -1$$
$$x = 1$$

QUESTIONS

1 Identify the matrix L and U for the equation below by using Doolittle Method

$$s + 4t - 2u = 3$$
$$3s - 2t + 5u = 14$$
$$2s + 3t + u = 11$$

Answer : $L = \begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 2 & \frac{5}{14} & 1 \end{bmatrix}$ $U = \begin{bmatrix} 1 & 4 & -2 \\ 0 & -14 & 11 \\ 0 & 0 & \frac{15}{14} \end{bmatrix}$

2 Solve the following equations by using Doolittle Method :

$$x_1 + 3x_2 + 3x_3 = 4$$
$$2x_1 - 3x_2 - 2x_3 = 2$$
$$3x_1 + x_2 + 2x_3 = 5$$

Answer : $x_1 = 7, \quad x_2 = 14, \quad x_3 = -15$

3 Solve the linear equation below by using Doolittle Method :

$$6x - 12y + 10z = 12$$
$$-8y + 6z = 4$$
$$4x + 8y - 8z = 10$$

Answer : $x = 1.625, \quad y = -3.3125, \quad z = -3.75$

CROUT METHOD

CROUT METHOD

- The Crout Method is a numerical approach for performing LU decomposition of a matrix. In this technique, a matrix is factored into two triangular matrices: a lower triangular matrix L and an upper triangular matrix U,with the diagonal entries of U fixed at 1.

- This decomposition, written as $A=LU$, simplifies the process of solving a s ystem of linear equations $AX=B$ Instead of tackling the full system directly, it is divided into two easier steps.
  - Solve $LY=B$ for Y
  - Solve $UX=Y$ for X

- By systematically calculating the entries of L and U,the Crout Method provides an efficient and organized way to find the unknowns in X.

- In the Crout Method, the upper triangular matrix U has unit diagonal elements (1s), while L contains the general lower triangular entries.

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = \begin{bmatrix} l_{11} & 0 & 0 \\ l_{21} & l_{22} & 0 \\ l_{31} & l_{32} & l_{33} \end{bmatrix} \begin{bmatrix} 1 & u_{12} & u_{13} \\ 0 & 1 & u_{23} \\ 0 & 0 & 1 \end{bmatrix}$$

$$\boldsymbol{A} = \boldsymbol{L} \quad \boldsymbol{U}$$

CROUT METHOD STEP BY STEP

- Crout Method is summarized by the following steps :

STEP 1 Write the system in matrix form AX =B

STEP 2 Construct the matrices A=LU and find matrices L and U

STEP 3 Let LY=B to solve for y using back substitution

STEP 4 Let UX=Y to solve for x using back substitution

EXAMPLE 1

Solve the linear equation below by using Crout Method :

$$2x - y = 3$$
$$-x + 2y - z = -3$$
$$2y + 5z = 9$$

Step 1 : Write the system in matrix form AX = B

$$\begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & 2 & 5 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ -3 \\ 9 \end{bmatrix}$$

Step 2 : Construct the matrices A=LU and find matrices L and U

Let ***A = LU*** where ***L*** is the **lower triangular matrix** and ***U*** is the **upper triangular matrix** assume that the diagonal entries U is equal to 1

$$\underset{A}{\begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & 2 & 5 \end{bmatrix}} = \underset{L}{\begin{bmatrix} l_{11} & 0 & 0 \\ l_{21} & l_{22} & 0 \\ l_{31} & l_{32} & l_{33} \end{bmatrix}} \underset{U}{\begin{bmatrix} 1 & u_{12} & u_{13} \\ 0 & 1 & u_{23} \\ 0 & 0 & 1 \end{bmatrix}}$$

$$\underset{A}{\begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & 2 & 5 \end{bmatrix}} = \underset{L}{\begin{bmatrix} a & 0 & 0 \\ b & d & 0 \\ c & e & f \end{bmatrix}} \underset{U}{\begin{bmatrix} 1 & g & h \\ 0 & 1 & i \\ 0 & 0 & 1 \end{bmatrix}}$$

You may replace the symbols l_{11}, l_{21}, l_{31} with other alphabets like a, b, c...

U

L

			1	g	h
			0	1	i
			0	0	1
a	0	0	a	ag	ah
b	d	0	b	bg+d	bh+di
c	e	f	C	cg+e	ch+ei+f

$a = 2$	$ag = -1$ $(2)g = -1$ $g = -\frac{1}{2}$	$ah = 0$ $(2)h = 0$ $h = 0$
$b = -1$	$bg + d = 2$ $(-1)(-\frac{1}{2})+d = 2$ $d = \frac{3}{2}$	$bh + di = -1$ $(-1)(0) + (= \frac{3}{2})i$ $i = -\frac{2}{3}$
$c = 0$	$cg + e = 2$ $(0)\left(-\frac{1}{2}\right) + e = 2$ $e = 2$	$ch + ei + f = 5$ $(0)(0) + (2)(-\frac{2}{3}) + f = 5$ $f = \frac{19}{3}$

Matrix L and matrix U are written as follows :

$$\underset{L}{\begin{bmatrix} a & 0 & 0 \\ b & d & 0 \\ c & e & f \end{bmatrix}} \underset{U}{\begin{bmatrix} 1 & g & h \\ 0 & 1 & i \\ 0 & 0 & 1 \end{bmatrix}}$$

$$L = \begin{bmatrix} 2 & 0 & 0 \\ -1 & \frac{3}{2} & 0 \\ 0 & 2 & \frac{19}{3} \end{bmatrix} \qquad U = \begin{bmatrix} 1 & -\frac{1}{2} & 0 \\ 0 & 1 & -\frac{2}{3} \\ 0 & 0 & 1 \end{bmatrix}$$

Step 3 : Let Ly=B to solve for y using back substitution

Let Ly = B, solve for y's

$$\begin{bmatrix} 2 & 0 & 0 \\ -1 & \frac{3}{2} & 0 \\ 0 & 2 & \frac{19}{3} \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 3 \\ -3 \\ 9 \end{bmatrix}$$

$$2y_1 = 3$$
$$y_1 = \frac{3}{2}$$

$$-y_1 + \frac{3}{2}y_2 = -3$$
$$\left(-\frac{3}{2}\right) + \frac{3}{2}y_2 = -3$$
$$y_2 = -1$$

$$2y_2 + \frac{19}{3}y_3 = 9$$
$$(2)(-1) + \frac{19}{3}y_3 = 9$$
$$y_3 = \frac{33}{19}$$

Step 4 : Let Ux=Y to solve for x using back substitution

Let Ux = Y, solve for the variable vectors x

$$\begin{bmatrix} 1 & -\frac{1}{2} & 0 \\ 0 & 1 & -\frac{2}{3} \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} \frac{3}{2} \\ -1 \\ \frac{33}{19} \end{bmatrix}$$

$$x_3 = \frac{33}{19}$$

$$x_2 - \frac{2}{3}x_3 = -1$$
$$x_2 - \left(\frac{2}{3}\right)\left(\frac{33}{19}\right) = -1$$
$$x_2 = \frac{3}{19}$$

$$x_1 - \frac{1}{2}x_2 = \frac{3}{2}$$
$$x_1 - \left(\frac{1}{2}\right)\left(\frac{3}{19}\right) = \frac{3}{2}$$
$$x_1 = \frac{30}{19}$$

CLICK TO WATCH!

SCAN TO WATCH!

EXAMPLE 2

Solve the linear equation below by using Crout Method :

$$4y + x - z = -1$$
$$3x - 2y + z = 6$$
$$4x + 2y + 3 = 2z$$

Step 1 : Write the system in matrix form AX = B

$$\begin{bmatrix} 1 & 4 & -1 \\ 3 & -2 & 1 \\ 4 & 2 & -2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -1 \\ 6 \\ -3 \end{bmatrix}$$

Step 2 : Construct the matrices A=LU and find matrices L and U

Let ***A = LU*** where ***L*** is the **lower triangular matrix** and ***U*** is the **upper triangular matrix** assume that the diagonal entries U is equal to 1

$$\underset{A}{\begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & 2 & 5 \end{bmatrix}} = \underset{L}{\begin{bmatrix} l_{11} & 0 & 0 \\ l_{21} & l_{22} & 0 \\ l_{31} & l_{32} & l_{33} \end{bmatrix}} \underset{U}{\begin{bmatrix} 1 & u_{12} & u_{13} \\ 0 & 1 & u_{23} \\ 0 & 0 & 1 \end{bmatrix}}$$

$$\underset{A}{\begin{bmatrix} 1 & 4 & -1 \\ 3 & -2 & 1 \\ 4 & 2 & -2 \end{bmatrix}} = \underset{L}{\begin{bmatrix} a & 0 & 0 \\ b & d & 0 \\ c & e & f \end{bmatrix}} \underset{U}{\begin{bmatrix} 1 & g & h \\ 0 & 1 & i \\ 0 & 0 & 1 \end{bmatrix}}$$

You may replace the symbols l_{11}, l_{21}, l_{31} with other alphabets like a, b, c...

U

			1	g	h
			0	1	i
L			0	0	1
a	0	0	a	ag	ah
b	d	0	b	bg+d	bh+di
c	e	f	C	cg+e	ch+ei+f

Multiply L and U

$a = 1$	$ag = 4$ $(1)g = 4$ $g = 4$	$ah = -1$ $(1)h = -1$ $h = -1$
$b = 3$	$bg + d = -2$ $(3)(4) + d = -2$ $12 + d = -2$ $d = -14$	$bh + di = 1$ $(3)(-1) + (-14)i = 1$ $-3 - 14i = 1$ $-14i = 4$ $i = -\frac{2}{7}$
$c = 4$	$cg + e = 2$ $(4)(4) + e = 2$ $16 + e = 2$ $e = -14$	$ch + ei + f = -2$ $(4)(-1) + (-14)\left(-\frac{2}{7}\right) + f = -2$ $-4 + 4 + f = -2$ $f = -2$

Matrix L and matrix U are written as follows :

$$\underset{L}{\begin{bmatrix} a & 0 & 0 \\ b & d & 0 \\ c & e & f \end{bmatrix}} \underset{U}{\begin{bmatrix} 1 & g & h \\ 0 & 1 & i \\ 0 & 0 & 1 \end{bmatrix}}$$

$$L = \begin{bmatrix} 1 & 0 & 0 \\ 3 & -14 & 0 \\ 4 & -14 & -2 \end{bmatrix} \qquad U = \begin{bmatrix} 1 & 4 & -1 \\ 0 & 1 & -\frac{2}{7} \\ 0 & 0 & 1 \end{bmatrix}$$

Step 3 : Let Ly=B to solve for y using back substitution

Let Ly = B, solve for y's

$$\begin{bmatrix} 1 & 0 & 0 \\ 3 & -14 & 0 \\ 4 & -14 & -2 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} -1 \\ 6 \\ -3 \end{bmatrix}$$

$$y_1 = -1$$

$$3y_1 - 14y_2 = 6$$
$$3(-1) - 14y_2 = 6$$
$$y_2 = -\frac{9}{14}$$

$$4y_1 - 14y_2 - 2y_3 = -3$$
$$4(-1) - 14\left(-\frac{9}{14}\right) - 2y_3 = -3$$
$$y_3 = 4$$

Step 4 : Let Ux=Y to solve for x using back substitution

Let Ux = Y, solve for the variable vectors x

$$\begin{bmatrix} 1 & 4 & -1 \\ 0 & 1 & -\frac{2}{7} \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -1 \\ -\frac{9}{14} \\ 4 \end{bmatrix}$$

$$z = 4$$

$$y - \frac{2}{7}z = -\frac{9}{14}$$
$$y - \frac{2}{7}(4) = -\frac{9}{14}$$
$$y = \frac{1}{2}$$

$$x + 4y - z = -1$$
$$x + 4\left(\frac{1}{2}\right) - 4 = -1$$
$$x = 1$$

QUESTIONS

1

Identify the matrix L and U for the equation below by using Crout Method

$$3x_1 + x_2 + x_3 = 4$$
$$x_1 + 2x_2 + 2x_3 = 3$$
$$2x_1 + x_2 + 3x_3 = 4$$

Answer : $L = \begin{bmatrix} 3 & 0 & 0 \\ 1 & \frac{5}{3} & 0 \\ 4 & \frac{1}{3} & 2 \end{bmatrix}$ $U = \begin{bmatrix} 1 & \frac{1}{3} & \frac{1}{3} \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$

2

Solve the following equations by using Crout Method :

$$2x + y + 4z = 12$$
$$8x - 3y + 2z = 20$$
$$4x + 11y_2 - z = 33$$

Answer : $x = 3, \quad y = 2, \quad z = 1$

3

Solve the linear equation below by using Crout Method :

$$3s + 2t + 7u = 4$$
$$2s + 3t + u = 5$$
$$3s + 4t + u = 7$$

Answer : $s = \frac{7}{8}, \quad t = \frac{9}{8}, \quad u = -\frac{1}{8}$

FIXED POINT ITERATION

FIXED POINT ITERATION METHOD

The fixed-point iteration method is an algorithm in numerical analysis for finding the roots of an equation by transforming the equation into the form $x = g(x)$ and then repeatedly applying the function g starting from an initial guess.The sequence of values, $x_0, x_1 = g(x_0), x_2 = g(x_1)$ and so on, is generated, and if the sequence converges, it will do so to a fixed point such that $p = g(p)$, which is also a root of the original equation.

- Given an equation of one variable, we use fixed point iteration as follows steps :

STEP 1 Convert the equation of $f(x) = 0$ into $x = g(x)$

STEP 2 Substituting the initial guess x_0 into $x = g(x)$

STEP 3 $|g'(x)| < 1$ iterative sequence converges to the root
$|g'(x)| > 1$ iterative method fails

STEP 4 Calculating the root – the root will be obtained when the value of x are keep repeating

EXAMPLE 1

Solve $2x^3 - 3x = 1$ using Fixed Point Iteration method. Given $x_0 = 1.5$. Give your answer in 3 decimal places.

Step 1: Convert the equation of $f(x) = 0$ into $g(x)$

Make x as a subject : $2x^3 - 3x - 1 = 0$

$$x(2x^2 - 3) = 1$$

$$x = \frac{1}{2x^2-3}, \text{ means } g(x) = \frac{1}{2x^2-3}$$

Step 2 : Substituting the initial guess x_0 into $|g'(x)|$

Substituting the initial guess x_0 into $|g'(x)|$

$$g'(x) = \frac{-4x}{(2x^2-3)^2}$$

$$\text{So, } |g'(1.5)| = \frac{-4(1.5)}{[2(1.5)^2-3]^2} = 2.67$$

Step 3 : $|g'(x)| < 1$ iterative sequence converges to the root
$|g'(x)| > 1$ iterative method fails

Check the value of $|g'(x)|$.

- If $|g'(x)| < 1$, the iterative sequence converge to the root, we can proceed with the chosen iterative function.
- If $|g'(x)| > 1$, the iterative method fails. In this case, a new iterative function must be formulated by repeating the earlier step.
- Since we obtained $|g'(x)|$ =2.67 which is $|g'(x)| > 1$, we must find an alternative iterative function

Repeat Step 1

Make x as a subject : $2x^3 - 3x - 1 = 0$

$$x = (\frac{3x+1}{2})^{\frac{1}{3}}$$

means, $g(x) = (\frac{3x+1}{2})^{\frac{1}{3}}$ @ $\sqrt[3]{\frac{3x+1}{2}}$

Repeat Step 2

Substituting the initial guess x_0 into $|g'(x)|$

$$g'(x) = \frac{1}{2}(\frac{3x+1}{2})^{-\frac{2}{3}}$$

So, $|g'(1.5)| = \frac{1}{2}(\frac{3(1.5)+1}{2})^{-\frac{2}{3}} = 0.255$

Repeat Step 3

Check the value of $|g'(x)|$.

- If $|g'(x)| < 1$, the iterative sequence converge to the root, we can proceed with the chosen iterative function.
- If $|g'(x)| > 1$, the iterative method fails. In this case, a new iterative function must be formulated by repeating the earlier step.

- Since we obtained $|g'(x)|$=0.255 which $|g'(x)| < 1$ is this rearrangement is a valid iterative function.

Step 4 : Calculating the root

The root is obtained when the values of x begin to stabilize and repeat.

Using the iteration function $g(x) = \sqrt[3]{\frac{3x+1}{2}}$ with the initial guess $x_0 = 1.5$

x	$x_{n+1} = \sqrt[3]{\frac{3x+1}{2}}$	iteration
$x_0 = 1.5$	$x_1 = \sqrt[3]{\frac{3x_0+1}{2}}$	$x_1 = \sqrt[3]{\frac{3(1.5)+1}{2}}$ $= 1.401$
$x_1 = 1.401$	$x_2 = \sqrt[3]{\frac{3x_1+1}{2}}$	$x_2 = \sqrt[3]{\frac{3(1.401)+1}{2}}$ $= 1.375$
$x_2 = 1.375$	$x_3 = \sqrt[3]{\frac{3x_2+1}{2}}$	$x_3 = \sqrt[3]{\frac{3(1.375)+1}{2}}$ $= 1.368$
$x_3 = 1.368$	$x_4 = \sqrt[3]{\frac{3x_3+1}{2}}$	$x_4 = \sqrt[3]{\frac{3(1.368)+1}{2}}$ $= 1.367$
$x_4 = 1.367$	$x_5 = \sqrt[3]{\frac{3x_4+1}{2}}$	$x_5 = \sqrt[3]{\frac{3(1.367)+1}{2}}$ $= 1.366$
$x_5 = 1.367$	$x_6 = \sqrt[3]{\frac{3x_4+1}{2}}$	$x_6 = \sqrt[3]{\frac{3(1.367)+1}{2}}$ $= 1.366$

EXAMPLE 2

Determine the root for equation $x^3 - 6x - 4 = 0$ by using Fix Point Iteration method. Give answer correct to 3 decimal places.

Step 1 : Convert the equation of $f(x) = 0$ into $x = g(x)$

Solution :

Lets' find $g(x)$ ⟶

$$x^3 = 6x - 4$$

$$x = \sqrt[3]{6x + 4}$$

$$g(x) = \sqrt[3]{6x + 4} \text{ @ } (6x + 4)^{1/3}$$

Step 2 : Substituting the initial guess x_0 into $|g'(x)|$

Since x_0 was not given. Let's find by using false position method. From the given equation let $y = x^3 - 6x - 4$. Choose any value of x_1 and x_2

Let $x_1 = 0$

$$y_1 = 0 - 6(0) - 4$$

$$y_1 = -4$$

Let $x_2 = 1$

$$y_2 = (1)^3 - 6(1) - 4$$

$$y_2 = -9$$

Insert x_1, x_2, y_1 and y_2 into $x_0 = \frac{1}{y_2 - y_1}\begin{vmatrix} x_1 & y_1 \\ x_2 & y_2 \end{vmatrix}$

Then $x_0 = \frac{1}{-9-(-4)}\begin{vmatrix} 0 & -4 \\ 1 & -9 \end{vmatrix}$

$$x_0 = \frac{1}{-5}(4)$$

$$= -0.8 \text{ (3 d.p)}$$

Substituting the initial guess x_0 into $|g'(x)|$.

$$g'(x) = \frac{1}{3}(6x + 4)^{-\frac{2}{3}}(6)$$

$$g'(x) = \frac{2}{(6x + 4)^{\frac{2}{3}}}$$

$$|g'(1.333)| = \frac{2}{[6(-0.8)+4]^{\frac{2}{3}}} = 2.321$$

Step 3 : $|g'(x)| < 1$ iterative sequence converges to the root
$|g'(x)| > 1$ iterative method was failed

Since we got $|g'(x)| = 2.321$, which is $|g'(x)| > 1$ we need to find another iterative function.

Let's get back to earlier step like before – find another $g(x)$ from the given equation.

$$x^3 - 4 = 6x$$

$$x = \frac{x^3 - 4}{6}$$

$$g(x) = \frac{x^3 - 4}{6}$$

$$g'(x) = \frac{3x^2}{6} \quad @\frac{x^2}{2}$$

$$g'(x) = \frac{(-0.8)^2}{2}$$

$$|g'(-0.8)| = 0.32 < 1$$ (valid iteration function)

Step 4 : Calculating the root

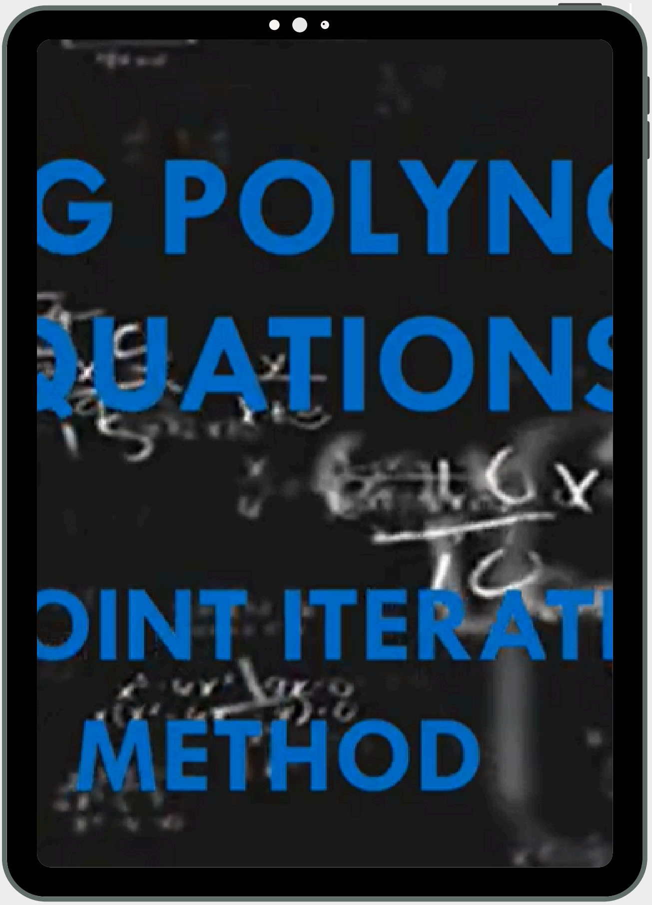
The root will be obtained when the value of x are keep repeating.

By using Iteration fuction of $x_{n+1} = \frac{x^3-4}{6}$ with initial guess of $x_0 = 1$

x	$x_{n+1} = \frac{x_0{}^3-4}{6}$	iteration
$x_0 = -0.8$	$x_1 = \frac{x_1{}^3-4}{6}$	$x_1 = \frac{(-0.8)^3-4}{6}$ $= -0.752$
$x_1 = -0.752$	$x_2 = \frac{x_2{}^3-4}{6}$	$x_2 = \frac{(-0.752)^3-4}{6}$ $= -0.738$
$x_2 = -0.738$	$x_3 = \frac{x_3{}^3-4}{6}$	$x_3 = \frac{(-0.738)^3-4}{6}$ $= -0.734$
$x_3 = -0.734$	$x_4 = \frac{x_4{}^3-4}{6}$	$x_4 = \frac{(-0.734)^3-4}{6}$ $= -0.732$
$x_4 = -0.732$	$x_5 = \frac{x_5{}^3-4}{6}$	$x_5 = \frac{(-0.732)^3-4}{6}$ $= -0.732$

Since the value of x is repeated (refer x_4 and x_5) then the root is $x = -0.732$

CLICK TO WATCH!



SCAN TO WATCH!

EXAMPLE 3

How about if initial guess or first approximate (x_0) is not given?
In this case, we can use False Position Method,

Step 1 : Get the value of x_1, x_2, y_1 and y_2

Consider the same equation, but x_0 was not given. Let's find by using false position method.
From the given equation let , $y = 2x^3 - 3x - 1$ .Choose any value of x_1 and x_2

Let $x_1 = 0$

$y_1 = 2(0)^3 - 3(0) - 1$

$y_1 = 1$

Let $x_2 = 1$

$y_2 = 2(1)^3 - 3(1) - 1$

$y_2 = -2$

Insert x_1, x_2, y_1 and y_2

$$x_0 = \frac{1}{y_2 - y_1}\begin{vmatrix} x_1 & y_1 \\ x_2 & y_2 \end{vmatrix}$$

Then $x_0 = \frac{1}{-2-1}\begin{vmatrix} 0 & 1 \\ 1 & -2 \end{vmatrix}$

$x_0 = \frac{1}{-3}(0 - 1)$

$= 0.333$ (3 d.p)

Step 2 : Use iteration formula to calculate the root

The iteration formula that we have find in example before is

$$x_{n+1} = \sqrt[3]{\frac{3x+1}{2}}$$

By using initial guess of

$$x_0 = 0.333$$

x	$x_{n+1} = \sqrt[3]{\frac{3x+1}{2}}$	iteration
$x_0 = 0.333$	$x_1 = \sqrt[3]{\frac{3x_0+1}{2}}$	$x_1 = \sqrt[3]{\frac{3(0.333)+1}{2}}$ $= 0.991$
$x_1 = 0.991$	$x_2 = \sqrt[3]{\frac{3x_1+1}{2}}$	$x_2 = \sqrt[3]{\frac{3(0.991)+1}{2}}$ $= 1.257$
$x_2 = 1.257$	$x_3 = \sqrt[3]{\frac{3x_2+1}{2}}$	$x_3 = \sqrt[3]{\frac{3(1.257)+1}{2}}$ $= 1.336$
$x_3 = 1.336$	$x_4 = \sqrt[3]{\frac{3x_3+1}{2}}$	$x_4 = \sqrt[3]{\frac{3(1.336)+1}{2}}$ $= 1.358$
$x_4 = 1.358$	$x_5 = \sqrt[3]{\frac{3x_4+1}{2}}$	$x_5 = \sqrt[3]{\frac{3(1.358)+1}{2}}$ $= 1.364$
$x_5 = 1.364$	$x_6 = \sqrt[3]{\frac{3x_4+1}{2}}$	$x_6 = \sqrt[3]{\frac{3(1.364)+1}{2}}$ $= 1.365$
$x_6 = 1.365$	$x_7 = \sqrt[3]{\frac{3x_3+1}{2}}$	$x_7 = \sqrt[3]{\frac{3(1.365)+1}{2}}$ $= 1.366$
$x_7 = 1.366$	$x_8 = \sqrt[3]{\frac{3x_4+1}{2}}$	$x_8 = \sqrt[3]{\frac{3(1.366)+1}{2}}$ $= 1.366$

Since the value of x is repeated (refer to x_7 and x_8) then the root is $x = 1.366$ (3 d.p)

EXAMPLE 4

Iteration formula was already given

In other case, Iteration formula was already given. For example, consider the following question. Obtain the root for $f(x) = x^2 - 5x + 1$.

Given that the iterative function is $x_{n+1} = \frac{x_0^2+1}{5}$ with the initial guess of $x_0 = 1$. Give answer in 3 d.p.

x	$x_{n+1} = \frac{x_0^2+1}{5}$	iteration
$x_0 = 1$	$x_1 = \frac{x_0^2+1}{5}$	$x_1 = \frac{(1)^2+1}{5}$ $= 0.4$
$x_1 = 0.4$	$x_2 = \frac{x_1^2+1}{5}$	$x_2 = \frac{(0.4)^2+1}{5}$ $= 0.232$
$x_2 = 0.232$	$x_3 = \frac{x_2^2+1}{5}$	$x_3 = \frac{(0.232)^2+1}{5}$ $= 0.211$
$x_3 = 0.211$	$x_4 = \frac{x_3^2+1}{5}$	$x_4 = \frac{(0.211)^2+1}{5}$ $= 0.209$
$x_4 = 0.209$	$x_5 = \frac{x_4^2+1}{5}$	$x_5 = \frac{(0.209)^2+1}{5}$ $= 0.209$

Since the value of x is repeated (refer to x_4 and x_5) then the root is $x = 0.209$ (3 d.p)

QUESTIONS

1 Identify the root for equation $x^3 - 5x - 16 = 0$ where $x_0 = 3.5$ by using Fix Point Iteration method. Give answer correct to 4 decimal places.

Answer : $x = 3.1698$

2 Given $x^3 + 3x^2 - 1 = 0$ has an approximate root $x_0 = 1$. By using Fix Point Iteration Method, determine the root of the equation correct to 3 d.p. Given that the Iteration function,

$$x_{n+1} = \sqrt{\frac{1}{x_0 + 3}}$$

Answer : $x = 0.532$

3 Determine the real root of $x^5 - x = 7$. Give the answer correct to 4 decimal places.

Answer : $x = 1.5355$

NEWTON RAPHSON

NEWTON RAPHSON METHOD

The Newton-Raphson method is another numerical method in solving equations of the form $f(x) = 0$ and satisfies the consistency condition as below.

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

Where :

x_n = value of the root at iteration n

x_{n+1} = a revised value of the root at iteration $n+1$

$f(x_n)$ = value of the function at iteration n

$f'(x_n)$ = derivative of $f(x)$

NEWTON RAPHSON STEP BY STEP

STEP 1

Compute $f'(n)$ and apply Newton Raphson formula $x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$

STEP 2

Starting with x_0 , perform the iterative step calculation to find the root. Repeat the calculation until get the root.

EXAMPLE 1

Find the root for the equation $x^3 - 6x + 4 = 0$ with initial guess $x_0 = 2$ by using Newton - Raphson Method.

Step 1 : Compute an approximate value of $f'(x_n)$

$$f(x) = x^3 - 6x + 4$$
$$f'(x) = 3x^2 - 6$$

Step 2 : Use Newton-Raphson formula

$$x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)}$$

$$= x_i - \frac{x^3 - 6x + 4}{3x^2 - 6}$$

$$x_0 = 0.5$$

$$x_1 = 0.5 - \frac{(0.5)^3 - 6(0.5) + 4}{3(0.5)^2 - 6}$$

$$= 0.5 - \frac{1.125}{-5.25}$$

$$= 0.71429$$

$$x_2 = 0.71429 - \frac{(0.71429)^3 - 6\,(0.71429) + 4}{3(.71429)^2 - 6}$$

$$x_2 = 0.71429 - \frac{0.078698}{-4.46937}$$

$$= 0.73190$$

$$x_3 = 0.73190 - \frac{(0.73190)^3 - 6\,(0.73190) + 4}{3(0.73190)^2 - 6}$$

$$= 0.73190 - \frac{6.62443 \times 10^{-4}}{-4.39297}$$

$$= 0.73205$$

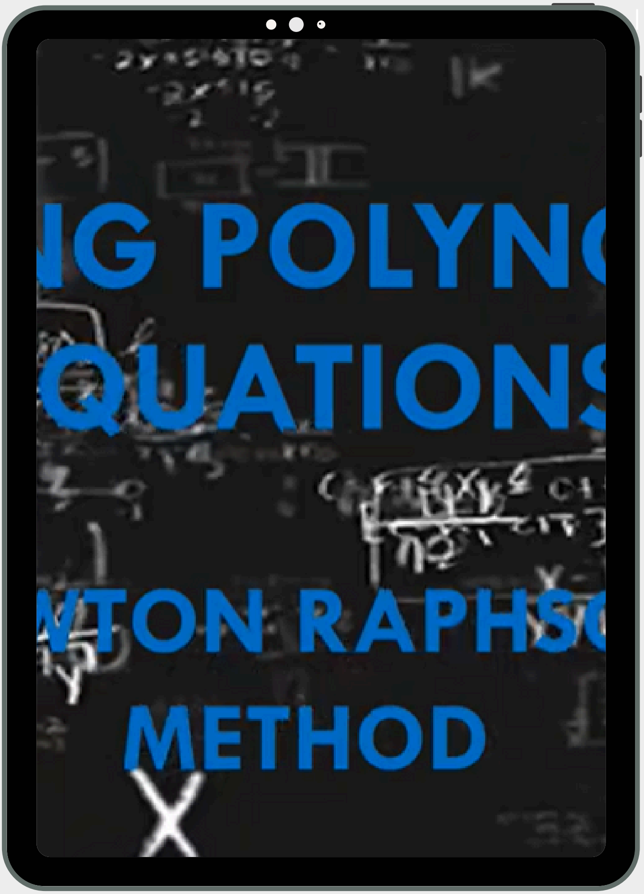
$$x_4 = 0.73205 - \frac{(0.73205)^3 - 6\,(0.73205) + 4}{3(0.73205)^2 - 6}$$

$$= 0.73205 - \frac{3.54709 \times 10^{-6}}{-4.39231}$$

$$= 0.73205$$

Since the value of x_3 and x_4 is the same, therefore the root is $x = 0.73205$

CLICK TO WATCH!



SCAN TO WATCH!

EXAMPLE 2

Find the root for the equation $f(x) = 2x^3 - 3x - 6$ using the Newton Raphson and give the correct answer in three decimal places.

Step 1 : Compute an approximate value of $f'(x_n)$

$$f(x) = 2x^3 - 3x - 6$$

$$f(x) = 6x^2 - 3$$

Step 2 : Use Newton-Raphson formula

$$x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)}$$

$$= x_i - \frac{2x^3 - 3x - 6}{6x^2 - 3}$$

$$x_0 = 1.5$$

$$x_1 = 1.5 - \frac{2(1.5)^3 - 3(1.5) - 6}{6(1.5)^2 - 3}$$

$$= 1.5 - \frac{-3.75}{10.5}$$

$$= 1.8571$$

$$x_2 = 1.8571 - \frac{2(1.8571)^3 - 3\,(1.8571) - 6}{6(1.8571)^2 - 3}$$

$$= 1.8571 - \frac{1.2383}{17.6929}$$

$$= 1.7871$$

$$x_3 = 1.7871 - \frac{2(1.7871)^3 - 3\,(1.7871) - 6}{6(1.7871)^2 - 3}$$

$$= 1.7871 - \frac{0.05372}{16.1624}$$

$$= 1.7838$$

$$x_4 = 1.7838 - \frac{2(1.7838)^3 - 3\,(1.7838) - 6}{6(1.7838)^2 - 3}$$

$$= 1.7838 - \frac{4.9785 \times 10^{-4}}{16.0917}$$

$$= 1.7838$$

Since the value of x_3 and x_4 is the same, therefore the root is $x = 1.7838$

QUESTIONS

1

By using Newton Raphson Method, determine the root for $x^4 - 2x^3 - x + 1 = 0$

Give the answer correct to three decimal places. Assume the first approximation as 1.5

Answer : $x = 0.641$

2

Approximate the real root correct to 4 decimal places of $x^3 + x - 3 = 0$ by using Newton Raphson

Answer : $x = 1.2134$

3

Solve the equation $x^3 + 3x^2 - 2 = 0$ by using Newton Raphson Method. Give the answer correct to three decimal places with an initial guess of $x_0 = 1$

Answer : $x = 0.732$

REFERENCES

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