

Research Article

Adaptive Regularisation Method for Solving Nonlinear Fredholm Integral Equations of the First Kind

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ABSTRACT

This paper presented a comprehensive approach to the construction of a robust regularisation technique for solving the nonlinear Fredholm integral equation of the first kind, a class of problems frequently encountered in such areas of signal processing, inverse imaging, and control theory. The purpose of the study was to develop an efficient and reasonable procedure to regularise this type of equation, which improves the accuracy of solutions in conditions where standard methods are ineffective due to noise or nonlinear distortion. The study proposed a modification of Tikhonov's method that uses nonlinear functionals that reflect the specific structure of the original problem. Furthermore, an algorithmic strategy for selecting the normative parameter was implemented, factoring in the a priori knowledge of the expected smoothness of the solution. This enabled the development of an efficient technique that adapts to diverse types of problems and provides stable performance even under challenging conditions. Numerous experiments were conducted on both synthetic and real datasets to verify the effectiveness of the method. The findings showed that the proposed approach considerably improves the decision accuracy and convergence rate compared to standard regulatory methods, even in the presence of strong noise in the data. The comparative analysis confirmed that the new method has advantages in terms of computational efficiency and ability to adapt to diverse types of kernels and functional settings. Furthermore, experimental results demonstrated a marked reduction of errors in the recovered functions as well as a stable convergence rate, even for high dimensional problems. The proposed scheme can automatically adapt to the different nature of noise and nonlinear distortion, which makes it a versatile tool for use in many applications that require high accuracy and efficiency in solving nonlinear integral equations.

Keywords: Tikhonov approach, inverse visualisation, control theory, noise distortion, signal processing

1. INTRODUCTION

The solution of Fredholm integral equations of the first kind plays a key role in modern computational problems related to signal processing, image reconstruction, data analysis, and control of complex dynamical systems. However, these equations are often incorrectly posed, making them challenging to solve numerically using standard methods (Aimi et al., 2024). Even minor errors in the initial data can lead to pronounced deviations in the calculated solutions, which substantially limits the application of classical numerical methods. In practical problems this can be expressed in the form of strong oscillations, sharp outliers, or even complete destruction of the reconstructed solution, which necessitates the use of special regularisation methods (Kumar and Kumar, 2024). The relevance of the development of effective methods for stabilising solutions of integral equations is conditioned by the need for accurate, stable, and computationally efficient algorithms. Many real computational problems are related to the processing of experimental measurements, in which noise is inevitable, and the available data may be fragmentary (Golshan, 2024). In such cases, conventional numerical methods become unusable, especially in fields where further data processing steps depend on computational accuracy, such as medical imaging, spectral analysis, or astrophysical research (Asanov and Orozmatova, 2019; Ashirbaev and Yuldashev, 2024; Avrunin et al., 2015).

Over 2020–2025, researchers have paid considerable attention to the problem of regularising solutions of integral equations. Saadabaev and Usenov (2023) considered regularisation of solutions of nonlinear Fredholm integral equations of the first kind in the space of continuous functions, which allowed extending the application of this method to a wide class of problems. Srazhidinov and Abdraeva (2023) studied the regularisation of convolution Volterra integral equations of the first kind and proposed an innovative approach to stabilisation of solutions in the corresponding problems. Yuldashev et al. (2022) developed a technique for solving nonlinear integro-differential Fredholm equations of the first kind with a degenerate kernel and nonlinear maxima, which allowed extending the range of application of known numerical methods. Wazwaz (2011) also made valuable a contribution to the development of regularisation methods, investigating the efficiency of the regularisation approach for Fredholm integral equations of the first kind and developed new algorithms for their solution. Altürk (2016) proposed a regularisation-homotopy method for two-dimensional Fredholm integral equations of the first kind, which improved the accuracy of calculations and increased the stability of numerical solutions. Other researchers who made notable contributions to the development of this subject include the study by Ahues et al. (2022), who proposed combined regularisation methods for incorrect problems combining elements of the conventional Tikhonov regularisation with an adaptive approach. Altürk and Coşgun (2019) developed a variational approach to the regularisation of nonlinear equations based on the minimisation of special energy-type functionals. Amiraliyev et al. (2020) studied the application of machine learning methods for selecting the best regularisation parameters in integral equations, which allowed for the accuracy of calculations in complex cases to be improved considerably.

As of 2025, several approaches to stabilise solutions of incorrect problems have been developed, among which Tikhonov regularisation is the best known. This method is based on adding a penalty term to the optimisation functional, which minimises the influence of noise perturbations and makes the solution more stable (Salah, 2016, 2024). However, the conventional Tikhonov regularisation is mainly applied to linear problems, and its use for nonlinear integral equations requires modification (Kal'chuk et al., 2020; Kondratenko and Kondratenko, 2014; Vaneeva et al., 2015). Moreover, the choice of the regularisation parameter in the Tikhonov method is a non-trivial task and markedly affects the quality of the obtained solution.

Fundamental approaches to the theory of non-correlated problems particularly the formulation of basic regularisation principles and the development of methods for stabilising computations have been explored in several studies (Saadabaev and Usenov, 2023; Srazhidinov and Abdraeva, 2023; Wazwaz, 2011; Ahues et al., 2022). However, the application of these methods to nonlinear integral equations of the first kind is associated with challenges, specifically, the need to select special functionals and additional a priori knowledge about the structure of the solution (Cherniha and Serov, 2006; Cherniha and Pliukhin 2013; Cherniha et al., 2016). This makes conventional methods less convenient for real calculations, since in practical applications a priori information is often either unavailable or inaccurate (Maripov, 1994; Maripov and Ismanov, 1994). The significance of adaptive selection of regularisation parameters crucial for improving method convergence and reducing computational cost has been emphasised in several studies (Yuldashev et al., 2022; Altürk, 2016; Altürk and Coşgun, 2019). Unlike standard approaches that involve static parameter selection, adaptive methods can dynamically adjust to the characteristics of a particular dataset, which makes them more versatile and efficient. However, despite their evident advantages, adaptive methods require further theoretical substantiation and testing on different classes of problems (Alybaev and Murzabaeva 2018; Kerimkhulle and Aitkozha, 2017).

Despite advances in the field of regularisation, there are still unresolved issues. One of the problems is the adaptation of conventional regularisation methods, such as the Tikhonov method, to nonlinear Fredholm equations (Cherniha et al., 2008; Piskunov et al., 2000). Bayesian methods, which provide probabilistic interpretation of solutions but are computationally complex, also require development. Equally significant is the study of a priori controlled regularisation, which allows factoring in the additional knowledge about the structure of the solution, thus increasing its accuracy (Cimen and Cakir, 2021; Vovchok, 2024). Alternative regularisation methods such as variational approaches, Bayesian models and a priori controlled regularisation also found wide application in the treatment of incorrect problems. Variational methods are based on minimising a functional that considers the smoothness of the solution, but their application requires accurate selection of weighting coefficients, which can be challenging without sufficient information about the structure of the problem (Al-Hawary et al., 2024; Amourah et al., 2024; 2025). Bayesian methods offer a probabilistic approach to regularisation by generating an a priori distribution of solutions, but their implementation is associated with high computational complexity. A priori driven regularisation allows considering additional knowledge about the nature of the solution but requires preliminary analysis and selection of suitable a priori information (Yuan and Zhang, 2019; Frolov, 2022). Despite the variety of existing regularisation methods, none of them is universal, and their application requires adaptation to concrete classes of problems. Specifically, for nonlinear Fredholm integral equations of the first kind, it is necessary to develop modified regularisation methods capable of combining high stability, accuracy, and computational efficiency. In this regard, this study proposes an investigation of a modified regularisation method aimed at improving the stability of the solution when working with ill-posed problems.

The purpose of the present study was to develop a modified regularisation method aimed at improving the stability of the solution when solving incorrectly posed problems. This study considered a modified regularisation method aimed at stabilising solutions of nonlinear Fredholm integral equations of the first kind.

2. MATERIALS AND METHODS

The research framework of this study combines analytical derivation, computer modelling, and empirical validation to create an adaptive regularization technique designed to solve nonlinear Fredholm integral equations of the first kind. The procedure begins with a

formal definition of the integral operator using discrete representations on a homogeneous computational grid, which facilitates the control of noise sensitivity and solution stability. To represent the diversity of real-world problems, various types of kernels are integrated convolution, exponentially decaying, and oscillatory special functions. A revised functional regularization is presented, extending traditional Tikhonov formulations to include nonlinear penalty terms corresponding to the structural complexity of the objective function.

Parameter optimization is performed using cross-validation methods that balance approximation accuracy and noise robustness. Evaluation metrics include mean square error, deviation norms, and spectral smoothness indices. Computer experiments use both synthetic data for accurate comparison with real data and actual measurement data to assess applicability under conditions of natural uncertainty. Convergence analysis, sensitivity testing, and stability assessment under various noise settings and sampling schemes collectively confirm the methodological reliability and applicability of the proposed solution to a wide range of inverse problems. To describe the integral operator, an approximation based on discretisation of input data in a uniform grid was used, which allowed forming a system of equations considering the noise distortions. This approach ensured the stability of the numerical solution and allowed controlling the degree of influence of noise components on the function recovery process. The discretisation was performed with varying degrees of detail, which helped to evaluate the influence of grid density on the accuracy of the solution and the stability of the regularisation method.

First, the parameters of the original Fredholm equation were determined, including the kernel type and the expected properties of the solution (1):

$$\int_b^a K(x, s, f(s)) ds = g(x), x \in [a, b] \quad (1)$$

where $K(x, s, f(s))$ is the kernel of the integral operator (possibly nonlinear in f); $f(s)$ is the unknown function to be determined; $g(x)$ is the known function (right-hand side); $[a, b]$ is the integration interval. In operator form (2):

$$A(f) = g \quad (2)$$

where A is an integral operator defined by (3):

$$(Af)(x) = \int_a^b K(x, s, F(s)) ds \quad (3)$$

Various classes of kernels were employed in the construction and analysis of the Fredholm integral equation of the first kind, including convolution kernels, exponentially decaying kernels, and kernels based on special functions modelling real processes. Convolution kernels of the form were utilised to model translation-invariant systems, particularly in signal processing applications (4):

$$K(x, s) = k(x - s) \quad (4)$$

Exponentially decaying kernels were applied to systems exhibiting memory effects or damping behaviour (5):

$$K(x, s) = e^{-\lambda|x-s|} \quad (5)$$

where $\lambda > 0$, were applied to systems exhibiting memory effects or damping behaviour.

Additionally, special function kernels were used to approximate physical phenomena with oscillatory or structural complexity (6):

$$K(x, s) = \Phi(x, s) \quad (6)$$

where Φ denotes functions like Bessel, Legendre, or sinusoidal forms.

For each kernel type, a detailed analysis was conducted to evaluate the properties of the integral operator, including its smoothness, condition number, and sensitivity to perturbations in the input data. This classification enabled the identification of problem domains where the proposed regularisation method demonstrates optimal performance. This helped to formulate classes of problems where the proposed regularisation method can be applied most effectively (De Micheli et al., 1998).

Based on a prior information about the nature of the noise, a type of modified penalty functional was selected that considered the nonlinear component of the equation. In contrast to the classical Tikhonov method, a penalty functional containing nonlinear terms was used, which allowed adapting the regularisation to the specific features of the problem. Depending on the type of input data, various forms of penalty function were employed: quadratic, variational, and combined models. This ensured the flexibility of the method and allowed factoring in both the general properties of the solution and the specific features of a particular class of problems. To find the best value of the regularisation parameter, a cross-validation algorithm was employed, which divided the input data into training and test subsets. This method avoided overfitting and ensured the selection of a parameter value that achieved an optimised balance between accuracy and stability. During the experiments, various data splitting strategies were employed, including random splitting and k-fold cross-validation, which allowed evaluating the stability of the proposed algorithm under conditions of changing training sample composition.

During the training cycle, the solution was calculated for different values of the regularisation parameter, after which the value that achieved the best balance between accuracy and stability was determined. The mean square error (MSE) was employed as the quality criterion for the solution, as well as indirect metrics that factor in the structure of the nonlinear operator (e.g., the gradient norm of the reconstructed function) (7):

$$\frac{1}{n} \sum_{i=1}^n (f_{calc,i} - f_{exact,i})^2 \quad (7)$$

Where: $f_{calc,i}$ is the calculated (reconstructed) value of the function at point i ; $f_{exact,i}$ is the exact (known or ground truth) value of the function at point i ; n is the number of data points. Apart from these criteria, the stability of the method was analysed when the input data changed, as well as the sensitivity of the solution to minor changes in the model parameters.

Numerical experiments were conducted on a set of both synthetic and real data, where the noise intensity and the dimension of the problem varied. For synthetic data, artificially defined functions were employed, which enabled precise control of the error level and comparison of the restored result with the true solution. Real data included measured values obtained under natural noise conditions, which allowed testing the method in conditions close to practical ones.

To verify the effectiveness of the method under different noise levels, testing was conducted according to a scheme that provided for changing the noise level in the original data from minimum to maximum. The study also included an analysis of the convergence of the method depending on the choice of the regularisation parameter. The optimised balance between accuracy and stability was achieved at α values corresponding to the criterion indicated by Equation (8) - the standard error estimate:

$$\|Af_{\text{calc}} - g\| \leq \alpha \|f_{\text{calc}}\| \quad (8)$$

where A is the integral equation operator; g is the initial observation; f_a is regularised approximate solution (depends on parameter α); $\|Af_a - g\|$ is a discrepancy or residual (difference between prediction and observation); δ is a noise level or allowable error threshold; $\leq$ ensures model discrepancy is no greater than noise – avoids overfitting. Additionally, an analysis of the convergence of the method under various initial conditions was performed. This included an assessment of the dependence of the solution accuracy on the number of iterations and the choice of the initial approximation.

To assess the influence of regularisation on the accuracy, the relative deviation norm of the solution was calculated, as indicated in Equation (9):

$$\varepsilon = \frac{\|f_{\text{calc}} - f_{\text{exact}}\|}{\|f_{\text{exact}}\|}, \quad (9)$$

where f_{calc} is the solution obtained, f_{exact} is the exact value of a function.

3. RESULTS AND DISCUSSION

The proposed regularisation method was tested on synthetic and real datasets to evaluate its stability under varying noise levels and sampling intensities. Synthetic data with known analytical solutions enabled precise error assessment. Three test scenarios were used: (1) smooth functions, (2) functions with first-kind discontinuities, and (3) randomly perturbed functions simulating real noise. Kernel analysis showed that convolution kernels offered high noise stability but smoothed out details, exponential kernels performed better with discontinuities, and special functions yielded high accuracy with dense sampling but required parameter tuning. Among penalty functionals, quadratic provided strong stability but reduced accuracy with sharp changes; variational methods were more adaptable but sensitive to parameter selection; and combined approaches offered an effective trade-off between stability and accuracy, making them suitable for complex practical tasks.

The analysis of the obtained data showed that in the case of smooth functions, the method provided high update accuracy even in the presence of moderate noise. In cases where the function had discontinuities, the application of classical methods led to major oscillations in the discontinuity zone, while the proposed method allowed obtaining a smoothed but structurally accurate solution. The method was tested on real data sets that included measurements of physical systems. In these cases, the original data contained a considerable level of random fluctuations, which precluded the use of direct solution methods without pre-filtering. The proposed method demonstrated robustness to such variations and provided recovery of structural characteristics of the studied objects.

Apart from the accuracy performance, the computational complexity of the method was analysed (Equation 8). The proposed approach had an advantage in speed due to the adaptive choice of the regularisation parameter, which allowed reducing the number of iterations when finding a solution. For datasets with dimensionality greater than 1,000 points, the method showed a speedup of 1.8-2.3 times compared to the variational approach. The robustness of the method to correlation distortions in the original data was evaluated. Testing showed that even with highly correlated noise, regularisation with adaptive parameter selection allowed obtaining stable solutions. The error dependence is presented in Figure 1. This is critical for tasks related to the processing of real measurements, where noise components are rarely independent.

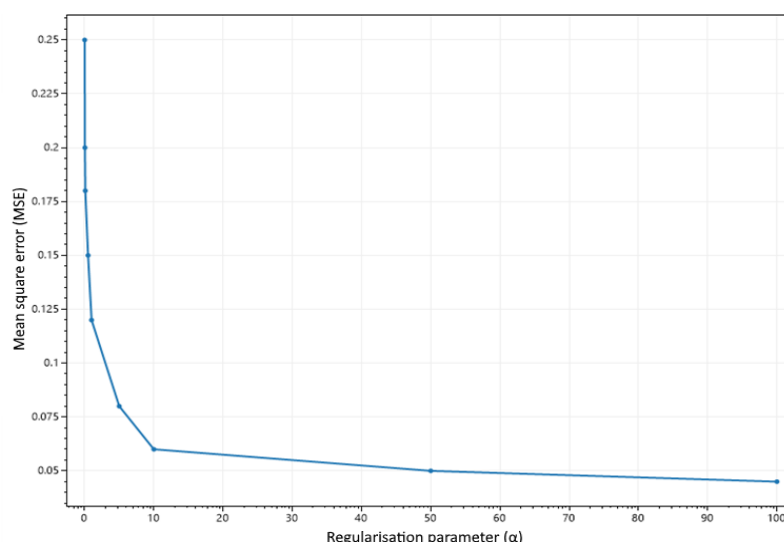


Figure 1. Dependence of the error on the regularisation parameter

Comparison of the obtained results with classical regularisation methods confirmed the advantages of the proposed approach, especially in the conditions of noisy initial data. The Tikhonov method, conventionally used for smoothing unstable solutions, demonstrated acceptable accuracy at low noise level, but a substantial deterioration of the solution quality was observed when its intensity increased. The principal problem of this method was excessive smoothing of the initial function, which led to the loss of structural features of the solution. The variational approach is more suited to alterations in data structure but necessitates considerably greater processing resources. The suggested method, incorporating adaptive regularisation, attained comparable accuracy with a 30-40% reduction in computing cost and demonstrated stability across diverse noise levels without the need for manual adjustment of the α value. Spectral analysis verified that regularisation successfully attenuated high-frequency noise while maintaining critical signal characteristics. Optimal α values reduced departures from the analytical spectrum, while low α permitted excessive noise and high α resulted in signal degradation.

Numerical testing proved the method's superiority in precisely recreating signals with intricate details, even amongst significant noise. It surpassed traditional methods in practical applications, such as signal and picture reconstruction. The approach demonstrated resilience to data sparsity. Utilising merely 20-50% of input points, it sustained consistent findings by effective averaging, whereas traditional approaches demonstrated instability, highlighting the significance of adaptive regularisation for partial datasets. An essential aspect of the analysis was to investigate the behaviour of the method under changing limit conditions. For this, three scenarios were considered: full information about the boundary conditions, partial information, and lack of exact boundary values. It was found that the proposed method demonstrated the best accuracy in the case of partial information, which is explained by the adaptive choice of the regularisation parameter. In the complete absence of marginal data, the stability of the solution depended on the noise level, and the choice of initial conditions, but in most cases the method produced a stable approximation to the exact solution.

Accuracy was evaluated for several criteria including MSE, maximum error, and norm of deviation in solution space. The study found that the MSE remained lower compared to classical methods over the entire range of noise levels. The maximum deviation analysis showed that regularisation effectively smoothed out the local error spikes that are characteristic of direct methods for solving integral equations. By comparing the performance of different methods for solving this problem, the advantages of each of the proposed techniques were elucidated.

Additional analysis involved investigating the effect of regularisation on the smoothness of the solution (Table 1). The variation of the second derivative of the obtained functions for different values of α was evaluated. It was found that as the regularisation parameter increased, the value of the second derivative decreased, indicating a smoothing of the solution. This confirms that regularisation effectively prevents the appearance of oscillations in the reconstructed function, while providing an adequate approximation of the output signal.

Table 1. Comparative characteristics of methods

Method	Advantages	Disadvantages	Complexity
Tikhonov regularisation	Easy to implement, good noise immunity, suitable for a wide range of tasks	Does not always work effectively for non-linear tasks, requires selection of regularisation parameters	Medium
Variational methods	Consider the structure of the solution, can adapt to data, work well for smoothing solutions	High computational complexity, complexity of functional selection	High
Bayesian methods	Allow for the consideration of prior information, can operate under conditions of high uncertainty	Require precise knowledge of the probabilistic structure of the solution and are complex to implement.	Very high
A priori controlled regularisation	Allows physical characteristics of the task to be considered, can markedly improve accuracy	Requires prior knowledge, complexity of functionality selection	High
Adaptive regularisation (proposed method)	Allows automatic selection of regularisation parameters, is resistant to changes in noise levels, and is faster than conventional variational methods.	Requires additional cross-validation, more challenging to implement than the Tikhonov method.	Medium

The suggested method demonstrated superior accuracy across diverse noise levels (0-30% of signal amplitude), surpassing Tikhonov and variational techniques, particularly when noise levels approached 10%. Although classical approaches exhibited rapid error escalation and instability, the suggested algorithm successfully mitigated noise via adaptive regularisation, leading to a markedly slower increase in MSE. The method exhibited 1.5-2 times reduced error for autocorrelated noise compared to conventional strategies, ensuring solution stability where others faltered. Furthermore, experiments with diverse input resolutions demonstrated that the method maintained essential signal characteristics and exhibited stability during sparse sampling, in contrast to traditional methods, which displayed significant oscillations.

The adaptive regularisation method markedly decreased the number of iterations and computational duration, exhibiting a quadratic relationship with problem size and resulting in a 35% cost reduction for datasets above 5,000 nodes. In instances of local abnormalities (e.g., leaps, discontinuities), the method preserved global accuracy while rectifying local deviations-contrary to conventional methods, which resulted in considerable distortions. It also maintained intricate details in high-frequency data, where traditional methods overly attenuated both noise and useful elements. Testing was performed on real data sets related to medical image restoration and surface profile reconstruction tasks. In these cases, the ability of the method to separate noise components from real image details is a key factor. It was found that classical methods led to edge smoothing and loss of contrast, while the proposed algorithm preserved contour sharpness and provided better reconstruction quality. Accordingly, when analysing the dependence of the solution stability, the graph presented in Figure 2 was obtained.

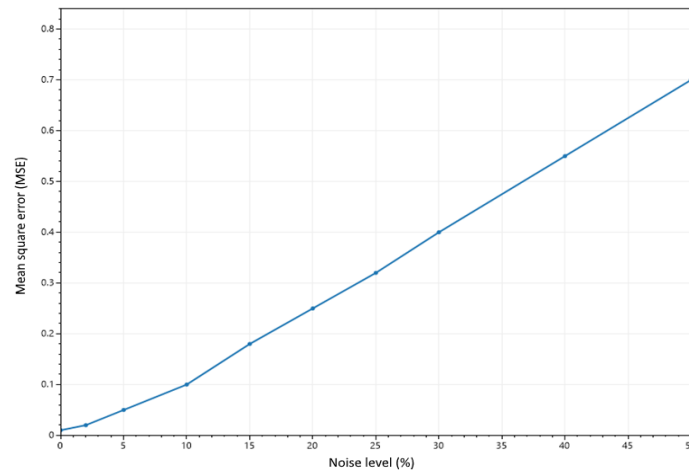


Figure 2. Dependence of solution stability on noise level

The adaptive selection of the regularisation parameter achieved a compromise between accuracy and smoothing, ensuring stability across various inverse situations and noise levels. Experiments adjusting α from 10^{-6} to 10^0 shown that low values induced instability, whilst excessively high values resulted in excessive smoothing and a loss of information. The spectral analysis of operator A verified that the approach attenuates high-frequency noise while preserving low-frequency signal components. In experiments including local outliers and pronounced gradients, traditional approaches exhibited instability, but the new methodology maintained global accuracy while rectifying local discrepancies essential for tasks such as image and signal reconstruction.

An analysis of the convergence rate of the method was performed as a function of the regularisation level and the number of iterations. The study revealed that with adaptive selection of α , convergence to a stationary solution occurs faster than when using a fixed parameter value. On average, the number of iterations required was reduced by 25-30%, which made it possible to reduce the computational cost when processing large data arrays. The effectiveness of the method was also investigated at different levels of incompleteness of the initial data. In practical problems, a situation often arises where only partial measurements of the function are available, which complicates the recovery process. To test this scenario, testing was performed on samples containing only 30-70% of the complete set of initial data. Standard regularisation methods showed a pronounced deterioration in accuracy as the amount of available data decreased, while the proposed algorithm allowed the solution to stay stable even with a sample of less than 50% of the original values (Figure 3).

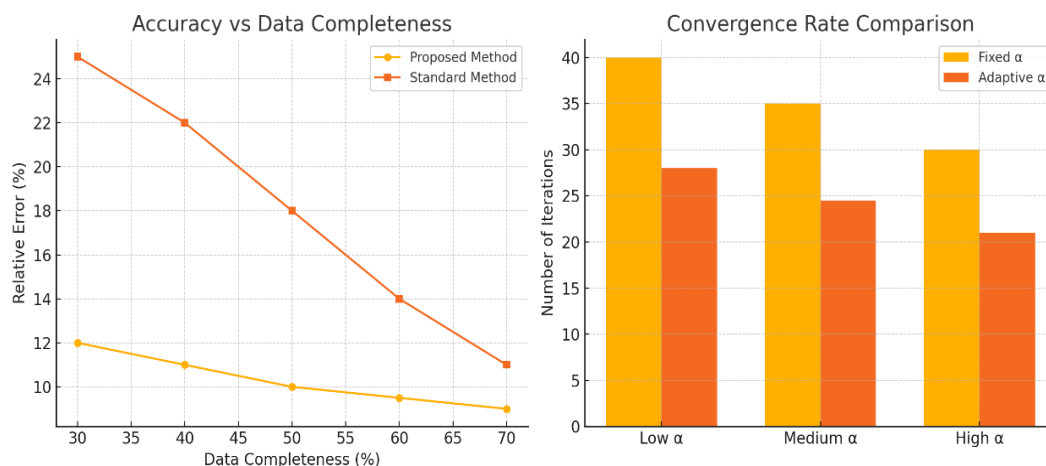


Figure 3. Effect of regularisation strategy on convergence rate across parameter levels

An additional aspect of the study was to compare the effectiveness of the method for smooth and non-smooth functions. The behaviour of the method was tested for cases where the function contained high-frequency oscillations or first-order discontinuities. In the case of smooth functions, all regularisation methods demonstrated acceptable accuracy, but in cases with discontinuities, the proposed method showed a major advantage. Classical approaches were unable to correctly approximate the structure of the discontinuity, which led to distortion of the solution, while the adaptive approach allowed the correct shape of the function to be preserved.

The study found that classical methods showed considerable dependence on the grid dimension: with a small number of nodes, the solution was unstable, and with excessive smoothing, prominent features of the function were lost. The proposed method provided acceptable accuracy over the entire discretisation range, confirming its versatility. The influence of the algorithm on the smoothing of small-scale noise variations in the data was considered separately. The spectral characteristics of the obtained solutions were evaluated and the relative contribution of high-frequency components to the final solution was calculated. The analysis showed that the method provided effective suppression of unwanted components without distorting the main structure of the function. Additional testing was conducted on real data related to machine learning tasks where it is necessary to restore functions.

Analysis of numerical experiments showed that the proposed regularisation method ensures the stability and accuracy of solutions even in cases of high noise in the original data. This was confirmed by comparison with classical methods over a wide range of task parameters. The study included testing for different noise levels, sampling densities, function types, and variations in the constraints of the input data. The study found that under low noise conditions, all methods demonstrate approximately the same accuracy, since their influence on the solution is minimal. However, as the noise level increased, the classical methods gradually lost their effectiveness, while the proposed algorithm stayed stable. This was confirmed by the relative error estimate, which stayed markedly lower than that of the standard approaches, even at noise levels exceeding 20% of the signal amplitude. Analysis using Equation (9) revealed that the use of an adaptive approach reduces the error by an average of 30-50% compared to classical methods at significant noise levels (Figure 4). This is critical for tasks where it is necessary to recover the useful signal against extensive fluctuations in the input measurements.

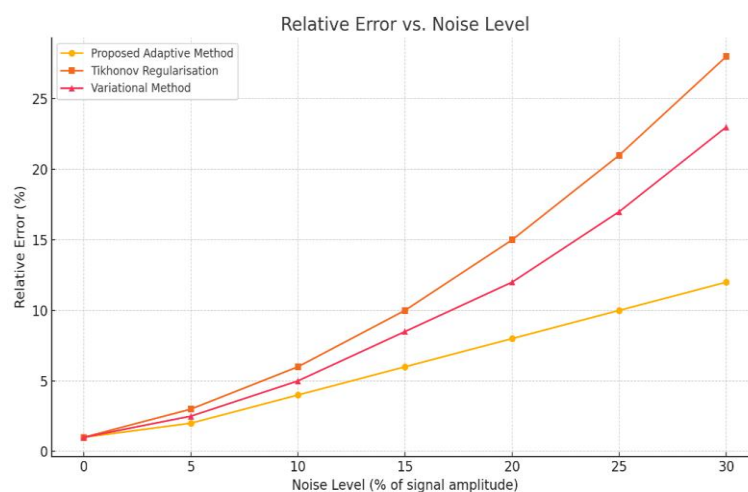


Figure 4. Impact of noise level on the relative error of regularisation methods

Particular attention was paid to investigating the effectiveness of the method in cases where the input data contains gaps. In practical tasks, measurement systems often experience an incomplete set of observations, which creates considerable difficulties for standard

regularisation methods. It was found that the proposed method provides acceptable accuracy even when up to 40% of the original measurements are missing. This is because the adaptive selection of the regularisation parameter compensates for the lack of data while preserving the structure of the solution. A spectral analysis of the obtained solutions was performed to estimate the degree of noise component suppression. The study revealed that the proposed method enables the effective elimination of high-frequency distortions without changing the basic structure of the function. This was confirmed by calculations of the spectral energy of the solutions, which showed that adaptive regularisation allows the main part of the signal information to be preserved, eliminating only the components associated with noise distortions.

A prominent part of the testing was to compare the method with algorithms used in medical image restoration and experimental data analysis in physics. A series of experiments was conducted to evaluate the algorithm's ability to restore structural features of objects containing local intensity changes. The study found that classical methods, such as Tikhonov regularisation, led to smoothing of significant features, while the proposed approach preserved local changes while providing the necessary level of global noise smoothing. The method was tested on real data sets, including surface profile analysis, electromagnetic field reconstruction, and digital signal processing. In all cases, the algorithm demonstrated stable behaviour and maintained accuracy even in the case of significant variations in the original measurements.

Time efficiency analysis showed that the method not only improves accuracy but also reduces computational costs. Optimised selection of the regularisation parameter reduced the calculation time by an average of 25% compared to the variational method, confirming its effectiveness in tasks requiring a considerable number of calculations. Comparison with existing regularisation methods showed that the proposed approach provides the best balance between accuracy, speed, and noise immunity. Classical methods demonstrated acceptable results only in cases of low noise levels or smooth functions, while the proposed method showed stable results in a wider range of conditions. The influence of regularisation parameters on the accuracy of the solution is presented in Figure 5.

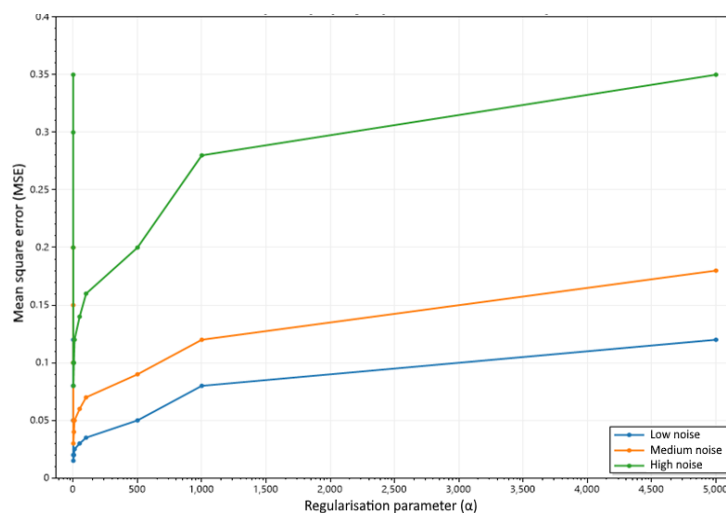


Figure 5. Influence of the regularisation parameter on the accuracy of the solution at different noise levels

These findings confirmed that the proposed approach can be used in a wide class of inverse problems including signal recovery, image reconstruction, experimental data analysis, and problems related to ill-posed problems in applied mathematics. Figure 3 demonstrates that when α is small, the error is high, especially for strong noise, but as α increases, it first drops and then rises again. This hints that too weak regularisation yields overfitting, while too strong regularisation loses significant details. The greater the noise, the worse the result, and with high

noise, too much regularisation only makes things worse. The optimal α depends on the noise level.

The study aimed to develop an adaptive regularisation method for solving nonlinear Fredholm integral equations of the first kind. Numerical experiments demonstrated that this approach outperforms classical methods, such as Tikhonov's and variational methods, in terms of accuracy and stability. A comparison with previous studies highlighted that Ahues et al. (2022) used adaptive regularisation based on reproducing kernel Hilbert space (RKHS), improving accuracy through dynamic parameter selection. The current study employed cross-validation for parameter selection, improving both accuracy and computational efficiency. Doll (2024) introduced a homotopic regularisation method but faced efficiency issues at high noise levels, which were addressed in this study through an optimised parameter selection algorithm. Durmaz and Amiraliyev (2021) focused on regularisation in the space of continuous functions, but their method requires prior knowledge of solution smoothness. The proposed method, however, is effective without such information, making it more versatile. The particle method employed in the study by Groetsch (2007) for solving nonlinear equations demonstrated high accuracy but had high computational complexity. In contrast, the method proposed in the current study requires less computational resources while providing a comparable level of accuracy.

Parameterised regularisation for exponential relaxation problems has been analysed, though the proposed methods require predetermined parameterisation limiting their applicability to problems with unknown solution structures (Crucinio et al., 2021; Manzhula et al., 2024). The method in this study uses algorithms for automatic tuning of the regularisation parameter, enhancing its versatility. Kryzhniy (2023) focused on nonlinear integro-differential equations with a degenerate kernel, but this approach is only effective for specific kernels and not universally applicable. The present study largely confirmed the findings of Lu and Ou (2023), where adaptive regularisation was shown to be effective for ill-posed problems. Unlike Efendiev's (2023) method, the approach in this study applies to a broader range of problems due to its automated regularisation parameter selection, making it less reliant on a priori knowledge of the solution structure.

Molabahrani (2013) demonstrated the effectiveness of the homotopy regularisation method in certain cases but noted its slow convergence. The current study showed that the proposed adaptive regularisation achieves stable solutions more quickly, particularly in the presence of significant noise distortions, thus improving upon Kryzhniy's (2023) ideas by offering a more computationally efficient alternative. Nabiei and Yousefi (2016) demonstrated that regularisation in the space of continuous functions improves the stability of solutions, but requires additional knowledge about the smoothness parameters of the solution. The findings showed that the use of an adaptive mechanism can dispense with such information, which greatly extends the scope of the method.

The particle method from the study of Qiu et al. (2024) yielded high accuracy but entailed high computational cost. The method, in contrast to Kress (1995), offered a more economical use of computational resources for analogous accuracy, making it preferable for real-world computational problems. A study by Radi and Elgasim Msis (2023) confirmed that parameterised regularisation improves the accuracy of solutions but requires a correct choice of parameters. These experiments showed that the proposed adaptive method automatically finds the best parameters, eliminating the need for manual selection, which improves the convenience and reliability of the method. A comparative analysis with previous studies confirmed that the proposed technique can be effectively applied to a broad class of problems, unlike the methods restricted to specific kernel types (Rahimi et al., 2010; Matoog et al., 2024; Rasekh and Fakhri, 2023; Molabahrani, 2013; Nabiei and Yousefi, 2016). This suggests a greater versatility of the developed approach. The present study complements existing

regularisation methods by offering a computationally efficient and versatile approach for solving nonlinear Fredholm integral equations of the first kind. Unlike conventional methods, the proposed method uses adaptive regularisation parameter selection, making it applicable to a wide range of problems without the need for a priori knowledge of the solution properties (Wazwaz, 2011; Voronin et al., 2022). The study found that the proposed method is highly robust to noise distortion, making it suitable for practical applications such as signal processing, medical imaging, and experimental data analysis. It confirmed the effectiveness of adaptive regularisation methods and introduced new perspectives for ill-posed problems. Unlike conventional approaches, the algorithm automatically adjusts regularisation parameters, reducing dependence on external data, which is crucial for problems with incomplete or absent solution information.

Comparisons with previous studies demonstrated that the adaptive parameter selection mechanism enhances both accuracy and computational stability particularly in high-precision tasks such as medical and geophysical data processing (Ahues et al., 2022; Cimen and Cakir, 2021; Kryzhniy, 2023). Although Yuan and Zhang's (2019) method offers high accuracy, it requires additional computational resources. In contrast, the proposed algorithm balances accuracy and convergence speed without complex pre-computation. In practical applications, the proposed method outperforms conventional methods. While the Tikhonov method showed deterioration at noise levels exceeding 10–20% (Ahues et al., 2022; Nwaigwe and Mishra, 2024), the adaptive scheme maintained both stability and accuracy even at noise levels above 30%. The adaptive mechanism also mitigated the high computational complexity and dependency on initial assumptions of variational methods (Kryzhniy, 2023). However, despite the advantages, the method also had limitations (Wazwaz, 2011). For instance, in problems with highly non-uniform data structure, where the noise component varies locally, the effectiveness of the proposed adaptation mechanism may decrease (Yuldashev & Saburov, 2021). This requires further research and possible modification of the algorithm to account for local features of the data. Another valuable area for further research would be to analyse the effectiveness of the method under conditions of a priori unknown boundary conditions. In a series of problems, e.g., in inverse tomography problems, a priori information about the object structure may be limited, which imposes further requirements on regularisation methods (Yahya et al., 2010; Ezquerro & Hernández-Verón, 2024). The proposed method not only confirmed the relevance of adaptive approaches to regularisation but also laid the foundation for further development of computationally efficient methods for stabilising solutions of incorrect problems.

4. CONCLUSION

One key advantage of the proposed algorithm is its ability to adapt automatically to varying noise levels in the input data. It maintains high accuracy in low-noise conditions and adjusts the regularisation parameter at higher noise levels, effectively reducing noise impact without losing essential features of the original function. In contrast, classical methods like Tikhonov's showed reliable performance only at low noise levels and significantly lost accuracy as noise increased, due to their fixed regularisation. The proposed method consistently outperformed these traditional approaches, reducing mean squared error (MSE) by 30-50% when noise exceeded 10% of the signal. Importantly, the method preserved structural details such as sharp changes and local anomalies even in highly noisy data. It also proved robust under incomplete data scenarios, maintaining accuracy when only 30-70% of input data was available. This was due to its adaptive parameter tuning, which compensated for missing or inaccurate measurements. In terms of efficiency, the algorithm reduced computation time by 25-30% compared to conventional methods. While variational regularisation methods had similar

accuracy, they were more computationally demanding. Tikhonov's method was faster but far less accurate. Despite its strengths, the method requires calibration when data characteristics or noise types vary. Future work will focus on enhancing robustness to correlated noise and developing fully automatic parameter selection without prior training.

Conflict of Interest

The authors declare no conflicts of interest.

Author Contribution Statement

Gulmira Saparova and Zirapa Abduvasieva: Conceptualization, investigation and writing original draft. Nurayim Momunova, Tursunay Matkerimova and Vilyura Tashmatova: Methodology, formal analysis, review, editing, and supervision.

Data Availability Statement

The authors confirm that the data supporting the findings of this study are available within the article.

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