



A new fractional order 6D chaotic model: Study of model dynamics, system structure graph, electronic circuit realization and fractional control

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ABSTRACT

A new fractional 6D chaotic model is constructed in this paper. The new fractional 6D chaotic model has six positive parameters plus the fractional order with eight nonlinear terms. The complicated chaotic dynamics of the new fractional 6D model is presented and analyzed. The basic properties of this model are studied and its chaotic attractors, dissipative feature, symmetry, equilibrium points, Lyapunov Exponents are investigated. The new dynamics of the 6D fractional model is numerically simulated using Matlab software. In addition, utilizing the graph theory tools certain structural characteristics are calculated. An electrical circuit is built to implement the new 5.4 fractional order 6D model. Finally, an active fractional order controller is proposed to control the new model at different fractional orders. The chaos of the new model is very useful and can be used to produce random keys for data encryption.

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1. Introduction

In [1], the first 3D autonomous chaotic system was found by Lorenz when he investigated atmospheric convection. Lorenz system has been taken as a first model for studying chaos. After that, in [2], Rössler built a simpler chaotic 3D system. It is noteworthy that during the past twenty years, the chaos of technological paradigms, like nonlinear electric circuits, has progressively shifted from a pure science to a promising subject that has important applications. It is observed that intentionally generating chaos is a vital case in several technical implementations. In this direction, Chen built a three-dimensional autonomous chaos model [3,4] based on an engineering approach to feedback control, after that Lü has built his system [5] and general model combining all cases as a special case has built and named the unified model [6]. Following Vanecek and Celikovský's classification of canonical form [7], in the 3D autonomous paradigms that have quadratic nonlinearities, considering the linear term ($B = [b_{ij}]$), the Lorenz model

fulfils $b_{12}b_{21} > 0$, the Chen model fulfils $b_{12}b_{21} < 0$, and the Lü model fulfils $b_{12}b_{21} = 0$ although they aren't topologically equivalent. With this direction, they form together a full collection of general Lorenz dynamical models. From another point of view, the Rössler system contains a quadratic cross product term and is not part of the general family of Lorenz systems mentioned above. Recently, during the study of a 3D Rössler-type autonomous chaotic model with quadratic cross product terms, Liu and Chen [8] constructed a chaotic model with a quadratic product term in every kernel, that could give two attractors at the same time each of which has a double-scroll.

The concept of fractional calculus has been recognized since the evolution of the ordinary one, and its origin probably relates to a 1695 communication of Leibniz and L' Hospital. They discussed the concept of 1.5 order derivative [9]. Despite the long age of the fractional derivation, its applications in physics and engineering are still recent. Much research has been devoted to the creation of chaos using autonomous fractional order nonlinear models. In [10], fractional order chaos and hyper chaos were examined for the Rössler model. In [11], system synchronization of fractional order was performed. Control and stability for a collection of fractional-order nonlinear models have been investigated with Caputo

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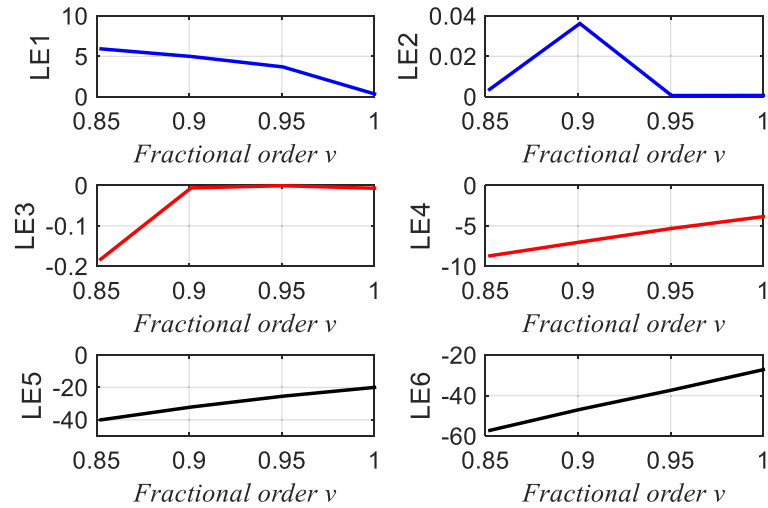


Fig. 1. Lyapunov exponents spectrum versus the fractional order ν where the model parameters are fixed as $a = 30$, $b = 10$, $c = 1$, $d = 10$.

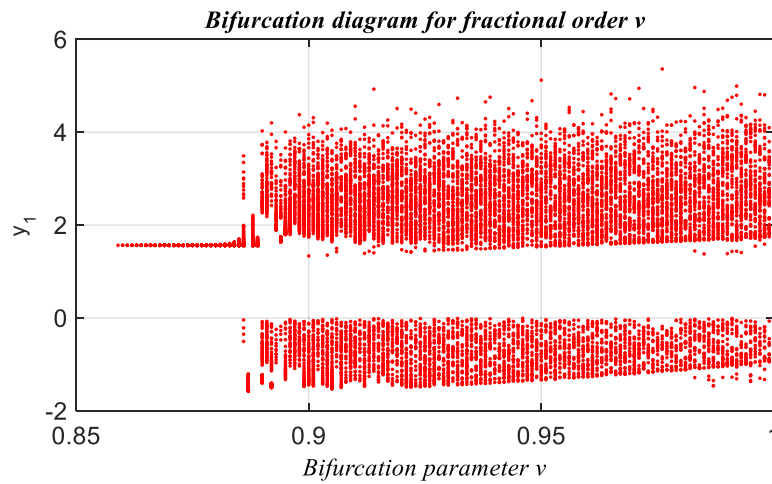


Fig. 2. Bifurcation map of y_1 related to the fractional order ν where the model parameters are fixed as $a = 30$, $b = 10$, $c = 1$, $d = 10$.

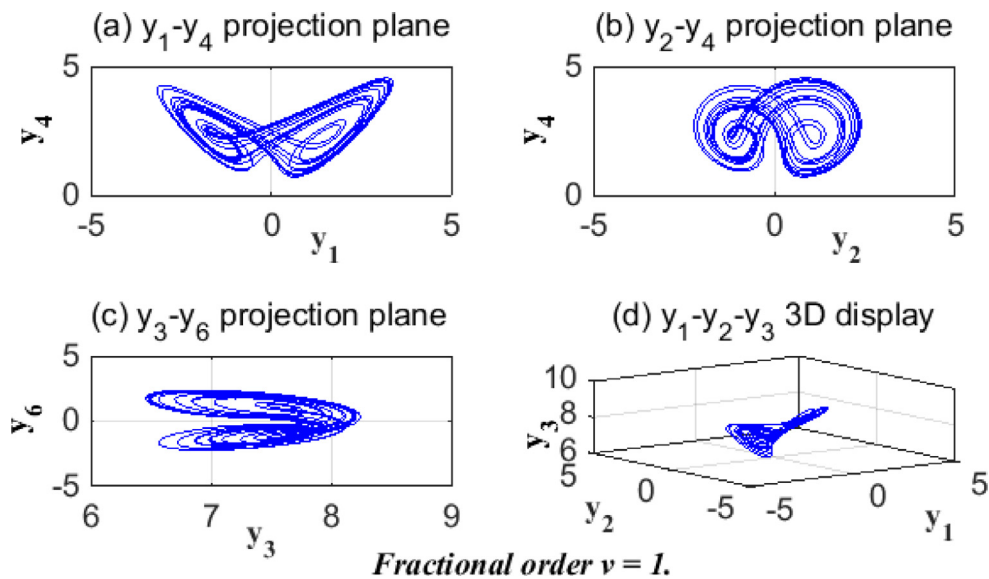


Fig. 3. Hyperchaotic attractor of the proposed model (18) at fractional order $\nu = 1$ and $a = 30$, $b = 10$, $c = 1$, $d = 10$.

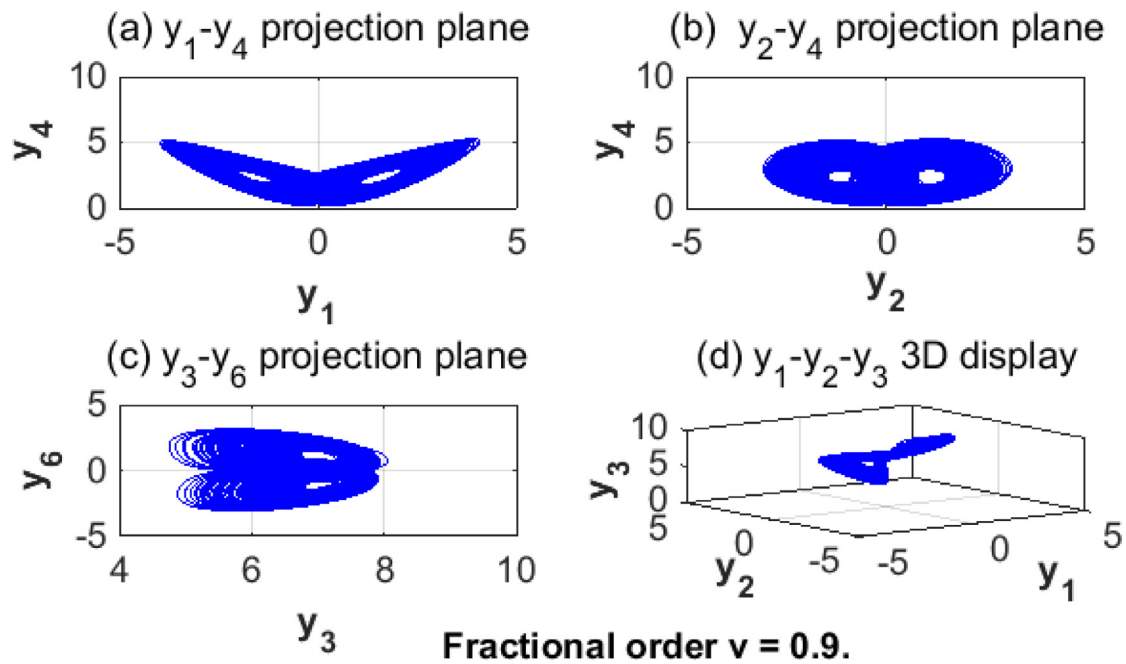


Fig. 4. Hyperchaotic attractor of the proposed model (18) at fractional order $\nu = 0.9$ and $a = 30$, $b = 10$, $c = 1$, $d = 10$.

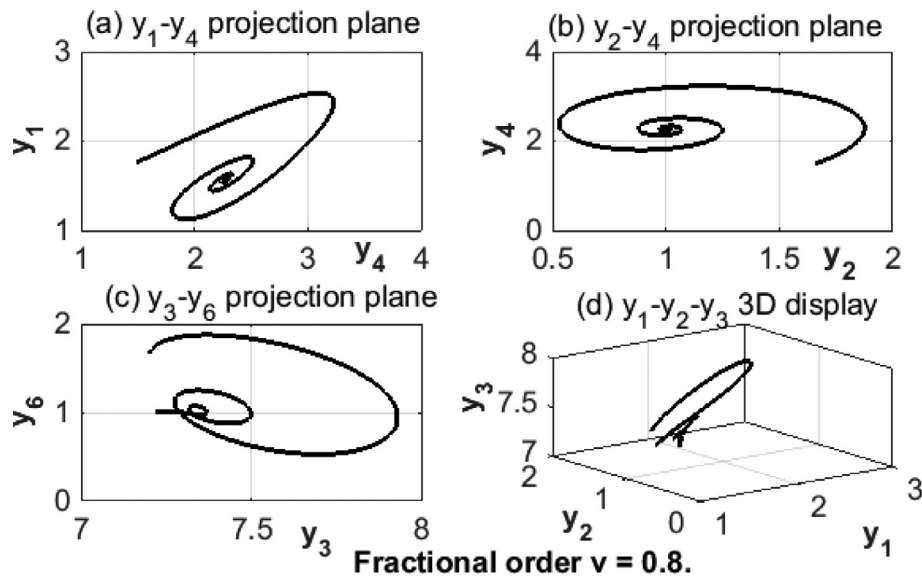


Fig. 5. Fixed point behavior of the proposed model (18) at fractional order $\nu = 0.8$ and $a = 30$, $b = 10$, $c = 1$, $d = 10$.

derivative in [12]. The fractional order derivative provides many features to the dynamical system. It takes the history of the system dynamics into account. And the order of the derivative increases the number of parameters of the chaotic model. So, we can get new chaotic attractors and increases the dimension of parameters' space which increase its efficiency in secure communication applications. Also, in modelling, the fractional order can be used as a tuner parameter for the response to meet the modelled phenomena. But the classical derivatives and the models based on them do not consider any memories and hereditary effects. Extra fractional and integer order chaotic models and applications can be found in [13–20] and [30–33,39–41].

Dynamical systems, especially the chaotic ones are widely utilized to model Ocean and atmospheric dynamics. Ocean engineering is interested with large scale wave motions in the Ocean (for example, wind waves, tsunami waves) and air along with the tem-

perature and density of water in the Ocean. Ocean and atmosphere are equally significant in transferring energy between positions in the form of wave propagation. Dynamical systems, especially the chaotic ones help to model various phenomena in Oceans. Several research works have highlighted the values of eddies and chaos inside the Ocean in controlling the response to forcing from climate change. Certain physical processes in the Ocean produce highly non-linear or chaotic fluctuations. In [34], Jianmin Yang and Wenye Lu numerically studied the generation and evolution of the super-rogue waves. In [35], the reader can see the importance of dynamical systems in modelling and evaluation of dynamics of jack-up platform system under wave, wind, earthquake, and tsunami loads. More numerical and computational studies of dynamical systems can be found in [36–38].

Encouraged by the previous studies, a new fractional chaotic 6D model is constructed which can be used to model may Oceanic

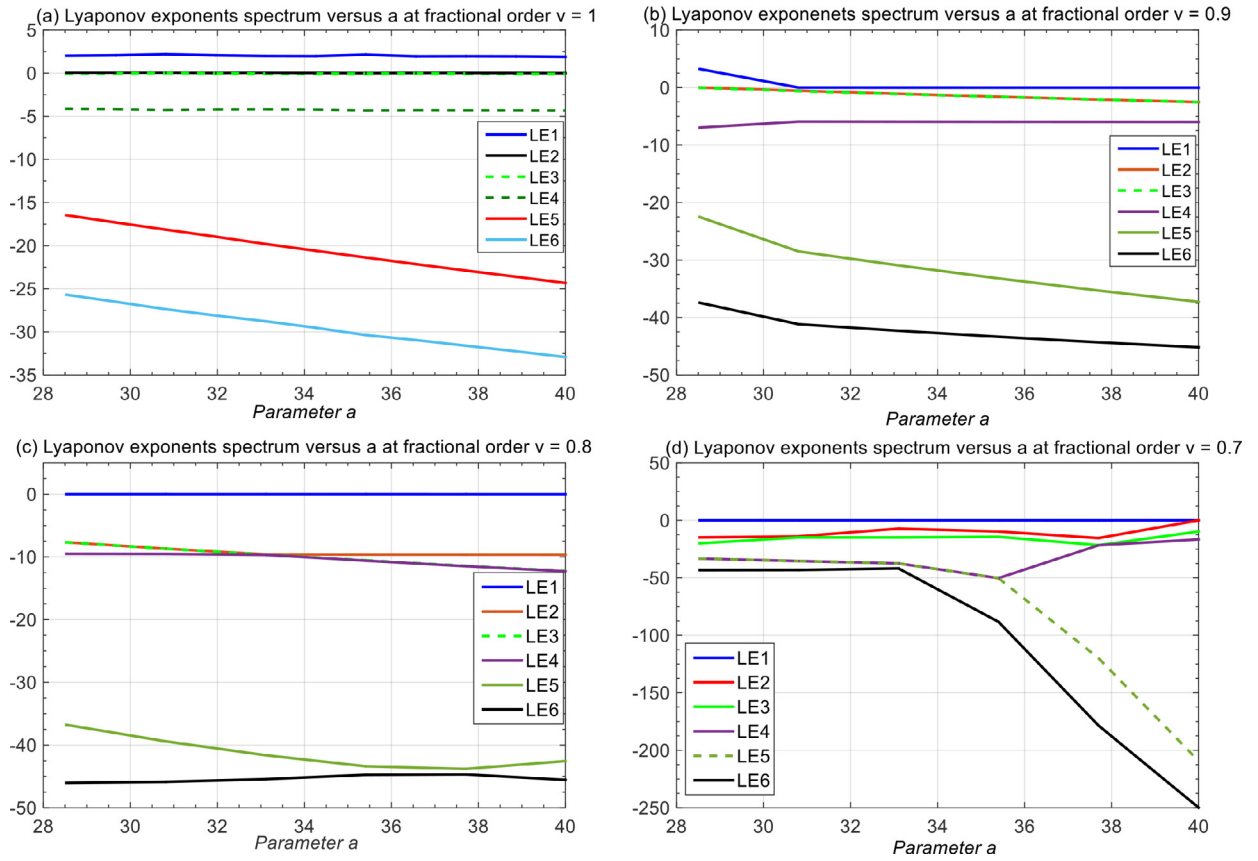


Fig. 6. Lyapunov exponents spectrum versus the parameter a where other parameters are fixed as $b = 10$, $c = 1$, $d = 10$ at different fractional orders $\nu = 1, 0.9, 0.8, 0.7$.

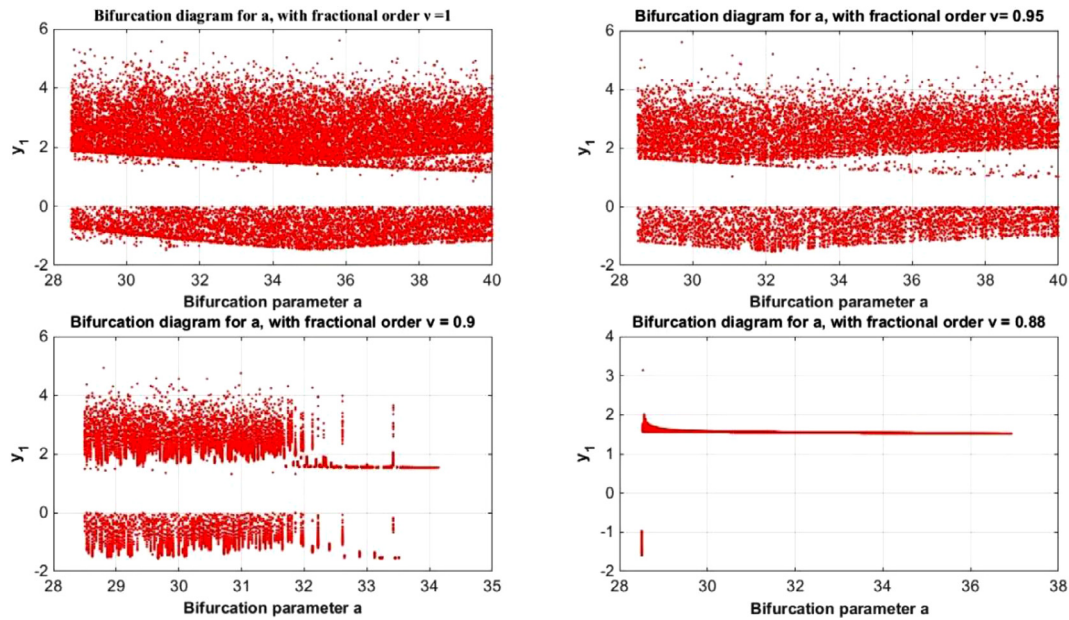
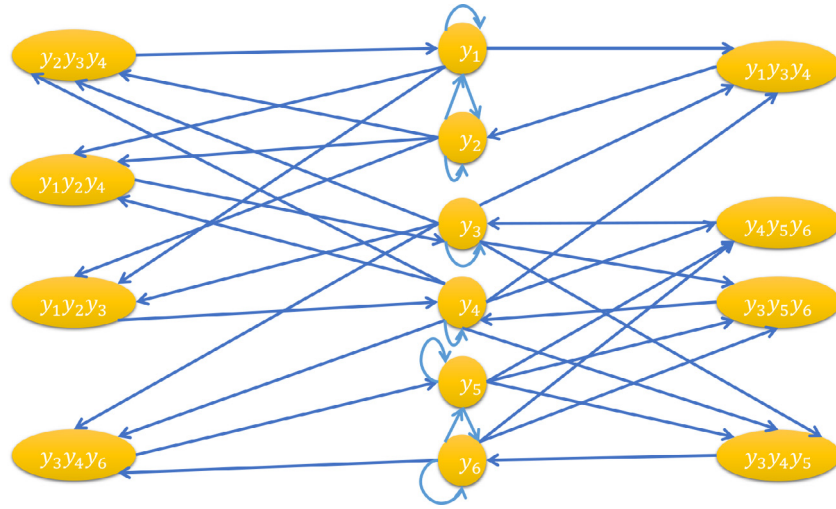
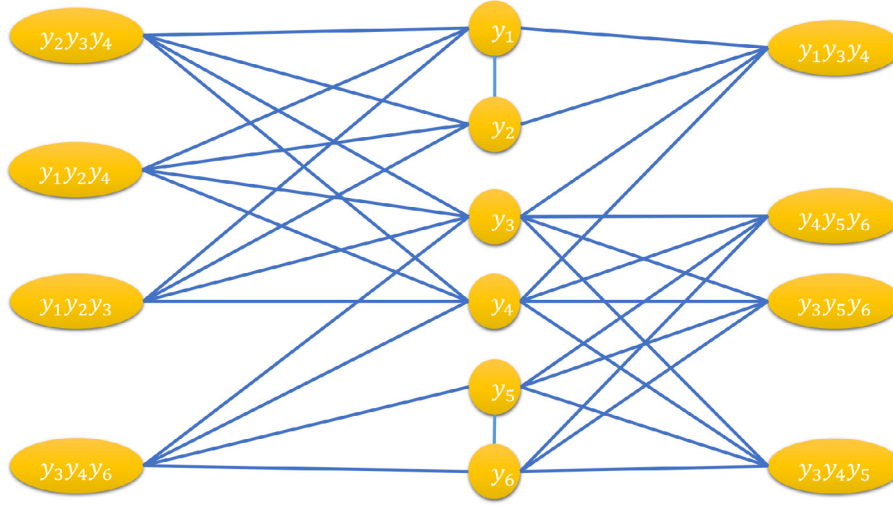


Fig. 7. Bifurcation maps of y_1 related to parameter a where other parameters are selected as $b = 10$, $c = 1$, $d = 10$ at different fractional orders $\nu = 1, 0.95, 0.9, 0.88$.

phenomena. We proved the chaos nature of the proposed new fractional 6D model numerically by displaying its dynamics related to the order of the fractional differentiation. The Lyapunov exponents and bifurcation maps of the proposed model is presented. The impact of the order of the fractional differentiation on the Lyapunov exponent are presented. The structural properties of the proposed 6D model are studied via certain graph theory tools. In

addition, the proposed 6D 5.4-fractional order chaotic model is realized by constructing an electronic circuit and we present its response. Also, an active fractional controller is produced to control the presented 6D fractional chaotic model.

The remaining of the manuscript is arranged as: the preliminaries and basic definitions of fractional differentiation are given in [Section 2](#). The 6D chaotic model is constructed and its basic

Fig. 8. The digraph D of the studied chaotic model.Fig. 9. The underlying graph G of the model graph D .

characteristics are given in Section 3. The new attractors are displayed in Section 4. Certain structural features of the proposed 6D chaotic model are presented using graph theory tools in Section 5. Section 6 contains the electronic circuit realization of the proposed fractional 6D chaotic model. An active fractional controller is designed and applied for the studied chaotic model in Section 7. The conclusions of our study are put in Section 8.

2. Fractional derivative and its preliminaries and basic definitions

This section briefly describes the fractional derivative and its corresponding fractional integral. Several notations were used in the new definition of fractional derivative. Through this article, we use ${}_l D_t^\nu Y(t)$ and ${}_l I_t^\nu Y(t)$ respectively for the fractional differentiation and its corresponding integration, where l signifies the integration lower limit and ν is the order of the fractional derivative or fractional integral. For each $\nu \in \mathbb{C}$, and for any function $Y : \mathbb{C} \rightarrow \mathbb{C}$, ${}_l D_t^\nu Y(t)$ and ${}_l I_t^\nu Y(t)$ should meet the following criteria [21]:

- If $Y(\cdot)$ is analytic function, then the operator ${}_l D_t^\nu Y(t)$ and ${}_l I_t^\nu Y(t)$ are analytic w.r.t. ν and t .

- The fractional differentiation operators satisfy the following linearity rules:

$${}_l D_t^\nu (c_1 Y_1(t) + c_2 Y_2(t)) = c_1 {}_l D_t^\nu Y_1(t) + c_2 {}_l D_t^\nu Y_2(t) \quad (1)$$

$${}_l I_t^\nu (c_1 Y_1(t) + c_2 Y_2(t)) = c_1 {}_l I_t^\nu Y_1(t) + c_2 {}_l I_t^\nu Y_2(t) \quad (2)$$

- The fractional operators with zero order must have no effect on $Y(t)$, i.e.,

$${}_l D_t^0 Y(t) = {}_l I_t^0 Y(t) = Y(t). \quad (3)$$

- If ν is a positive integer, the fractional operator should give the same result as a normal derivative or integral.

In the coming parts, the Caputo fractional operators of differentiation and integration are introduced in its common sense which defined as following [22] and shall be used to construct the proposed new chaotic model.

$${}_l^C D_t^\nu Y(t) = \frac{1}{\Gamma(n-\nu)} \int_l^t Y^{(n)}(\theta) (t-\theta)^{n-\nu-1} d\theta, \quad (4)$$

$${}_l^C I_t^\nu Y(t) = \frac{1}{\Gamma(\nu)} \int_l^t Y(\theta) (t-\theta)^{\nu-1} d\theta. \quad (5)$$

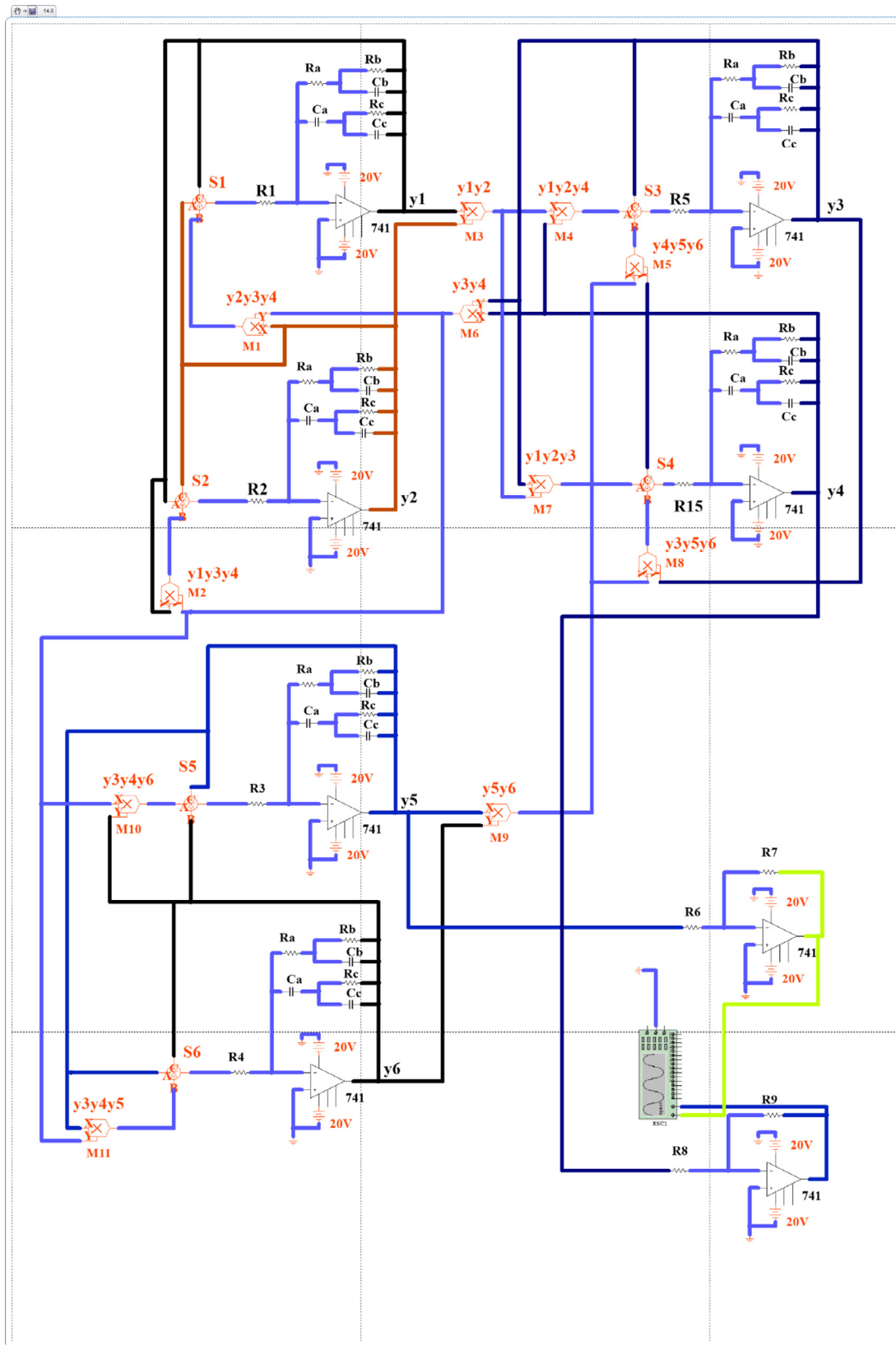


Fig. 10. Electronic circuit implementing the 5.4-fractional-order 6D new chaotic model (18).

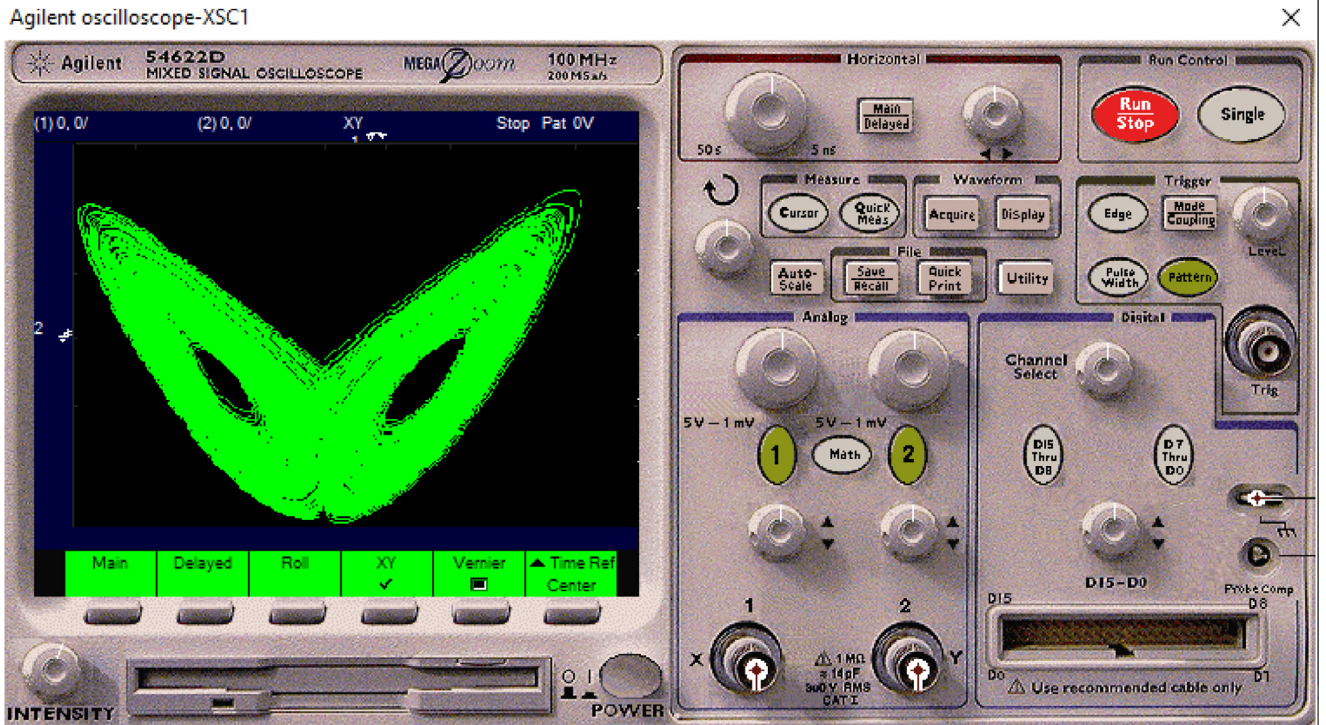


Fig. 11. y_4 - y_1 phase plane from the electronic circuit of the 5.4-fractional-order 6D new chaotic model (18).

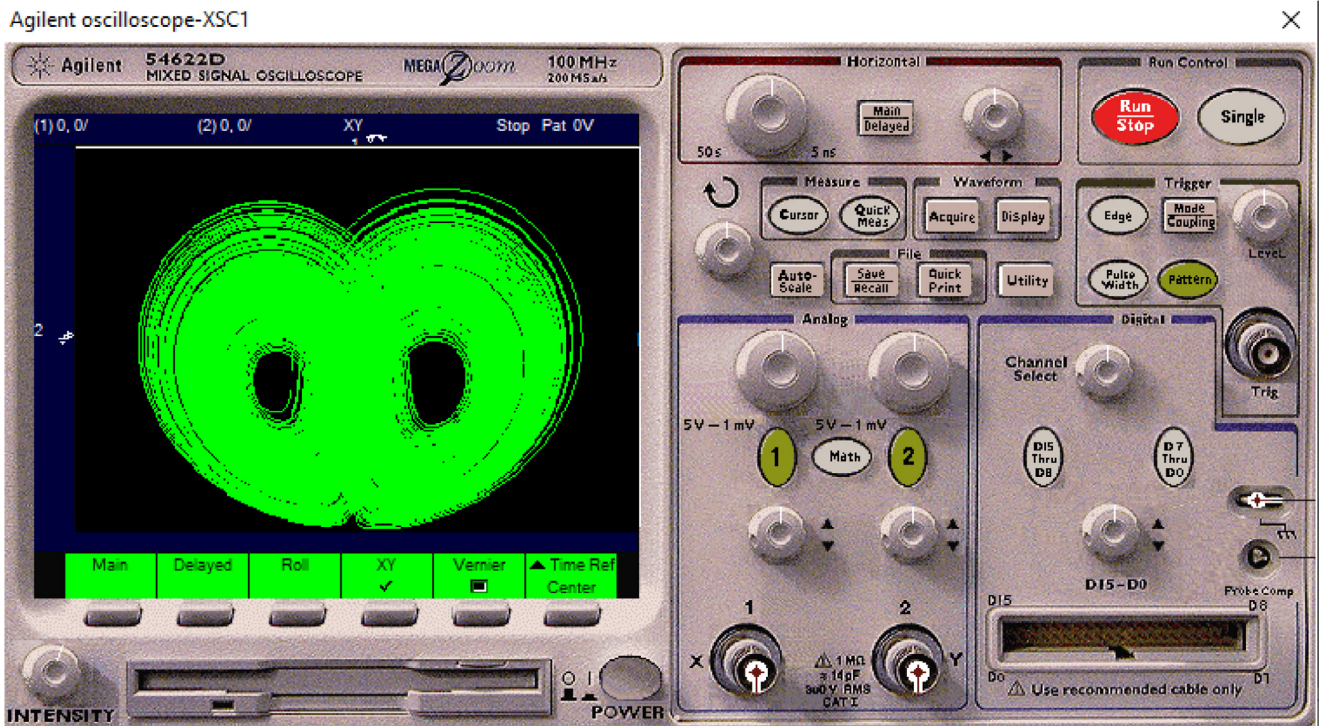


Fig. 12. y_4 - y_2 phase plane from the electronic circuit of the 5.4-fractional-order 6D new chaotic model (18).

For more details and other definitions as Atangana-Baleanu derivative and Caputo-Fabrizio derivative with its possible applications can be found in [23].

Definition 1. The following formula represents the Laplace transformation of Caputo fractional order operator

$$L\{D^{(\varepsilon)} f(t)\} = s^\varepsilon F(s) - \sum_{i=0}^{k-1} s^{\varepsilon-i-1} f^{(i)}(0);$$

$$k-1 < \varepsilon < k, k \in \mathbb{N}; F(s) = L\{f(t)\}.$$

Definition 2. For the graph G , define the matrix $A = A(G) = [a_{ij}]$, such that $a_{ij} = 1$ if u_i is adjacent to u_j in G and $a_{ij} = 0$ otherwise. A

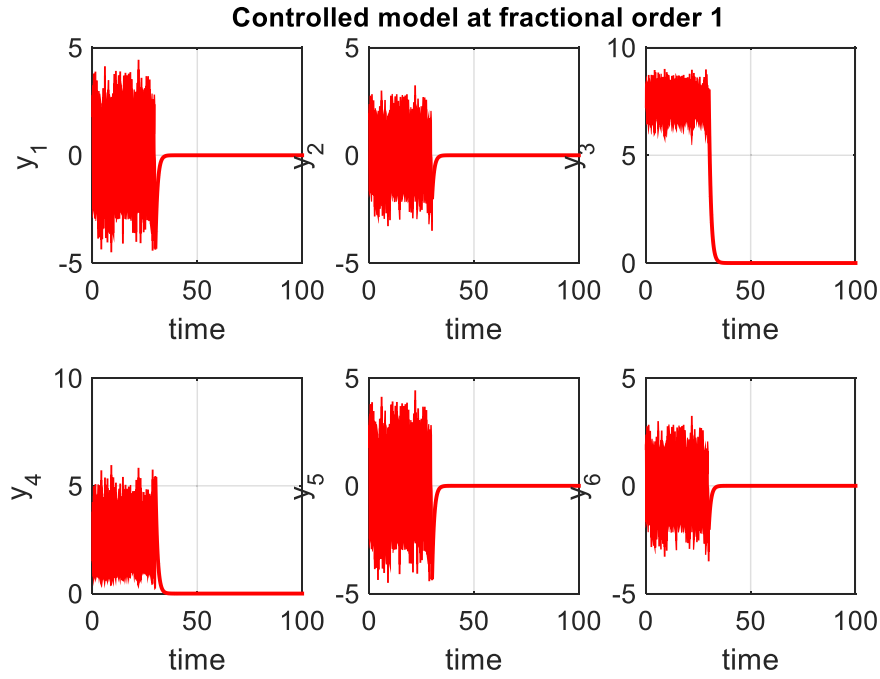


Fig. 13. Dynamics of the new fractional 6D chaotic model before and after control at fractional order $\nu = 1$, with all gains $G_i = -1$.

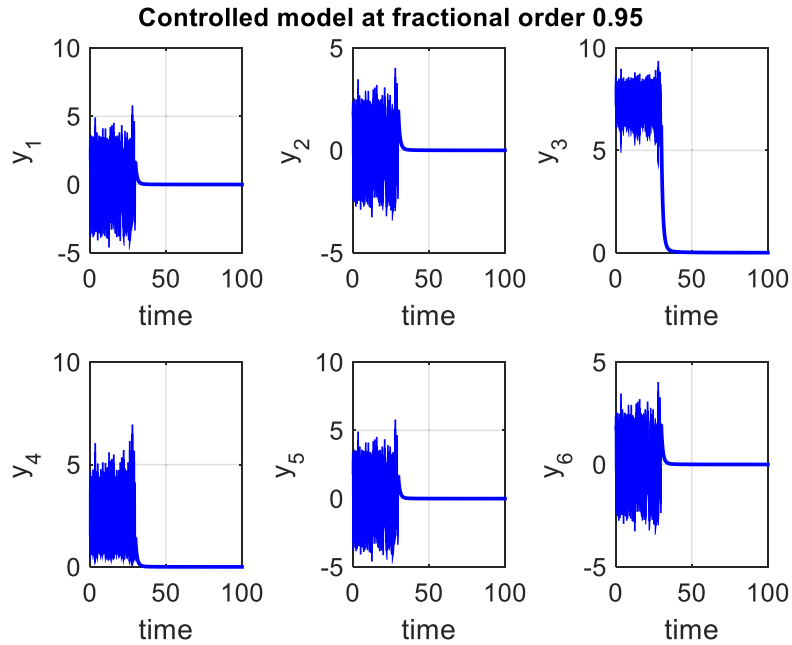


Fig. 14. Dynamics of the new fractional 6D chaotic model before and after control at fractional order $\nu = 0.95$, with all gains $G_i = -1$.

is called adjacency matrix of G .

Definition 3. The Hermitian matrix of a digraph D is defined as:

$$H_{uv}(D) = \begin{cases} 1 & \text{if } (u, v) \text{ and } (v, u) \text{ are edges in } D; \\ i & \text{if } (u, v) \text{ is an edge in } D \text{ but } (v, u) \text{ is not}; \\ -i & \text{if } (v, u) \text{ is an edge in } D \text{ but } (u, v) \text{ is not}; \\ 0 & \text{otherwise.} \end{cases}$$

The basics of the graph theory and its applications can be found for example in [24]

3. The novel 6D chaotic model and its basic characteristics

In the following, let us write ${}_0^C D_t^\nu$ as D^ν . The 6D nonlinear fractional autonomous novel system is described by

$$\begin{aligned} D^\nu y_1 &= \alpha_1(y_2 - y_1) + y_2 y_3 y_4, \\ D^\nu y_2 &= \beta_1(y_2 + y_1) - y_1 y_3 y_4, \\ D^\nu y_3 &= -\gamma y_3 + y_1 y_2 y_4 + y_5 y_6 y_4, \\ D^\nu y_4 &= -\delta y_4 + y_1 y_2 y_3 + y_5 y_6 y_3, \\ D^\nu y_5 &= \alpha_2(y_6 - y_5) + y_6 y_3 y_4, \\ D^\nu y_6 &= \beta_2(y_6 + y_5) - y_5 y_3 y_4. \end{aligned} \quad (6)$$

where y_i ($i = 1, 2, 3, \dots, 6$) are the state variables of the proposed

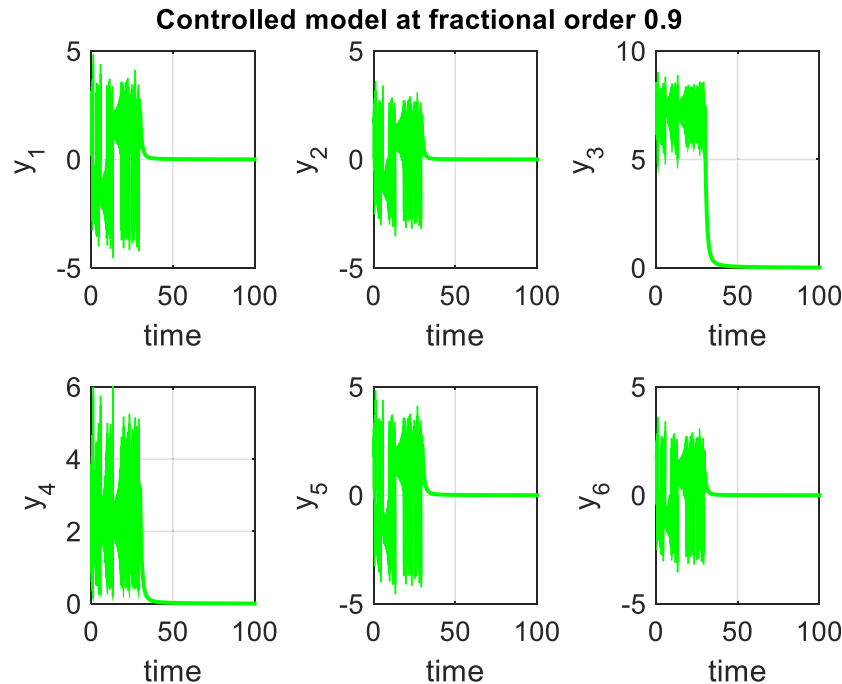


Fig. 15. Dynamics of the newfractional 6D chaotic model before and after control at fractional order $\nu = 0.9$, with all gains $G_i = -1$.

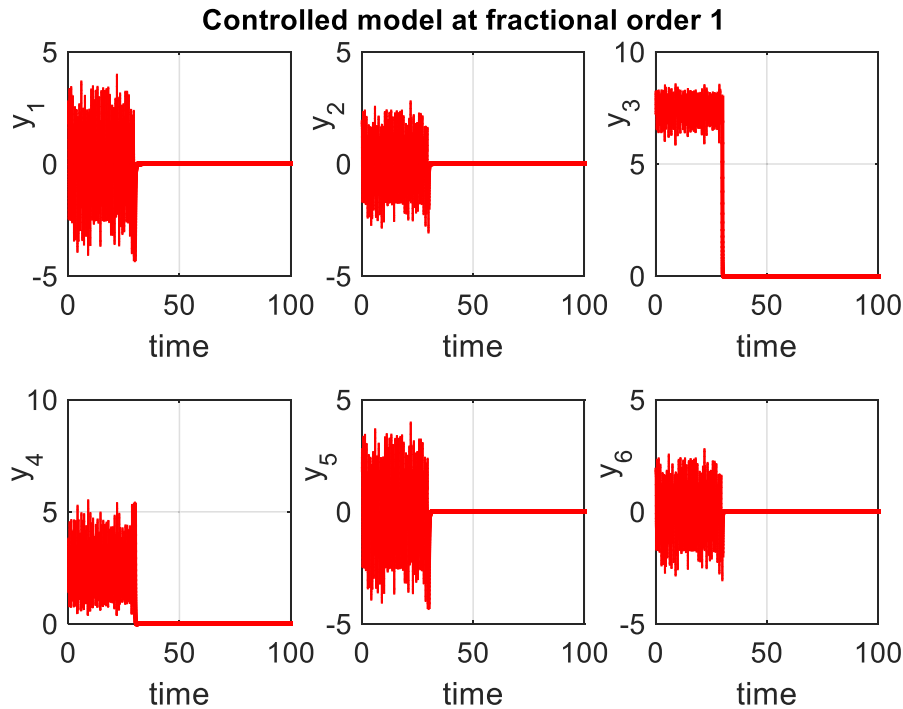


Fig. 16. Dynamics of the newfractional 6D chaotic model before and after control at fractional order $\nu = 1$, with all gains $G_i = -5$.

Table 1
Lyapunov exponents of the proposed system (18) for various values of fractional order ν .

Fractional order ν	LE1	LE2	LE3	LE4	LE5	LE6
0.8515	5.9389	0.0031	-0.1850	-8.7262	-40.1658	-57.4287
0.9010	4.9808	0.0362	-0.0060	-6.9942	-32.1153	-46.8727
0.9505	3.6936	0.0006	-0.0006	-5.3108	-25.4198	-37.3238
1	0.3514	0.0007	-0.0068	-3.8473	-19.9668	-27.2099

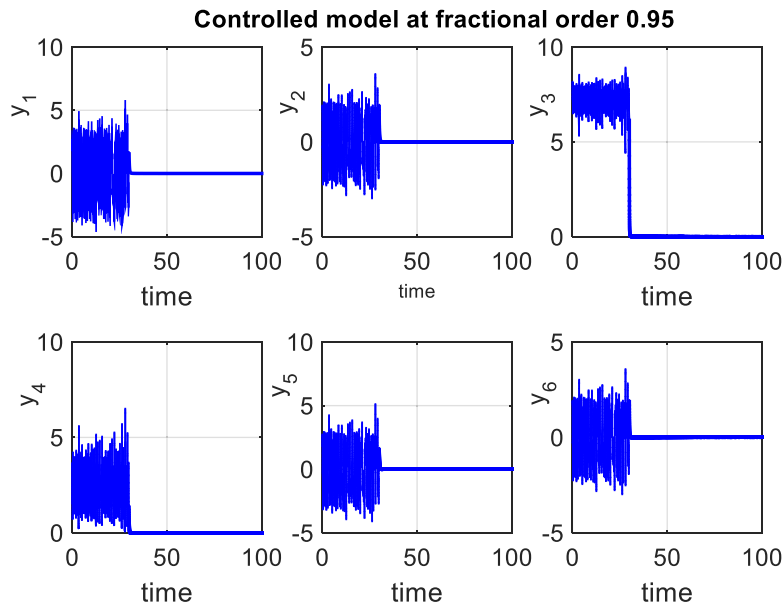


Fig. 17. Dynamics of the newfractional 6D chaotic model before and after control at fractional order $\nu = 0.95$, with all gains $G_i = -5$.

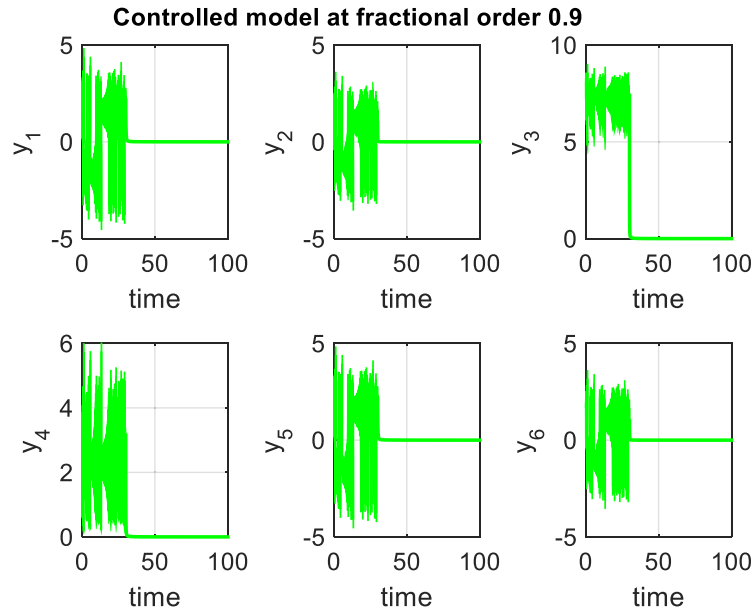


Fig. 18. Dynamics of the newfractional 6D chaotic model before and after control at fractional order $\nu = 0.9$, with all gains $G_i = -5$.

Table 2
The tree shape $1/s^{0.9}$ (2dB) integrator: values of resistors (c.f. [27]).

ν	R_a	R_b	R_c
0.9	$1.55M\Omega$	$61.54M\Omega$	$2.5K\Omega$

Table 3
The tree shape $1/s^{0.9}$ (2dB) integrator: values of capacitors (c.f. [27]).

ν	C_a	C_b	C_c
0.9	$0.73\mu F$	$0.52\mu F$	$1.1\mu F$

Table 4
The used voltage summing devices': gains for inputs (A, B, C) and output O.

summer symbol	A	B	C	O
S1	-30	-1	30	0.28
S2	-10	1	10	0.28
S3	-1	-1	1	0.015
S4	-1	-1	10	0.015
S5	-1	-30	30	0.28
S6	-10	1	-10	0.28

model, and $\alpha_1, \alpha_2, \beta_1, \beta_2, \gamma, \delta$ are the model parameters and all of them are positive real constants.

Constructing this model and determining the values of its parameters such that the model has chaotic dynamic follow some common concepts of chaotification [25,26], to build an autonomous chaotic paradigm or to chaotic a non-chaotic autonomous paradigm, the coming framework rules need to be satisfied:

- i) Dissipative of the model, in other words, the energy of the model is diminishing (except if Hamiltonian models are taken into consideration).
- ii) The model has an unstable equilibrium. In other words, the equilibrium-evaluated Jacobian value has unstable eigenvalues.
- iii) The model consists of one cross product term at least. That is, the dynamic effects between different variables can be considered.
- iv) All system trajectories are bounded. In other words, dynamic equilibrium is maintained by increasing and decreasing system energy.

The method of chaotifying the discrete models have some systematic steps [25], but there are no general techniques for non-discrete cases. In the latter case, the above analytical conditions usually need to be combined with trial-and-error computer experiments to accomplish the needed chaotic job. System (6) is also has such a shape. Now, some basic features of (6) are examined.

In the following subsections, some basic characteristics of the proposed system (6) are examined.

3.1. Dissipative feature for the proposed model (6)

The 6D nonlinear fractional autonomous novel system is described by

$$\begin{aligned} D^\nu y_1 &= \alpha_1(y_2 - y_1) + y_2 y_3 y_4 = v_1, \\ D^\nu y_2 &= \beta_1(y_2 + y_1) - y_1 y_3 y_4 = v_2, \\ D^\nu y_3 &= -\gamma y_3 + y_1 y_2 y_4 + y_5 y_6 y_4 = v_3, \\ D^\nu y_4 &= -\delta y_4 + y_1 y_2 y_3 + y_5 y_6 y_3 = v_4, \\ D^\nu y_5 &= \alpha_2(y_6 - y_5) + y_6 y_3 y_4 = v_5, \\ D^\nu y_6 &= \beta_2(y_6 + y_5) - y_5 y_3 y_4 = v_6. \end{aligned} \quad (7)$$

For this model (7), let $V = [v_1, v_2, \dots, v_6]^T$, we have

$$\nabla \cdot V = \sum_{i=1}^6 \frac{\partial}{\partial y_i} (D^\nu y_i) = (\beta_1 + \beta_2) - (\alpha_1 + \alpha_2 + \delta + \gamma) = D. \quad (8)$$

Then, system (7) is dissipative, if

$$(\beta_1 + \beta_2) - (\alpha_1 + \alpha_2 + \delta + \gamma) < 0 \quad (9)$$

Suppose that $\Theta(t)$ is a suitable region in R^6 which has a smooth boundary and let $\Theta(t) \in \Lambda(t)$ such that $\Lambda(t)$ is the flow of the vector field $V(t)$. Plus, let that the volume of $\Theta(t)$ is $V(t)$. Referring to Liouville's theorem as in [13], we get

$$\dot{V}(t) = \int_{\Theta(t)} (\nabla \cdot V) \prod_{k=1}^6 dv_k \quad (10)$$

Then from (7) and (9), we have

$$\dot{V}(t) = \int_{\Theta(t)} (D) \prod_{k=1}^6 dv_k = DV(t) \quad (11)$$

Integrating (11), then

$$V(t) = V(0)e^{Dt} \quad (12)$$

from (12), we conclude that the orbit of this dynamic model (6) forces all contained volumes to be reduced to zero. Therefore, there are attractors that attract all the orbits of the new model. From this, we can show that the newly proposed chaos model (6) is dissipative in the following condition is met:

$$(\beta_1 + \beta_2) - (\alpha_1 + \alpha_2 + \delta + \gamma) < 0 \quad (13)$$

3.2. Model (6) equilibrium points

The equilibrium points of the proposed model (6) can be found by simultaneously solving the next equations:

$$\begin{aligned} \alpha_1(y_2 - y_1) + y_2 y_3 y_4 &= 0, \quad \beta_1(y_2 + y_1) - y_1 y_3 y_4 \\ &= 0, \quad -\gamma y_3 + y_1 y_2 y_4 + y_5 y_6 y_4 = 0, \quad -\delta y_4 + y_1 y_2 y_3 + y_5 y_6 y_3 \\ &= 0, \quad \alpha_2(y_6 - y_5) + y_6 y_3 y_4 = 0, \quad \beta_2(y_6 + y_5) - y_5 y_3 y_4 = 0. \end{aligned} \quad (14)$$

Clearly, $P_0 = (0, 0, 0, 0, 0, 0)$ is one of the equilibrium points. The system is very complex and has many other equilibrium points that can be calculated using Matlab. Here, the stability of the zero equilibrium P_0 is examined only. The Jacobian can be obtained by linearizing system (6) at P_0 as:

$$J = \begin{pmatrix} -\alpha_1 & \alpha_1 & 0 & 0 & 0 & 0 \\ \beta_1 & \beta_1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -\gamma & 0 & 0 & 0 \\ 0 & 0 & 0 & -\delta & 0 & 0 \\ 0 & 0 & 0 & 0 & -\alpha_2 & \alpha_2 \\ 0 & 0 & 0 & 0 & \beta_2 & \beta_2 \end{pmatrix} \quad (15)$$

That has the following eigenvalues:

$$\lambda_{1,2} = \frac{1}{2}\beta_1 - \frac{1}{2}\alpha_1 \pm \frac{1}{2}\sqrt{\alpha_1^2 + 6\alpha_1\beta_1 + \beta_1^2}, \quad (16)$$

$$\lambda_3 = -\delta, \quad \lambda_4 = -\gamma,$$

$$\lambda_{5,6} = \frac{1}{2}\beta_2 - \frac{1}{2}\alpha_2 \pm \frac{1}{2}\sqrt{\alpha_2^2 + 6\alpha_2\beta_2 + \beta_2^2}. \quad (17)$$

Since all model parameters are positive real numbers, it is easy to find that $\lambda_2, \lambda_6 > 0$, proving the instability of the equilibrium P_0 . The other nonzero equilibria, it is still possible to numerically evaluate their stabilities. Since two unstable equilibria has been got at P_0 , this long numerical calculation becomes not needed then it is not examined further here.

4. Displaying of new chaotic attractors

In the following, to simplifying the study, the produced model parameters are taken as: $\alpha_1 = \alpha_2 = a, \beta_1 = \beta_2 = b, \gamma = c$ and $\delta = d$. Then proposed model is rewritten as follows.

$$\begin{aligned} D^\nu y_1 &= a(y_2 - y_1) + y_2 y_3 y_4, \\ D^\nu y_2 &= b(y_2 + y_1) - y_1 y_3 y_4, \\ D^\nu y_3 &= -cy_3 + y_1 y_2 y_4 + y_5 y_6 y_4, \\ D^\nu y_4 &= -dy_4 + y_1 y_2 y_3 + y_5 y_6 y_3, \\ D^\nu y_5 &= a(y_6 - y_5) + y_6 y_3 y_4, \\ D^\nu y_6 &= b(y_6 + y_5) - y_5 y_3 y_4. \end{aligned} \quad (18)$$

Considering condition (12), with initial conditions $Y(0) = (1.77, 1.66, 7.2, 1.5, 1.77, 1.66)$ and fractional order $\nu = 0.8515, 0.9010, 0.9505, 1$, many numerical experiments have been executed, certain discoveries recorded as follows.

4.1. Varying the fractional order with $a = 30, b = 10, c = 1, d = 10$

Fig. 1 displays the spectrum of Lyapunov exponents versus the order ν of fractional derivative and parameters are chosen as $a = 30, b = 10, c = 1, d = 10$. From which one can note that LE1 and LE2 are positive for fractional order $\nu \in [0.8515, 1]$. In Table 1, some values of Lyapunov exponents are recoded against different values of ν . The proposed system (18) has a hyperchaotic behavior for the fractional order $\nu \in [0.8515, 1]$. Fig. 2 shows the bifurcation map of y_1 related to the fractional order ν where parameters are chosen as $a = 30, b = 10, c = 1, d = 10$. Figs. 3, 4, show two hyperchaotic attractors at fractional orders 0.9 and 1 respectively. For ν outside this period $\nu \in [0.8515, 1]$, the system has fixed point behavior as shown in Fig. 5.

From above experiments, the proposed system is hyperchaotic for fractional order $\nu \in [0.8515, 1]$ where model parameters are selected as $a = 30$, $b = 10$, $c = 1$, $d = 10$.

4.2. Vary the parameter a and fix $b = 10$, $c = 1$, $d = 10$

As clear from Fig. 6, the spectrum of Lyapunov-exponents show that the proposed model (18) is hyperchaotic for a wide range of the parameter a for fractional order $\nu \in [0.8515, 1]$ and the system lose its chaotic dynamical behavior outside this period. Fig. 7 displays the bifurcation maps of y_1 related to parameter a where other parameters are selected as $b = 10$, $c = 1$, $d = 10$ at different fractional orders.

5. Structural characteristic of proposed model (18)

In this section, we display the relations between the different terms of the proposed model via drawing its graph. The proposed model consists of pure state variables and nonlinear product terms. We draw the suggested graph as follows. The graph vertices are the six pure state variables plus the eight nonlinear product terms. There will be a directed edge between two vertices if the corresponding terms affect the other. For example, there will be a directed edge from the vertex corresponds to the term $y_2y_3y_4$ to the vertex corresponds to the term y_1 , but there no edge from the vertex corresponds to y_1 towards that corresponds to $y_2y_3y_4$, so model (18) can be displayed by the digraph D given in Fig. 8. Many structural characteristics of the model can be calculated from the adjacency and Hermitian matrices.

The underlying graph G (shown in Fig. 9) of a digraph D is constructed by neglecting the direction of the edges and the loops (edges with same ends). Following Definitions 2, 3, respectively, the adjacency matrix $A(G)$ for the underlying graph of D and the Hermitian matrix $H(D)$ for the model digraph D are constructed as follows in (19) and (20).

$$A(G) = \begin{matrix} & \begin{matrix} y_1 & y_2 & y_3 & y_4 & y_5 & y_6 & y_2y_3y_4 & y_1y_3y_4 & y_1y_2y_4 & y_4y_5y_6 & y_1y_2y_3 & y_3y_5y_6 & y_3y_4y_6 & y_3y_4y_5 \end{matrix} \\ \begin{matrix} y_1 \\ y_2 \\ y_3 \\ y_4 \\ y_5 \\ y_6 \\ y_2y_3y_4 \\ y_1y_3y_4 \\ y_1y_2y_4 \\ y_4y_5y_6 \\ y_1y_2y_3 \\ y_3y_5y_6 \\ y_3y_4y_6 \\ y_3y_4y_5 \end{matrix} & \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \end{bmatrix} \quad (19)$$

$$H(D) = \begin{matrix} & \begin{matrix} y_1 & y_2 & y_3 & y_4 & y_5 & y_6 & y_2y_3y_4 & y_1y_3y_4 & y_1y_2y_4 & y_4y_5y_6 & y_1y_2y_3 & y_3y_5y_6 & y_3y_4y_6 & y_3y_4y_5 \end{matrix} \\ \begin{matrix} y_1 \\ y_2 \\ y_3 \\ y_4 \\ y_5 \\ y_6 \\ y_2y_3y_4 \\ y_1y_3y_4 \\ y_1y_2y_4 \\ y_4y_5y_6 \\ y_1y_2y_3 \\ y_3y_5y_6 \\ y_3y_4y_6 \\ y_3y_4y_5 \end{matrix} & \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 & -i & i & i & 0 & i & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & i & -i & i & 0 & i & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & i & i & -i & -i & i & i & i & i \\ 0 & 0 & 0 & 1 & 0 & 0 & i & i & i & i & -i & -i & i & i \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & i & 0 & i & -i & i \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & i & 0 & i & i & -i \\ i & -i & -i & -i & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -i & i & -i & -i & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -i & -i & i & -i & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & i & -i & -i & -i & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -i & -i & -i & i & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -i & i & -i & -i & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -i & -i & i & -i & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -i & -i & -i & i & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \end{bmatrix} \quad (20)$$

The eigenvalues of $A(G)$ are:

$\{-4.755, -2.3723, -1, -1, 0, 0, 0, 0, 0, 0, 0, 0.6605, 3.3723, 5.0945\}$. And the eigenvalues of $H(D)$ are: $\{3.2361, 2, -2, -1.2361, 3.3723, 3, 2, -2.3723, -2, 2, 3.2361, 0, 0, -1.2361\}$.

The energy of the graph G is defined as $E(G) = \sum_{i=1}^{14} |\lambda_i| = 18.2546$, where 14 is the order of G and λ_i 's are the eigenvalues of its adjacency matrix. This idea was presented by I. Gutman in [24] and has several applications and it can be utilized to estimate the whole energy of graphs. The Hermitian energy is defined as $EH(D) = \sum_{i=1}^{14} |\mu_i| = 27.689$, where 14 is the order of D and μ_i 's are the eigenvalues of its Hermitian matrix.

6. Electric circuit realization of the studied fractional chaotic system

In this section, we show that the proposed chaotic model is applicable by constructing its electric circuit. By constructing an electric circuit for the proposed model, we prove its ability to be applicable in real world applications. Here, we consider the suggested 6D fractional order chaotic paradigm (18). First, we fix the fractional order $\nu = 0.9$ to design circuit to realize model (18). We follow the method published in [27] to realize $1/s^{0.9}$. With aid of NI Multisim 14.0 package, the electric circuit given in Fig. 10 is designed, where $S1, S2, \dots, S6$ are six summers all of them has 3 inputs and single output; $M1, M2, \dots, M11$ are AD633 voltage multiplier with 2 inputs and single output all of them are adjusted with identity gain; six LM741 op. amp's are utilized to simulate the six 0.9-fractional-order voltage integrators with outputs equal the states of the system y_1 to y_6 ; $R1, R2, R3, R4, R5, R15$ are $100K\Omega$ resistances and $R6, R7, R8, R9$ are $10K\Omega$ resistances plus a couple of LM741 op. amps. are used to construct two voltage inverters. The designed values of resistors and capacitors are given in Tables 2 and 3. In Table 4, we record input/output gain's utilized

in voltage summers. The electronic circuit implementing the new chaotic 6D model (18) is shown in Fig. 10. Figs. 11 and 12 display the response of 5.4-fractional-order chaotic 6D new model.

7. Design of fractional order active controller for the proposed chaotic model (18)

In this section, we propose an active control strategy to control the model states and convert it to a fixed point.

For the proposed system (18), the control rule is designed as follows:

$$\begin{aligned} C_1 &= G_1 y_1 - (a(y_2 - y_1) + y_2 y_3 y_4), \\ C_2 &= G_2 y_2 - b(y_2 + y_1) + y_1 y_3 y_4, \\ C_3 &= G_3 y_3 - c y_3 - y_1 y_2 y_4 - y_4 y_5 y_6, \\ C_4 &= G_4 y_4 - d y_4 - y_1 y_2 y_3 - y_3 y_5 y_6, \\ C_5 &= G_5 y_5 - a(y_6 - y_5) - y_3 y_4 y_6, \\ C_6 &= G_6 y_6 - b(y_6 + y_5) + y_3 y_4 y_5. \end{aligned} \quad (21)$$

Where G_i 's are control gains. The controlled model becomes:

$$\begin{aligned} D^\nu y_1 &= a(y_2 - y_1) + y_2 y_3 y_4 + C_1, \\ D^\nu y_2 &= b(y_2 + y_1) - y_1 y_3 y_4 + C_2, \\ D^\nu y_3 &= -c y_3 + y_1 y_2 y_4 + y_5 y_6 y_4 + C_3, \\ D^\nu y_4 &= -d y_4 + y_1 y_2 y_3 + y_5 y_6 y_3 + C_4, \\ D^\nu y_5 &= a(y_6 - y_5) + y_6 y_3 y_4 + C_5, \\ D^\nu y_6 &= b(y_6 + y_5) - y_5 y_3 y_4 + C_6. \end{aligned} \quad (22)$$

Substituting (21) into (22), the dynamics of the controlled model becomes:

$$\begin{aligned} D^\nu y_1 &= G_1 y_1, \\ D^\nu y_2 &= G_2 y_2, \\ D^\nu y_3 &= G_3 y_3, \\ D^\nu y_4 &= G_4 y_4, \\ D^\nu y_5 &= G_5 y_5, \\ D^\nu y_6 &= G_6 y_6. \end{aligned} \quad (23)$$

It is obvious that the controlled system (23) has $\hat{E}_0 = (0, 0, 0, 0, 0, 0)$ as an equilibrium point. The following theorem can be proved for the controlled fractional order model (23).

Theorem 1. If all controller gains G_i 's in the controlled model (23) are chosen to be negative values, then $\hat{E}_0 = (0, 0, 0, 0, 0, 0)$ of the controlled model (23) is fixed point.

Proof. Taking Laplace transformation for the controlled model (23), then

then

$$s^\nu Y_i(s) - s^{\nu-1} y_i(0) = G_i Y_i(s),$$

where $Y_i(s)$ is the Laplace transformation of $y_i(t)$, so

$$Y_i(s) = \frac{s^{\nu-1} y_i(0)}{s^\nu - G_i} \quad (24)$$

applying inverse taransformation of Laplace for (24), we get

$$y_i(t) = y_i(0) M_\nu(G_i t^\nu), \quad (25)$$

such that $M_\nu(\cdot)$ is the common function of Mittag-Leffler. Because $0 < \nu \leq 1$, so $M_\nu(G_i t^\nu) = \sum_{h=0}^{\infty} \frac{(G_i t^\nu)^h}{\Gamma(h\nu+1)}$ approaches zero for all values of $G_i t^\nu$ if all G_i 's are negatives, the reader can consult [28–29] and its cited references for more about function of Mittag-Leffler. Then, the states $y_i(t)$ of the model are all stable asymptotically. Figs. 13, 14 and 15 show the time response of the new fractional 6D chaotic model before and after control at fractional order $\nu = 1, 0.95$ and 0.9 respectively with all gains $G_i = -1$. The controller is applied after 30 secs. We can see that the model takes a transient time to reach its final state. Changing the controller gains to be $G_i = -5$, the transient time is approaches zero as shown in Figs. 16, 17 and 18.

8. Conclusions

The new fractional 6D chaotic model presented above has complicated behaviors and all characteristics of the chaotic system. The proposed fractional 6D chaotic system with different topology and structure than the existing 6D systems is constructed in this work. The basic features and dynamical response of the new model have been investigated to prove it's chaotic. Certain graph theory tools have been utilized to display some hidden properties of the proposed model. In addition, an electronic circuit realizing the proposed model has been designed. Also, an active fractional controller is proposed to control the proposed model. In the future work, we suggest utilizing this fractional 6D model to produce random keys for data encryption and in secure communication applications.

Data availability statement

Not applicable.

Author contributions

Conceptualization, M.H.; methodology, M.H., N.A.; software, M.H., S.M., A.A., N.A.; validation, M.H., S.M., A.A., N.A.; formal analysis, M.H., S.M., A.A., N.A.; investigation, M.H., S.M., A.A., N.A.; resources, M.H., S.M., A.A., N.A.; data curation: M.H.; Funding acquisition, M.H., S.M., A.A., N.A.; writing—original draft preparation, M.H.; writing—review and editing, M.H., S.M., A.A., N.A.; visualization, M.H.; supervision, M.H.; project administration, M.H., S.M., All authors have read and agreed to the published version of the manuscript.

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Declaration of Competing Interest

The authors declare no conflict of interests.

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