POLYTECHNIC VERSION

ENGINEERING MATHEMATICS 2

DIFFERENTIATION

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PREFACE

Alhamdulillah, puji dan syukur kehadrat Ilahi, kerana dengan izin dan limpah kurnia-Nya, eBook ini telah berjaya dihasilkan dan diterbitkan seperti yang telah dirancang.

Ebook **ENGINEERING MATHEMATICS 2 : DIFFERENTIATION** ini dihasilkan bagi memudahkan pensyarah serta pelajar yang mengambil kursus Engineering Mathematics 2 untuk mengikuti kursus ini secara berstruktur dan sistematik. Melalui eBook yang dihasilkan ini juga diharapkan supaya dapat menjadi sumber penyebaran ilmu berdasarkan bidang yang dipilih di samping meningkatkan kemantapan intelek penulis sendiri.

Akhir kata, kami memohon kemaafan di atas kekurangan yang terdapat dalam penghasilan eBook ini. Sekalung penghargaan kepada pihak Pengurusan Politeknik Tuanku Syed Sirajuddin kerana telah memberi ruang dan peluang dalam penghasilan eBook ini.

Harapan kami agar pensyarah dan pelajar mendapat ilmu yang bermanfaat dan dapat dijadikan sebagai rujukan.

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DIFFERENTIATION

1.0 DESCRIBE RULES OF DIFFERENTIATION

1.1 IDENTIFY BASIC RULES OF DIFFERENTIATION

Rule	
CONSTANT RULE	• $\frac{d}{dx}(k) = 0$, where k is constant
POWER RULE	• $\frac{d}{dx}(x^n) = nx^{n-1}$
CONSTANT MULTIPLE RULE	• $\frac{d}{dx}(ax^n) = nax^{n-1}$
SUM AND DIFFERENCE RULE	• Differentiate each terms separately

EXAMPLE

Differentiate the following functions with respect to x:

1. $y = 5$

$\frac{dy}{dx} = 0$ *Constant Rule*

2. $y = x^5$

$\frac{dy}{dx} = 5x^{5-1}$

$\frac{dy}{dx} = 5x^4$ *Power Rule*

3. $y = 3x$

$\frac{dy}{dx} = 3x^{1-1}$

$\frac{dy}{dx} = 3x^0$ → $x^0 = 1$

$\frac{dy}{dx} = 3$ *Constant Multiple Rule*

4. $y = -3x^7$

$\frac{dy}{dx} = (7)(-3)x^{7-1}$

$\frac{dy}{dx} = -21x^6$ *Constant Multiple Rule*

5. $y = (2x + 4)(-3x - 2)$

$y = -6x^2 - 4x - 12x - 8$

$y = -6x^2 - 16x - 8$ } *Expand and simplify*

$\frac{dy}{dx} = (2)(-6)x^{2-1} - 16 - 0$

$\frac{dy}{dx} = -12x - 16$

6. $y = \frac{-3}{x^2} + \frac{1}{\sqrt[3]{x^4}}$

$y = [-3x^{-2}] + [x^{-4/3}]$ *Sum /Difference Rule*

$\frac{dy}{dx} = [(-2)(-3)x^{-2-1}] + [\left(-\frac{4}{3}\right)x^{-\frac{4}{3}-1}]$

$\frac{dy}{dx} = 6x^{-3} - \frac{4}{3}x^{-\frac{7}{3}}$

7. $y = \frac{5x-3x^4}{x^3} = \frac{5x}{x^3} - \frac{3x^4}{x^3}$

$y = \frac{5}{x^2} - 3x$

$y = 5x^{-2} - 3x$

$\frac{dy}{dx} = (-2)(5)x^{-2-1} - 3$

$\frac{dy}{dx} = -10x^{-3} - 3$

8. $y = (x^3 + 7)^5$ → Differentiate $x^3 + 7$

$\frac{dy}{dx} = 5(x^3 + 7)^4 \times (3x^2)$

$\frac{dy}{dx} = 15x^2(x^3 + 7)^4$ *Extended Power Rule*

9. $y = \frac{2}{(x^3-5)^3}$

$y = 2(x^3 - 5)^{-3}$

$\frac{dy}{dx} = -3(2)(x^3 - 5)^{-4} \times (3x^2)$ ← Differentiate $x^3 - 5$

$\frac{dy}{dx} = -18x^2(x^3-5)^{-4}$

$\frac{dy}{dx} = \frac{-18x^2}{(x^3-5)^4}$

10. Given $y = \frac{1}{3}x^3 + 2x - 5x^{-2}$, find the value of

$\frac{dy}{dx}$ if $x = 2$.

$\frac{dy}{dx} = x^2 + 2 + 10x^{-3}$

Substitute $x = 2$

$\frac{dy}{dx} = (2)^2 + 2 + 10(2)^{-3}$

$\frac{dy}{dx} = 7.25$

11. If $f(x) = 4x^5 - 10x^3$, find the value of $f'(-3)$

$f'(x) = 20x^4 - 30x^2$

$f'(-3) = 20(-3)^4 - 30(-3)^2$

$f'(-3) = 1350$

FORMATIVE PRACTICE 1

Differentiate with respect to x :

1. $y = 4 - 3x^3$ — *Ans:* $\frac{dy}{dx} = -9x^2$
2. $y = 1 - 5x^2$ — *Ans:* $\frac{dy}{dx} = -10x$
3. $y = \frac{1}{2x^2}$ — *Ans:* $\frac{dy}{dx} = -\frac{1}{x^3}$
4. $y = -2x^3 + 3x - 4$ — *Ans:* $\frac{dy}{dx} = -\frac{1}{x^3}$
5. $y = \frac{3}{(7-3x)^2}$ — *Ans:* $\frac{dy}{dx} = \frac{18}{(-3x+7)^3}$
6. $y = (8x+3)(7x+5)$ — *Ans:* $\frac{dy}{dx} = 112x + 61$
7. $y = \frac{5\sqrt{x}-3x^2}{3}$ — *Ans:* $\frac{dy}{dx} = -2x + \frac{5}{6x^{1/2}}$
8. $y = (\sqrt{x}-3)(2x^2+5)$ — *Ans:* $\frac{dy}{dx} = -12x + 5x^{3/2} + \frac{5}{2x^{1/2}}$
9. $y = (5-x)^2$ — *Ans:* $\frac{dy}{dx} = -10 + 2x$
10. $y = \frac{3}{5}x^5 - 5x^{-2}$ — *Ans:* $\frac{dy}{dx} = 3x^4 + 10x^{-3}$

1.2 DESCRIBE THE TECHNIQUES OF DIFFERENTIATION IN SOLVING PROBLEMS

PRODUCT RULE

$$\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}$$

EXAMPLE

Find $\frac{dy}{dx}$ for the following functions.

1. $y = x^2(10x-2)$

$u = x^2$ $\quad v = 10x - 2$

$\frac{du}{dx} = 2x$ $\quad \frac{dv}{dx} = 10$

Applying the Product Rule

$\frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$

$\frac{dy}{dx} = x^2(10) + (10x-2)(2x)$

$\frac{dy}{dx} = 10x^2 + 20x^2 - 4x$

$\frac{dy}{dx} = 30x^2 - 4x$

2. $y = (x^5+1)(x-3)$

$= x^5 + 1$ $\quad v = x - 3$

$\frac{du}{dx} = 5x^4$ $\quad \frac{dv}{dx} = 1$

Applying the Product Rule

$\frac{dy}{dx} = (x^5+1)1 + (x-3)(5x^4)$

$\frac{dy}{dx} = x^5 + 1 + 5x^5 - 15x^4$

$\frac{dy}{dx} = 6x^5 - 15x^4 + 1$

3. $y = (2\sqrt{x}+1)(x^2-2)$

$u = (2\sqrt{x}+1)$ $\quad v = (x^2-2)$

$\frac{du}{dx} = x^{-\frac{1}{2}}$ $\quad \frac{dv}{dx} = 2x$

Applying the Product Rule

$$\frac{dy}{dx} = (2\sqrt{x} + 1)2x + (x^2 - 2)x^{-\frac{1}{2}}$$

$$\frac{dy}{dx} = 4x^{\frac{3}{2}} + 2x + x^{\frac{3}{2}} - 2x^{-\frac{1}{2}}$$

$$\frac{dy}{dx} = 5x^{\frac{3}{2}} + 2x - 2x^{-\frac{1}{2}}$$

$$\frac{dy}{dx} = 5x^{\frac{3}{2}} + 2x - \frac{2}{\sqrt{x}}$$

4. $y = (2 - x^2)(5x + 2)$

$u = (2 - x^2)$ $\quad v = (5x + 2)$

$\frac{du}{dx} = -2x$ $\quad \frac{dv}{dx} = 5$

$$\frac{dy}{dx} = (2 - x^2)5 + (5x + 2)(-2x)$$

$$\frac{dy}{dx} = 10 - 5x^2 - 10x^2 - 4x$$

$$\frac{dy}{dx} = -15x^2 - 4x + 10$$

5. $y = (4x - 2)(3x^2 + 7)$

$u = (4x - 2)$ $\quad v = (3x^2 + 7)$

$\frac{du}{dx} = 4$ $\quad \frac{dv}{dx} = 6x$

$$\frac{dy}{dx} = (4x - 2)6x + (3x^2 + 7)4$$

$$\frac{dy}{dx} = 24x^2 - 12x + 12x^2 + 28$$

$$\frac{dy}{dx} = 36x^2 - 12x + 28$$

6. $y = (2x - 5)(3\sqrt[3]{x} - 2)$

$u = (2x - 5)$ $\quad v = 3\sqrt[3]{x} - 2$ $\quad$ $3\sqrt[3]{x}$ is written as $3x^{\frac{1}{3}}$

$\frac{du}{dx} = 2$ $\quad \frac{dv}{dx} = x^{-\frac{2}{3}}$

$$\frac{dy}{dx} = (2x - 5)x^{-\frac{2}{3}} + \left(3x^{\frac{1}{3}} - 2\right)2$$

$$\frac{dy}{dx} = 2x^{\frac{1}{3}} - 5x^{-\frac{2}{3}} + 6x^{\frac{1}{3}} - 4$$

$$\frac{dy}{dx} = 8x^{\frac{1}{3}} - 5x^{-\frac{2}{3}} - 4$$

FORMATIVE PRACTICE 2

Differentiate the following functions.

1. $y = (-4x^2 + 3)(x - 1)$ $\quad \frac{dy}{dx} = -12x^2 + 8x + 3$
2. $y = 6x^{-2}(x^2 + 5)$ $\quad \frac{dy}{dx} = -60x^{-3}$
3. $y = (4x^3 + 5)(10x^2 + 3)$ $\quad \frac{dy}{dx} = 200x^4 + 36x^2 + 100x$
4. $y = (6x^4 - 2)(6x^4 + 2)$ $\quad \frac{dy}{dx} = 288x^7$
5. $y = (2x - 5)(3\sqrt{x} - 2)$ $\quad \frac{dy}{dx} = 9x^{1/2} - 4 - \frac{15}{2x^{1/2}}$

QUOTIENT RULE

$$\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$$

EXAMPLE

Find the first derivative of the following equations.

1. $y = \frac{x^2}{(10x-2)}$

$u = x^2$ $\quad v = 10x - 2$

$\frac{du}{dx} = 2x$ $\quad \frac{dv}{dx} = 10$

Applying the Quotient Rule

$$\frac{d}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$$

$$\frac{dy}{dx} = \frac{(10x-2)(2x) - x^2(10)}{(10x-2)^2}$$

$$\frac{dy}{dx} = \frac{20x^2 - 4x - 10x^2}{(10x-2)^2}$$

$$\frac{dy}{dx} = \frac{10x^2 - 4x}{(10x-2)^2}$$

2. $y = \frac{(x^5+1)}{(x-3)}$

$u = x^5 + 1$ $\quad v = x - 3$

$\frac{du}{dx} = 5x^4$ $\quad \frac{dv}{dx} = 1$

Applying the Quotient Rule

$$\frac{dy}{dx}=\frac{(x-3)(5x^4)-(x^5+1)(1)}{(x-3)^2}$$

$$\frac{dy}{dx}=\frac{5x^5-15x^4-x^5-1}{(x-3)^2}$$

$$\frac{dy}{dx}=\frac{4x^5-15x^4-1}{(x-3)^2}$$

3. $y=\frac{(2-x^2)}{(x+3)}$

$u=(2-x^2)$ $\quad\quad v=(x+3)$

$\frac{du}{dx}=-2x$ $\quad\quad \frac{dv}{dx}=1$

$$\frac{dy}{dx}=\frac{(x+3)(-2x)-(2-x^2)(1)}{(x+3)^2}$$

$$\frac{dy}{dx}=\frac{-2x^2-6x-2+x^2}{(x+3)^2}$$

$$\frac{dy}{dx}=\frac{-x^2-6x-2}{(x+3)^2}$$

4. $y=\frac{(3x+2)}{(1-2x)}$

$u=(3x+2)$ $\quad\quad v=(1-2x)$

$\frac{du}{dx}=3$ $\quad\quad \frac{dv}{dx}=-2$

$$\frac{dy}{dx}=\frac{(1-2x)(3)-(3x+2)(-2)}{(1-2x)^2}$$

$$\frac{dy}{dx}=\frac{3-6x+6x+4}{(1-2x)^2}$$

$$\frac{dy}{dx}=\frac{7}{(1-2x)^2}$$

5. $y=\frac{3-2x}{x-2}$

$u=3-2x$ $\quad\quad v=x-2$

$\frac{du}{dx}=-2$ $\quad\quad \frac{dv}{dx}=1$

$$\frac{dy}{dx}=\frac{x-2(-2)-(3-2x)(1)}{(x-2)^2}$$

$$\frac{dy}{dx}=\frac{-2x+4-3+2x}{(x-2)^2}$$

$$\frac{dy}{dx}=\frac{1}{(x-2)^2}$$

FORMATIVE PRACTICE 3

Differentiate the following functions.

1. $y=\frac{-4x^2+3}{x-1}$ $\quad\quad$ *Ans:* $\frac{dy}{dx}=-\frac{(2x-1)(2x-3)}{(x-1)^2}$
2. $y=\frac{6x}{x^2+5}$ $\quad\quad$ *Ans:* $\frac{dy}{dx}=-\frac{6(x^2-5)}{(x^2+5)^2}$
3. $y=\frac{4x^3+5}{10x^2+3}$ $\quad\quad$ *Ans:* $\frac{dy}{dx}=\frac{4x(10x^3+9x-25)}{(10x^2+3)^2}$
4. $y=\frac{6x^4-2}{6x^4+2}$ $\quad\quad$ *Ans:* $\frac{dy}{dx}=\frac{24x^3}{(3x^4+1)^2}$
5. $y=\frac{2x-5}{3\sqrt{x}-2}$ $\quad\quad$ *Ans:* $\frac{dy}{dx}=\frac{6x-8x^{1/2}+15}{2x^{1/2}(3x^{1/2}-2)^2}$

CHAIN RULE AND EXTENDED POWER RULE

EXAMPLE

Find $\frac{dy}{dx}$ for the following functions.

1. $y=(-2x-7)^4$

Method 1 : *Chain Rule*

$$\frac{dy}{dx}=\frac{du}{dx}\times\frac{dy}{du}$$

$y=(-2x-7)^4$

$u=-2x-7$ $\quad\quad y=u^4$

$\frac{du}{dx}=-2$ $\quad\quad \frac{dy}{du}=4u^3$

$\frac{dy}{dx}=-2\times 4u^3$

$\frac{dy}{dx}=-8u^3=-8(-2x-7)^3$

Method 2 : *Extended Power Rule*

$$\frac{dy}{dx}=n[u(x)]^{n-1}\frac{du}{dx}$$

$y=(-2x-7)^4$

$\frac{dy}{dx}=4(-2x-7)^3(-2)$

$\frac{dy}{dx}=-8(-2x-7)^3$

2. $y = (10 - 2x^2 + 3x)^3$

Method 1 : Chain Rule

$$\frac{dy}{dx} = \frac{du}{dx} \times \frac{dy}{du}$$

$y = (10 - 2x^2 + 3x)^3$

$u = 10 - 2x^2 + 3x$ $\qquad y = u^3$

$\frac{du}{dx} = -4x + 3$ $\qquad \frac{dy}{du} = 3u^2$

$\frac{dy}{dx} = (-4x + 3) \times 3u^2$

$\frac{dy}{dx} = (-12x + 9)(10 - 2x^2 + 3x)^2$

Method 2 : Extended Power Rule

$$\frac{dy}{dx} = n[u(x)]^{n-1}\frac{du}{dx}$$

$y = (10 - 2x^2 + 3x)^3$

$\frac{dy}{dx} = 3(10 - 2x^2 + 3x)^2(-4x + 3)$

$\frac{dy}{dx} = (-12x + 9)(10 - 2x^2 + 3x)^2$

3. $y = \frac{-3}{(6x^2-3)^3}$

Method 1 : Chain Rule

$$\frac{dy}{dx} = \frac{du}{dx} \times \frac{dy}{du}$$

$y = \frac{-3}{(6x^2-3)^3}$

$y = -3(6x^2 - 3)^{-3}$

$u = 6x^2 - 3$ $\qquad y = -3u^{-3}$

$\frac{du}{dx} = 12x$ $\qquad \frac{dy}{du} = 9u^{-4}$

$\frac{dy}{dx} = 12x \times 9u^{-4}$

$\frac{dy}{dx} = 108xu^{-4}$

$\frac{dy}{dx} = 108x(6x^2 - 3)^{-4}$

$\frac{dy}{dx} = \frac{108x}{(6x^2-3)^4}$

Method 2 : Extended Power Rule

$$\frac{dy}{dx} = n[u(x)]^{n-1}\frac{du}{dx}$$

$y = \frac{-3}{(6x^2-3)^3}$

$y = -3(6x^2 - 3)^{-3}$

$\frac{dy}{dx} = 9(6x^2 - 3)^{-4}(12x)$

$\frac{dy}{dx} = 108x(6x^2 - 3)^{-4}$

$\frac{dy}{dx} = \frac{108x}{(6x^2-3)^4}$

FORMATIVE PRACTICE 4

Find the derivative of the following functions by using Chain Rule and Extended Power Rule.

1. $y = 3(-4x^2 + 3)^2$ — *Ans:* $\frac{dy}{dx} = 192x^3 - 144x$
2. $y = \sqrt{7x^2 + 3}$ — *Ans:* $\frac{dy}{dx} = 7x(7x^2 + 3)^{-1/2}$
3. $y = -3(2x^2 - 10)^{-3}$ — *Ans:* $\frac{dy}{dx} = 36x(2x^2 - 10)^{-4}$
4. $y = \frac{5}{(-3x^4+3)^4}$ — *Ans:* $\frac{dy}{dx} = \frac{240x^3}{(-3x^4+3)^5}$
5. $y = \frac{-1}{\sqrt[3]{5x^2+3}}$ — *Ans:* $\frac{dy}{dx} = \frac{10x}{3(5x^2+3)^{1/2}}$

COMBINATION OF PRODUCT RULE/ QUOTIENT RULE AND EXTENDED POWER RULE

EXAMPLE

Find $\frac{dy}{dx}$ for the following functions.

1. $y = (2 - x^2)^2(5x + 2)^3$

$u = (2 - x^2)^2$ $\qquad$ $v = (5x + 2)^3$

$\frac{du}{dx} = 2(2 - x^2)^1 . -2x$ $\qquad$ $\frac{dv}{dx} = 3(5x + 2)^2 . 5$

Applying Product Rule

$\frac{dy}{dx} = (2 - x^2)^2 3(5x + 2)^2 . 5 + (5x + 2)^3 2(2 - x^2). -2x$

$\frac{dy}{dx} = 15(2 - x^2)^2(5x + 2)^2 + (-4x)(5x + 2)^3(2 - x^2)$

$\frac{dy}{dx} = (2 - x^2)^1(5x + 2)^2[15(2 - x^2) - 4x(5x + 2)]$

$\frac{dy}{dx} = (2 - x^2)(5x + 2)^2[30 - 15x^2 - 20x^2 - 8x]$

$\frac{dy}{dx} = (2 - x^2)(5x + 2)^2(30 - 35x^2 - 8x)$

2. $y = (4x - 2)^4(3x^2 + 7)^2$

$u = (4x - 2)^4$ $\qquad$ $v = (3x^2 + 7)^2$

$\frac{du}{dx} = 4(4x - 2)^3 \times 4$ $\qquad$ $\frac{dv}{dx} = 2(3x^2 + 7) \times 6x$

$\frac{dy}{dx} = (4x - 2)^4 2(3x^2 + 7).6x + (3x^2 + 7)^2 4(4x - 2)^3 . 4$

$\frac{dy}{dx} = 12x(4x - 2)^4(3x^2 + 7) + 16(3x^2 + 7)^2(4x - 2)^3$

$\frac{dy}{dx} = 4(4x - 2)^3(3x^2 + 7)[3x(4x - 2) + 4(3x^2 + 7)]$

$\frac{dy}{dx} = 4(4x - 2)^3(3x^2 + 7)(12x^2 - 6x + 12x^2 + 28)$

$\frac{dy}{dx} = 4(4x - 2)^3(3x^2 + 7)(24x^2 - 6x + 28)$

3. $y = (2x - 5)^3(3\sqrt[3]{x} - 2)^4$ $\qquad$ $\sqrt[3]{x}$ is written as $x^{\frac{1}{3}}$

$u = (2x - 5)^3$ $\qquad$ $v = (3\sqrt[3]{x} - 2)^4$

$\frac{du}{dx} = 3(2x - 5)^2 . 2$ $\qquad$ $\frac{dv}{dx} = 4(3x^{\frac{1}{3}} - 2)^3 . x^{-\frac{2}{3}}$

$\frac{du}{dx} = 6(2x - 5)^2$

$\frac{dy}{dx} = (2x - 5)^3 4(3x^{\frac{1}{3}} - 2)^3 . x^{-\frac{2}{3}} + (3x^{\frac{1}{3}} - 2)^4 6(2x - 5)^2$

$$\frac{dy}{dx} = 4x^{-\frac{2}{3}}(2x-5)^3(3x^{\frac{1}{3}}-2)^3 + 6(3x^{\frac{1}{3}}-2)^4(2x-5)^2$$

$$\frac{dy}{dx} = 2(2x-5)^2(3x^{\frac{1}{3}}-2)^3\left[2x^{-\frac{2}{3}}(2x-5)^1 + 3(3x^{\frac{1}{3}}-2)^1\right]$$

$$\frac{dy}{dx} = 2(2x-5)^2(3x^{\frac{1}{3}}-2)^3\left(4x^{\frac{1}{3}} - 10x^{-\frac{2}{3}} + 9x^{\frac{1}{3}} - 6\right)$$

$$\frac{dy}{dx} = 2(2x-5)^2(3x^{\frac{1}{3}}-2)^3\left(13x^{\frac{1}{3}} - 10x^{-\frac{2}{3}} - 6\right)$$

4. $y = \frac{(2-x^2)^3}{(x+3)}$

$u = (2-x^2)^3$ $\qquad$ $v = (x+3)$

$\frac{du}{dx} = 3(2-x^2)^2(-2x)$ $\qquad$ $\frac{dv}{dx} = 1$

Applying the Quotient Rule

$$\frac{dy}{dx} = \frac{(x+3)3(2-x^2)^2.(-2x)-(2-x^2)^3(1)}{(x+3)^2}$$

$$\frac{dy}{dx} = \frac{(-6x^2-18x)(2-x^2)^2-(2-x^2)^3}{(x+3)^2}$$

$$\frac{dy}{dx} = \frac{(2-x^2)^2[(-6x^2-18x)-(2-x^2)^1]}{(x+3)^2}$$

$$\frac{dy}{dx} = \frac{(2-x^2)^2(-5x^2-18x-2)}{(x+3)^2}$$

5. $y = \frac{(3x+2)^5}{(1-2x)^3}$

$u = (3x+2)^5$ $\qquad$ $v = (1-2x)^3$

$\frac{du}{dx} = 5(3x+2)^4.(3)$ $\qquad$ $\frac{dv}{dx} = 3(1-2x)^2.(-2)$

$$\frac{dy}{dx} = \frac{(1-2x)^3 5(3x+2)^4.(3)-(3x+2)^5 3(1-2x)^2.(-2)}{((1-2x)^3)^2}$$

$$\frac{dy}{dx} = \frac{15(1-2x)^3(3x+2)^4+6(3x+2)^5(1-2x)^2}{(1-2x)^6}$$

$$\frac{dy}{dx} = \frac{(1-2x)^2(3x+2)^4[15(1-2x)^1+6(3x+2)^1]}{(1-2x)^6}$$

$$\frac{dy}{dx} = \frac{(3x+2)^4[15-30x+18x+12]}{(1-2x)^4}$$

$$\frac{dy}{dx} = \frac{(3x+2)^4[-12x+27]}{(1-2x)^4}$$

6. $y = \frac{\sqrt{3-2x}}{(x-2)^3}$

$\sqrt{3-2x}$ is written as $(3-2x)^{\frac{1}{2}}$

$u = (3-2x)^{\frac{1}{2}}$ $\qquad$ $v = (x-2)^3$

$\frac{du}{dx} = \frac{1}{2}(3-2x)^{-\frac{1}{2}}.(-2)$ $\qquad$ $\frac{dv}{dx} = 3(x-2)^2.(1)$

$$\frac{dy}{dx} = \frac{(x-2)^3\frac{1}{2}(3-2x)^{-\frac{1}{2}}.(-2)-(3-2x)^{\frac{1}{2}}3(x-2)^2.(1)}{((x-2)^3)^2}$$

$$\frac{dy}{dx} = \frac{-(x-2)^3(3-2x)^{-\frac{1}{2}}-3(3-2x)^{\frac{1}{2}}(x-2)^2}{(x-2)^4}$$

$$\frac{dy}{dx} = \frac{(x-2)^2(3-2x)^{-\frac{1}{2}}[-(x-2)^1-3(3-2x)^1]}{(x-2)^4}$$

$$\frac{dy}{dx} = \frac{(3-2x)^{-\frac{1}{2}}[-x+2-9+6x]}{(x-2)^2}$$

$$\frac{dy}{dx} = \frac{5x-7}{\sqrt{3-2x}(x-2)^2}$$

FORMATIVE PRACTICE 5

Find the derivative of the following functions.

1. $y = (x-4)(x-3)^{\frac{1}{2}}$
2. $y = (5-x)^6\sqrt{1-5x^2}$
3. $y = \frac{(3x+2)^2}{1-3x}$
4. $y = \frac{(5x-7)^2}{(-2x^2+3)^4}$
5. $y = \frac{4}{(x^2-2x)^3}$

ANSWER

1. $\frac{dy}{dx} = \frac{3x-10}{2(x-3)^{1/2}}$
2. $\frac{dy}{dx} = \frac{(-6+30x^2)(5-x)^5-5x(5-x)^6}{\sqrt{1-5x^2}}$
3. $\frac{dy}{dx} = \frac{9x^2+12x+4}{1-3x}$
4. $\frac{dy}{dx} = \frac{25x^2-70x+49}{(-2x^2+3)^4}$
5. $\frac{dy}{dx} = \frac{24-24x}{(x^2-2x)^4}$

2.0 USE TRIGONOMETRIC, LOGARITHMIC AND EXPONENTIAL FUNCTIONS

2.1 EXPLAIN DIFFERENCE BETWEEN LAWS OF DIFFERENTIATION

2.2 USE VARIOUS DIFFERENTIATION TECHNIQUES TO SOLVE PROBLEMS

Derivative of a Trigonometric Function

$$\frac{d}{dx}(\sin x) = \cos x$$

$$\frac{d}{dx}(cos\ x) = -sin\ x$$

$$\frac{d}{dx}(tan\ x) = sec^2 x$$

$$\frac{d}{dx}[\sin(ax+b)] = \cos(ax+b) \times \frac{d}{dx}(ax+b)$$

$$\frac{d}{dx}[cos(ax+b)] = -sin(ax+b) \times \frac{d}{dx}(ax+b)$$

$$\frac{d}{dx}[tan(ax+b)] = sec^2(ax+b) \times \frac{d}{dx}(\mathrm{ax+b})$$

EXAMPLE

Differentiate the following functions :

1. $y = sin\ 2x$

$\frac{dy}{dx} = \cos 2x \ \times 2$ — Using Extended Power Rule

$\frac{dy}{dx} = 2\cos 2x$

Using Chain Rule

$u = 2x$ $\qquad$ $y = sin\ u$

$\frac{du}{dx} = 2$ $\qquad$ $\frac{dy}{du} = \cos u$

$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$

$\frac{dy}{dx} = \cos u \times 2$

$\frac{dy}{dx} = 2\cos u = 2\cos 2x$

2. $y = cos\ (\frac{2}{3}x - 5)$

Using Extended Power Rule

$\frac{dy}{dx} = -\sin(\frac{2}{3}x - 5)\ \times \frac{2}{3}$

$\frac{dy}{dx} = -\frac{2}{3}\sin(\frac{2}{3}x - 5)$

Using Chain Rule

$u = \frac{2}{3}x - 5$ $\qquad$ $y = cos\ u$

$\frac{du}{dx} = \frac{2}{3}$ $\qquad$ $\frac{dy}{du} = -\sin u$

$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$

$\frac{dy}{dx} = -\sin u \times \frac{2}{3}$

$\frac{dy}{dx} = -\frac{2}{3}\sin u$

$\frac{dy}{dx} = -\frac{2}{3}\sin(\frac{2}{3}x - 5)$

3. $y = tan\ (-3x^2 + 2x - 3)$

$\frac{dy}{dx} = sec^2(-3x^2 + 2x - 3) \times (-6x + 2)$

$\frac{dy}{dx} = (-6x + 2)\ sec^2(-3x^2 + 2x - 3)$

Using Chain Rule

$u = -3x^2 + 2x - 3$ $\qquad$ $y = tan\ u$

$\frac{du}{dx} = -6x + 2$ $\qquad$ $\frac{dy}{du} = \sec^2 u$

$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$

$\frac{dy}{dx} = \sec^2 u \times (-6x + 2)$

$\frac{dy}{dx} = (-6x + 2)\sec^2 u$

$\frac{dy}{dx} = (-6x + 2)\sec^2(-3x^2 + 2x - 3)$

Derivative of a Trigonometric Function (Involving Power)

$$\frac{d}{dx}[sin^n u] = nsin^{n-1} u \times \cos u \times \frac{du}{dx}$$

$$\frac{d}{dx}[cos^n u] = ncos^{n-1} u \times -\sin u \times \frac{du}{dx}$$

$$\frac{d}{dx}[tan^n u] = ntan^{n-1} u \times sec^2 u \times \frac{du}{dx}$$

$$\frac{d}{dx}[\cot x] = -cosec^2 x$$

$$\frac{d}{dx}[sec\, x] = \sec x\, tan\, x$$

$$\frac{d}{dx}[cosec\, x] = -cosec\, x \cot x$$

EXAMPLE

Differentiate the following functions:

1. $y = sin^3\, 2x$

Using Extended Power Rule

$$\frac{dy}{dx} = 3\sin^2 2x \times \cos 2x \times 2$$

$$\frac{dy}{dx} = 6\sin^2 2x \cos 2x$$

Using Chain Rule

$u = \sin 2x$ $\qquad y = u^3$

$\frac{du}{dx} = 2\cos 2x$ $\qquad \frac{dy}{du} = 3u^2$

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$$

$$\frac{dy}{dx} = 3u^2 \times 2\cos 2x$$

$$\frac{dy}{dx} = 3sin^2\, 2x \times 2\cos 2x$$

$$\frac{dy}{dx} = 6sin^2\, 2x \cos 2x$$

2. $y = cos^2\, (3x^2 - 4)$

Using Extended Power Rule

$$\frac{dy}{dx} = 2\cos(3x^2 - 4) \times -\sin\, (3x^2 - 4) \times 6x$$

$$\frac{dy}{dx} = -12x\, cos\, (3x^2 - 4) \sin\, (3x^2 - 4)$$

Using Chain Rule

$u = cos\, (3x^2 - 4)$ $\qquad y = u^2$

$\frac{du}{dx} = -\sin(3x^2 - 4) \times 6x$ $\qquad \frac{dy}{du} = 2u$

$$\frac{du}{dx} = -6x \sin(3x^2 - 4)$$

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$$

$$\frac{dy}{dx} = 2u \times -6x \sin(3x^2 - 4)$$

$$\frac{dy}{dx} = 2cos\, (3x^2 - 4) \times -6x \sin(3x^2 - 4)$$

$$\frac{dy}{dx} = -12x\, cos\, (3x^2 - 4) \sin(3x^2 - 4)$$

FORMATIVE PRACTICE 6

Differentiate the following functions using suitable method:

1. $y = \cos(x + \pi)$
2. $y = \sin(\frac{1}{2}x^2 - 3)$
3. $y = 2sin^3(5x^2 - 2x)$
4. $y = cos^2 3x$
5. $y = x\, tan^3\, (x^2 - 1)$

ANSWER

1. $\frac{dy}{dx} = -\sin(x + \pi)$
2. $\frac{dy}{dx} = x\cos(\frac{1}{2}x^2 - 3)$
3. $\frac{dy}{dx} = 6\sin(5x^2 - 2x)^2 cos(5x^2 - 2x)(10x - 2)$
4. $\frac{dy}{dx} = -6\cos 3x sin\, 3x$
5. $\frac{dy}{dx} = 6x^2 tan^2(x^2 - 1) sec^2(x^2 - 1) + tan^3(x^2 - 1)$

Derivative of Logarithm Function

$$\frac{d}{dx}(\ln|x|) = \frac{1}{x}$$

$$\frac{d}{dx}(\ln|ax+b|) = \frac{1}{ax+b} \times \frac{d}{dx}(ax+b)$$

EXAMPLE

Differentiate the following functions:

1. $y = \ln(2x^3 + x)$

Using Differentiation Rule

$$\frac{dy}{dx} = \frac{1}{2x^3+x} \times (6x^2+1)$$

$$\frac{dy}{dx} = \frac{6x^2+1}{2x^3+x}$$

Using Chain Rule

$u = 2x^3 + x$ $\quad$ $y = \ln u$

$\frac{du}{dx} = 6x^2 + 1$ $\quad$ $\frac{dy}{du} = \frac{1}{u}$

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$$

$$\frac{dy}{dx} = \frac{1}{u} \times (6x^2+1)$$

$$\frac{dy}{dx} = \frac{1}{2x^3+x}(6x^2+1)$$

2. $y = \ln(-3x+7)^3$

$y = 3\ln(-3x+7)$ *Applying Law of Logarithms*

$$\frac{dy}{dx} = 3.\frac{1}{-3x+7} \times (-3)$$

$$\frac{dy}{dx} = \frac{-9}{-3x+7}$$

Using Chain Rule

$y = 3\ln(-3x+7)$ *Applying Law of Logarithms*

$u = -3x + 7$ $\quad$ $y = 3\ln u$

$\frac{du}{dx} = -3$ $\quad$ $\frac{dy}{du} = \frac{3}{u}$

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$$

$$\frac{dy}{dx} = \frac{3}{u} \times (-3)$$

$$\frac{dy}{dx} = \frac{-9}{-3x+7}$$

3. $y = 2x^3 \ln x$

Applying Product Rule

$u = 2x^3$ $\quad$ $v = \ln x$

$\frac{du}{dx} = 6x^2$ $\quad$ $\frac{dv}{dx} = \frac{1}{x}$

$$\frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$$

$$\frac{dy}{dx} = 2x^3\left(\frac{1}{x}\right) + \ln x\,(6x^2)$$

$$\frac{dy}{dx} = 2x^2 + 6x^2 \ln x$$

4. $y = \frac{5\ln x}{x^3+2}$

Applying Quotient Rule

$u = 5\ln x$ $\quad$ $v = x^3 + 2$

$\frac{du}{dx} = \frac{5}{x}$ $\quad$ $\frac{dv}{dx} = 3x^2$

$$\frac{d}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$$

$$\frac{dy}{dx} = \frac{(x^3+2)(\frac{5}{x}) - 5\ln x(3x^2)}{(x^3+2)^2}$$

$$\frac{dy}{dx} = \frac{(x^3+2)(\frac{5}{x}) - 5\ln x(3x^2)}{(x^3+2)^2}$$

$$\frac{dy}{dx} = \frac{5x^2 + \frac{10}{x} - 15x^2\ln x}{(x^3+2)^2}$$

$$\frac{dy}{dx} = \frac{5x^3 + 10 - 15x^3\ln x}{x(x^3+2)^2}$$

FORMATIVE PRACTICE 7

Differentiate the following functions :

1. $y = \ln(x)^2$ $\quad$ *Ans*: $\frac{dy}{dx} = \frac{2}{x}$
2. $y = \ln(\frac{x}{2})$ $\quad$ *Ans*: $\frac{dy}{dx} = \frac{1}{x}$
3. $y = \frac{5-2\ln x}{x^2+3}$ $\quad$ *Ans*: $\frac{dy}{dx} = \frac{-12x^2-6+4x^2\ln x}{x^5+6x^3+9x}$
4. $y = \sqrt{x} - \ln(3x^3-1)$ $\quad$ *Ans*: $\frac{dy}{dx} = \frac{1}{2\sqrt{x}} - \frac{6x}{3x^3-1}$
5. $y = x^2\ln(\frac{x}{2})$ $\quad$ *Ans*: $\frac{dy}{dx} = x + 2x\ln\left(\frac{x}{2}\right)$

Derivative of Exponential Function

$$\frac{d}{dx}(e^x) = e^x$$

$$\frac{d}{dx}\left(e^{ax+b}\right) = e^{ax+b} \times \frac{d}{dx}(ax+b)$$

EXAMPLE

Differentiate the following functions with respect to x:

1. $y = 2e^{-3x+5}$

$$\frac{dy}{dx} = 2e^{-3x+5} \times -3$$

$$\frac{dy}{dx} = -6e^{-3x+5}$$

Using Chain Rule

$u = -3x + 5$ $\qquad$ $y = 2e^u$

$\frac{du}{dx} = -3$ $\qquad$ $\frac{dy}{du} = 2e^u$

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$$

$$\frac{dy}{dx} = 2e^u \times -3$$

$$\frac{dy}{dx} = -6e^{-3x+5}$$

2. $y = 7xe^{x^3}$

Applying Product Rule

$u = 7x$ $\qquad$ $v = e^{x^3}$

$\frac{du}{dx} = 7$ $\qquad$ $\frac{dv}{dx} = e^{x^3} \times 3x^2$

$\frac{dv}{dx} = 3x^2e^{x^3}$

$$\frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$$

$$\frac{dy}{dx} = 7x3x^2e^{x^3} + e^{x^3}(7)$$

$$\frac{dy}{dx} = 21x^3e^{x^3} + 7e^{x^3}$$

$$\frac{dy}{dx} = 7e^{x^3}(3x^3 + 1)$$

3. $y = \frac{\sqrt{x}}{e^{x+2}}$

Applying Quotient Rule

$u = x^{\frac{1}{2}}$ $\qquad$ $v = e^{x+2}$

$\frac{du}{dx} = \frac{1}{2}x^{-\frac{1}{2}}$ $\qquad$ $\frac{dv}{dx} = e^{x+2} \times 1$

$\frac{dv}{dx} = e^{x+2}$

$$\frac{d}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$$

$$\frac{dy}{dx} = \frac{(e^{x+2})(\frac{1}{2}x^{-\frac{1}{2}}) - x^{\frac{1}{2}}(e^{x+2})}{(e^{x+2})^2}$$

$$\frac{dy}{dx} = \frac{(e^{x+2})\left[(\frac{1}{2}x^{-\frac{1}{2}}) - x^{\frac{1}{2}}\right]}{(e^{x+2})^2}$$

$$\frac{dy}{dx} = \frac{\frac{1}{2}x^{-\frac{1}{2}} - x^{\frac{1}{2}}}{e^{x+2}}$$

$$\frac{dy}{dx} = \frac{\frac{1-2x}{2\sqrt{x}}}{e^{x+2}}$$

$$\frac{dy}{dx} = \frac{1-2x}{2\sqrt{x}(e^{x+2})}$$

FORMATIVE PRACTICE 8

Differentiate the following functions:

1. $y = -e^{10x+5}$ $\qquad$ *Ans*: $\frac{dy}{dx} = -10e^{10x+5}$
2. $y = \frac{3e^{2x}}{3x^3+2}$ $\qquad$ *Ans*: $\frac{dy}{dx} = \frac{6e^{3x}(3x^3+2) - 27x^2e^{2x}}{(3x^3+2)^2}$
3. $y = e^{-3x}(e^{2x} + 6)$ $\qquad$ *Ans*: $\frac{dy}{dx} = -e^{-x} - 18e^{-3x}$
4. $y = (1 - e^{2x})^2$ $\qquad$ *Ans*: $\frac{dy}{dx} = -4e^{2x} + 4e^{4x}$
5. $y = e^x(1 - e^x)$ $\qquad$ *Ans*: $\frac{dy}{dx} = e^x - 2e^{2x}$

3.0 APPLY SECOND ORDER DIFFERENTIATION

3.1 EXECUTE SECOND ORDER DIFFERENTIATION BY USING VARIOUS DIFFERENTIATION TECHNIQUES FOR VARIOUS FUNCTIONS

> Second order differentiation is the derivative of $\frac{dy}{dx}$
>
> It is denoted by $\frac{d^2y}{dx^2}$ or $f"(x)$

EXAMPLE

Find the second derivative for each of the following functions.

1. $y = -2x^3 + 3x^2 - 4x$

$$\frac{dy}{dx} = -6x^2 + 6x - 4$$

$$\frac{d^2y}{dx^2} = -12x + 6$$

2. $y = \sqrt{3 - 2x}$

$$y = (3 - 2x)^{\frac{1}{2}}$$

$$\frac{dy}{dx} = \frac{1}{2}(3 - 2x)^{-\frac{1}{2}} \times (-2)$$

$$\frac{dy}{dx} = -1(3 - 2x)^{-\frac{1}{2}}$$

$$\frac{d^2y}{dx^2} = \frac{1}{2}(3 - 2x)^{-\frac{3}{2}} \times (-2)$$

$$\frac{d^2y}{dx^2} = -(3 - 2x)^{-\frac{3}{2}}$$

3. $y = \frac{3x+1}{5-3x}$

$u = 3x + 1 \qquad v = 5 - 3x$

$\frac{du}{dx} = 3 \qquad \frac{dv}{dx} = -3$

$$\frac{dy}{dx} = \frac{(5-3x)(3)-(3x+1)(-3)}{(5-3x)^2}$$

$$\frac{dy}{dx} = \frac{15-9x+9x+3}{(5-3x)^2}$$

$$\frac{dy}{dx} = \frac{18}{(5-3x)^2}$$

$$\frac{dy}{dx} = 18(5 - 3x)^{-2}$$

$$\frac{d^2y}{dx^2} = -36(5 - 3x)^{-3}(-3)$$

$$\frac{d^2y}{dx^2} = 108(5 - 3x)^{-3}$$

$$\frac{d^2y}{dx^2} = \frac{108}{(5-3x)^3}$$

Given $f(x) = x^4 + 5x^3 - 6x$, find :

1. $f"(2)$

$$f(x) = x^4 + 5x^3 - 6x$$

$$f'(x) = 4x^3 + 15x^2 - 6$$

$$f"(x) = 12x^2 + 30x$$

$$f"(2) = 12(2)^2 + 30(2)$$

$$f"(2) = 108$$

2. $f"(-2)$

$$f(x) = x^4 + 5x^3 - 6x$$

$$f'(x) = 4x^3 + 15x^2 - 6$$

$$f"(x) = 12x^2 + 30x$$

$$f"(2) = 12(-2)^2 + 30(-2)$$

$$f"(2) = -12$$

FORMATIVE PRACTICE 9

Find the second derivative for each of the following functions.

1. $y = \sin 3x - 4e^{2x}$ *Ans:* $\frac{d^2y}{dx^2} = -9\sin 3x - 16e^{2x}$
2. $y = (4 - 3x)^2$ *Ans:* $\frac{d^2y}{dx^2} = 18$
3. $y = (7x^2 - x)(3 + 5x)^2$ *Ans:* $\frac{d^2y}{dx^2} = 2100x^2 + 1110x + 66$
4. $y = \frac{5}{2(6-3x)^2}$ *Ans:* $\frac{d^2y}{dx^2} = \frac{135}{(6-3x)^4}$

Find the second derivative for $p = (3q + 6)^3$ if $q = 1$ *Ans:* $\frac{d^2y}{dx^2} = 486$

4.0 DEMONSTRATE THE APPLICATION OF DIFFERENTIATION

4.1 USE DIFFERENTIATION TO FIND THE GRADIENT OF A CURVE

The gradient at a point on a curve

- $f'(x) = \frac{dy}{dx}$ is the gradient function of $y = f(x)$
- To find the gradient at a particular point on the curve $y = f(x)$, substitute the x-coordinate of that point into the derivative.

EXAMPLE

1. Find the gradient of the curve $y = x^2 - 5x - 7$ at the point on the curve whose x-coordinate is -2.

Solution

$y = x^2 - 5x - 7$

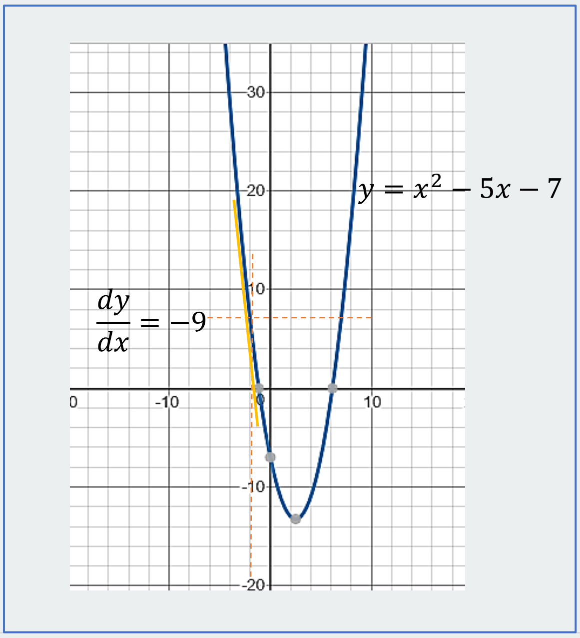
$\frac{dy}{dx} = 2x - 5$

Substitute $x = -2$ into $\frac{dy}{dx}$

$\frac{dy}{dx} = 2(-2) - 5$

$\frac{dy}{dx} = -9$

At point $(-2,7)$ the gradient is -9



2. Find the gradient of the curve $y = 3x^2 + 8x + 4$ at the point $(2,32)$

Solution

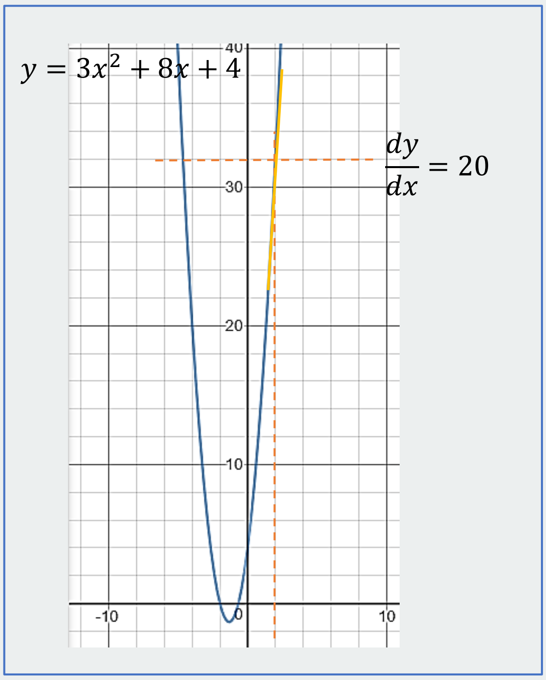
$y = 3x^2 + 8x + 4$

$\frac{dy}{dx} = 6x + 8$

Substitute $x = 2$ into $\frac{dy}{dx}$

$\frac{dy}{dx} = 6(2) + 8$

$\frac{dy}{dx} = 20$ at $(2,32)$



FORMATIVE PRACTICE 10

1. Find the gradient of the curve $= x^3 + 5x^2 + 5x + 5$ at the point on the curve whose x-coordinate is -5.
2. Find the gradient of the curve $y = -2x^3 + x^2 + 3$ at the point $(1,2)$

ANSWER

1. $\frac{dy}{dx} = 30$
2. $\frac{dy}{dx} = -4$

4.1 EXECUTE TURNING POINTS/STATIONARY POINTS

- Turning points / stationary points are the points on a function where its derivative is equal to zero, $\frac{dy}{dx} = 0$

  (Tangent to the curve is horizontal = no gradient)

- Types of stationary points

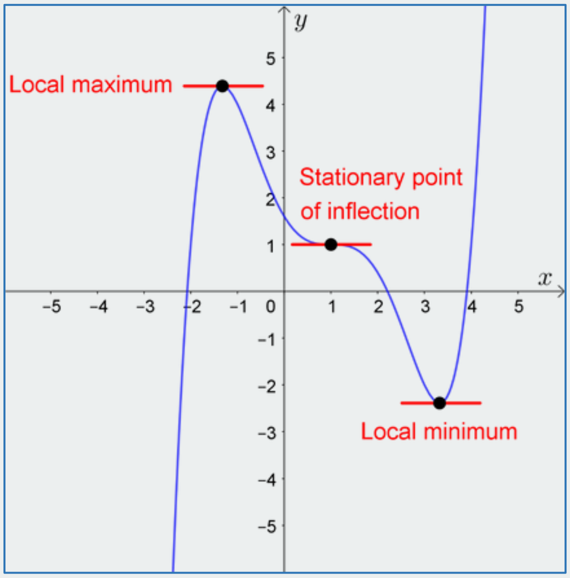
  There are three types of stationary points:

  I. Local maximums

  II. Local minimums

  III. Stationary points of inflection.

Each of these are illustrated on the graph shown.



Step to determine the coordinate of stationary point $y = f(x)$

- 1. Differentiate the function, $\frac{dy}{dx}$
- 2. Set the derivative equal to zero, $\frac{dy}{dx} = 0$
- 3. Solve for x
- 4. Substitute x back in to find y

EXAMPLE

1. Find the stationary point of $y = x^2 - 2x + 5$.

   Solution

   $y = x^2 - 2x + 5$

   $\frac{dy}{dx} = 2x - 2$

$\frac{dy}{dx} = 2x - 2 = 0$

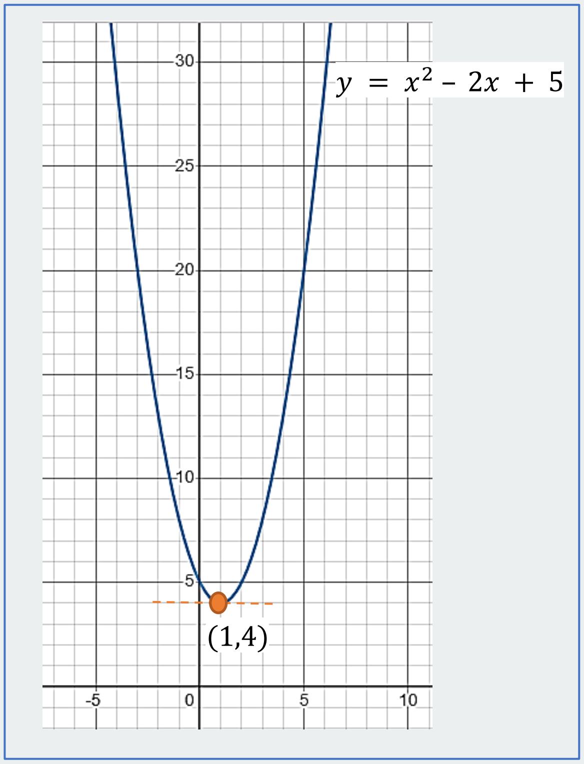
$2x = 2$

   $x = 1$ into y

$y = (1)^2 - 2(1) + 5$

$y = 4$

   ∴Stationary point $(1,4)$



2. Find the stationary points of $y = x^3 + 9x^2 + 15x + 3$.

   Solution

   $y = x^3 + 9x^2 + 15x + 3$

   $\frac{dy}{dx} = 3x^2 + 18x + 15$

$$\frac{dy}{dx} = 3x^2 + 18x + 15 = 0$$

$$(x+1)(x+5) = 0$$

$$x_1 = -1, \qquad x_2 = -5$$

Stationary point 1

$x_1 = -1$ into $y_1 = x^3 + 9x^2 + 15x + 3$

$$y_1 = (-1)^3 + 9(-1)^2 + 15(-1) + 3$$

$$y_1 = -4$$

$$SP_1 = (-1, -4)$$

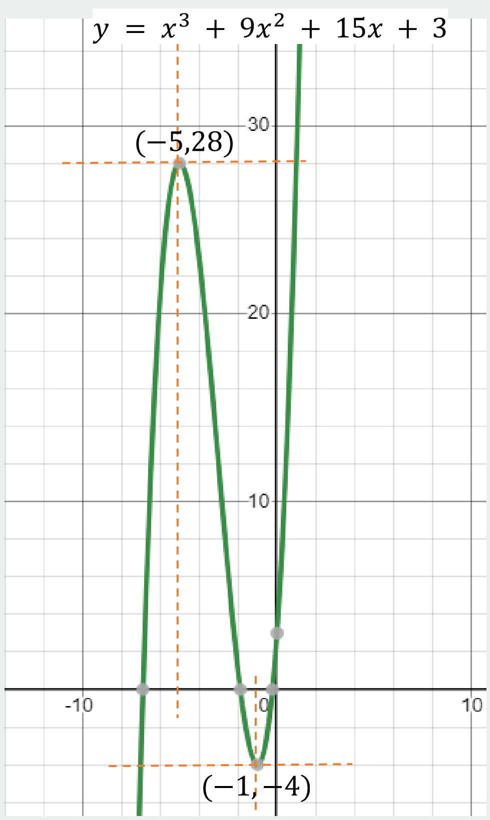
Stationary point 2

$x_2 = -5$ into $y_2 = x^3 + 9x^2 + 15x + 3$

$$y_2 = (-5)^3 + 9(-5)^2 + 15(-5) + 3$$

$$y_2 = 28$$

$$SP_2 = (-5, 28)$$



FORMATIVE PRACTICE 11

1. Find the stationary point on the curve,

   $y = 4x - x^2$

2. Find the stationary points of

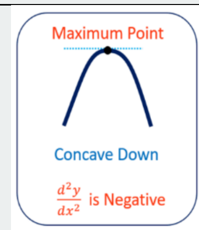
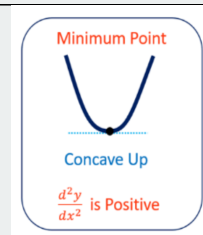
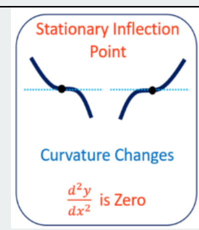
   $y = x^3 + 12x^2 + 21x - 5$

ANSWER

1. (2, 4)
2. $(-1, -15)$ and $(-7, 93)$

4.2 DESCRIBE MAXIMUM, MINIMUM AND POINT OF INFLECTION

- The nature of any stationary point can be determined by substituting the x coordinate of the stationary point into the second derivative of the function, $\frac{d^2y}{dx^2}$.
- Nature of stationary point:

Maximum Point (∩-shaped)	$\frac{d^2y}{dx^2} < 0$	
Minimum Point (∪-shaped)	$\frac{d^2y}{dx^2} > 0$	
Stationary Inflections Point, (curvature changes)	$\frac{d^2y}{dx^2} = 0$	

EXAMPLE

1. Find the stationary points on the curve $y = x^3 + 7x^2 - 5x - 4$ (including point of inflection) and determine their nature.

Solution

$y = x^3 + 7x^2 - 5x - 4$

$\frac{dy}{dx} = 3x^2 + 14x - 5$

$3x^2 + 14x - 5 = 0$

$x_1 = \frac{1}{3},\ x_2 = -5$

$\frac{d^2y}{dx^2} = 6x + 14$

When $x_1 = \frac{1}{3}$

$\frac{d^2y}{dx^2} = 6\left(\frac{1}{3}\right) + 14$

$\frac{d^2y}{dx^2} = 16 > 0$ (Minimum point)

$y_1 = \left(\frac{1}{3}\right)^3 + 7\left(\frac{1}{3}\right)^2 - 5\left(\frac{1}{3}\right) - 4 = -4\frac{23}{27}$

Thus $(\frac{1}{3}, -4\frac{23}{27})$is a minimum point

When $x_2 = -5$

$\frac{d^2y}{dx^2} = 6(-5) + 14$

$\frac{d^2y}{dx^2} = -16 < 0$ (Maximum point)

$y_2 = (-5)^3 + 7(-5)^2 - 5(-5) - 4 = 71$

Thus $(-5,71)$is a maximum point.

When $\frac{d^2y}{dx^2} = 0,\ 6x + 14 = 0$

$x_i = -\frac{7}{3}$

$y_i = \left(-\frac{7}{3}\right)^3 + 7\left(-\frac{7}{3}\right)^2 - 5\left(-\frac{7}{3}\right) - 4$

$y_i = 33\frac{2}{27}$

Thus $(-\frac{7}{3}, 33\frac{2}{27})$is a point of inflection.

2. Determine the nature of the stationary points for the function $y = 6 + 3x^2 - x^3$

Solution

$y = 6 + 3x^2 - x^3$

$\frac{dy}{dx} = 6x - 3x^2$

$6x - 3x^2 = 0$

$3x(2 - x) = 0$

$x_1 = 0,\ x_2 = 2$

$\frac{d^2y}{dx^2} = 6 - 6x$

When $x_1 = 0$

$\frac{d^2y}{dx^2} = 6 - 6(0)$

$\frac{d^2y}{dx^2} = 6 > 0$ (Minimum point)

$y_1 = 6 + 3(0)^2 - (0)^3$

$y_1 = 6$

Thus $(0,6)$ is a minimum point

When $x_2 = 2$

$\frac{d^2y}{dx^2} = 6 - 6(2)$

$\frac{d^2y}{dx^2} = -6 < 0$ (Maximum point)

$y_2 = 6 + 3(2)^2 - (2)^3$

$y_2 = 10$

Thus $(2,10)$ is a maximum point

FORMATIVE PRACTICE 12

1. Find the stationary points on the curve $y = 4x^3 - 3x^2 - 6x$ and determine the nature of the points:
2. Determine the nature of the stationary points for the function $y = 4x^3 + 3x^2 - 6x - 1$

ANSWER

1. (−0.5, 1.75) is a maximum point

   (1, −5) is a minimum point

2. $\left(\frac{1}{2}, -2\frac{3}{4}\right)$ is a minimum point

   $(-1,4)$ is a maximum point

4.3 SHOW THE GRAPH OF A CURVE

Curve sketching, the steps

I. Locate the stationary points using the criterion that at a stationary point

II. Determine the character of each stationary point using the criteria

III. Sketch the graph by marking on the stationary points first and their character. Between stationary points a function must be either always increasing or always decreasing.

EXAMPLE

1. A curve whose equation is $y = 2x^3 - 3x^2 - 12x + 4$ has stationary points at (−1, 11) and (2, −16). Determine the nature of these points. hence sketch the curve.

   Solution

   Nature of stationary point;

   $y = 2x^3 - 3x^2 - 12x + 4$

   $\frac{dy}{dx} = 6x^2 - 6x - 12$

   $\frac{d^2y}{dx^2} = 12x - 6$

   At $x = -1$

   $\frac{d^2y}{dx^2} = 12(-1) - 6$

   $\frac{d^2y}{dx^2} = -18 < 0$ (Maximum point)

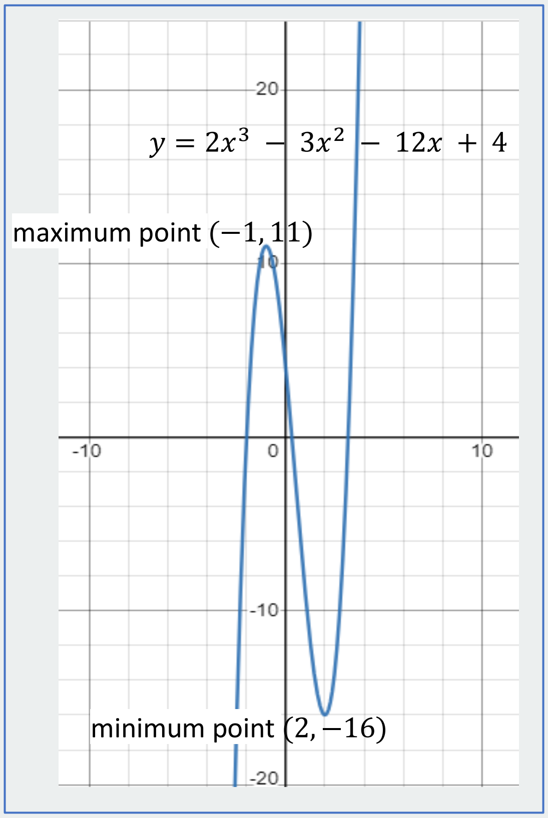
   Thus (−1, 11) is a maximum point

   At $x = 2$

   $\frac{d^2y}{dx^2} = 12(2) - 6$

   $\frac{d^2y}{dx^2} = 18 > 0$ (Minimum point)

   Thus (2, −16) is a minimum point



2. Given $y = 3x^4 - 4x^3$, find if exist the point of inflection, the maximum and the minimum point. Hence sketch the graph.

   Solution

   Find the stationary point

   $y = 3x^4 - 4x^3$

   $\frac{dy}{dx} = 12x^3 - 12x^2$

   $12x^3 - 12x^2 = 0$

   $x^2(x - 1) = 0$

   $x = 0, x = 1$

   $\frac{d^2y}{dx^2} = 36x^2 - 24x$

   When $x = 0$

$$\frac{d^2y}{dx^2} = 36(0)^2 - 24(0)$$

$\frac{d^2y}{dx^2} = 0$ is a point of inflection

$y = 3(0)^4 - 4(0)^3$

$y = 0$

Thus $(0,0)$ is a point of inflection.

When $x = 1$

$$\frac{d^2y}{dx^2} = 36(1)^2 - 24(1)$$

$\frac{d^2y}{dx^2} = 12 > 0$ (Minimum point)

$y = 3(1)^4 - 4(1)^3$

$y = -1$

Thus $(1, -1)$ is a minimum point 

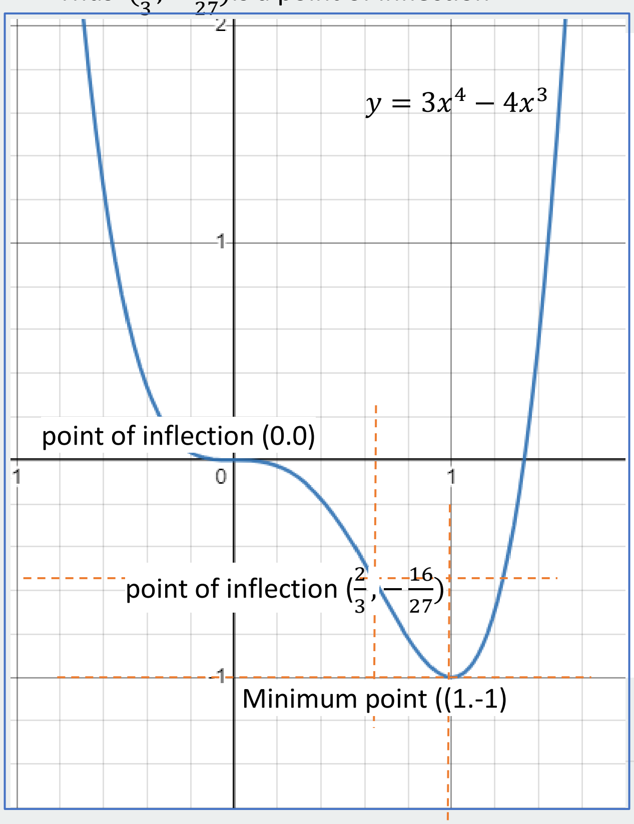
When $\frac{d^2y}{dx^2} = 0, \quad 36x^2 - 24x = 0$

$$x = \frac{2}{3}$$

$$y = 3\left(\frac{2}{3}\right)^4 - 4\left(\frac{2}{3}\right)^3$$

$$y = -\frac{16}{27}$$

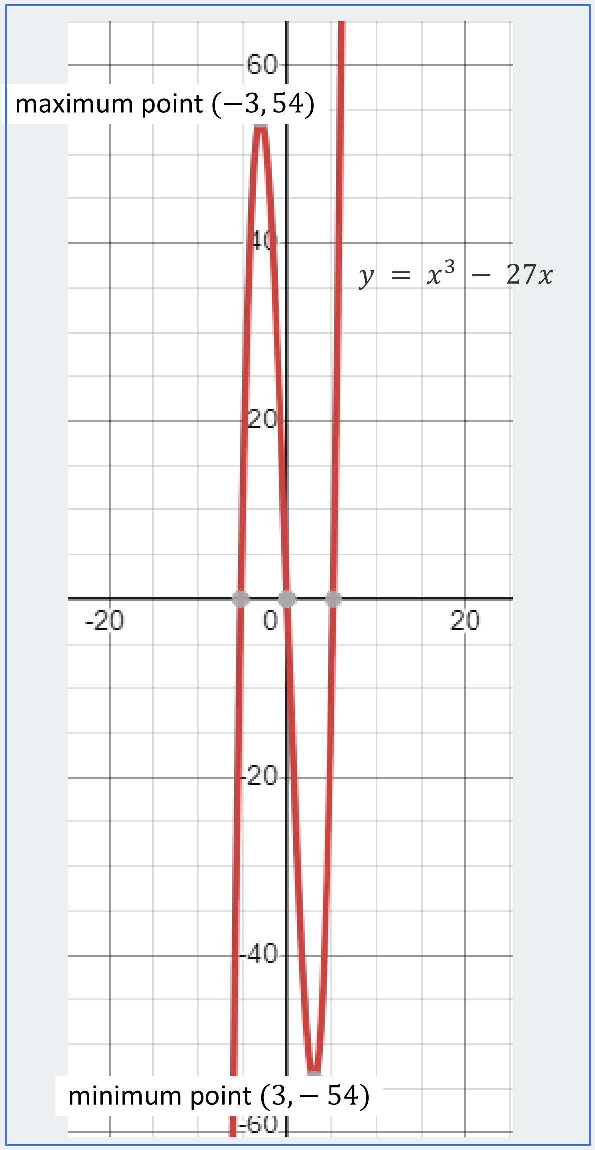
Thus $\left(\frac{2}{3}, -\frac{16}{27}\right)$ is a point of inflection



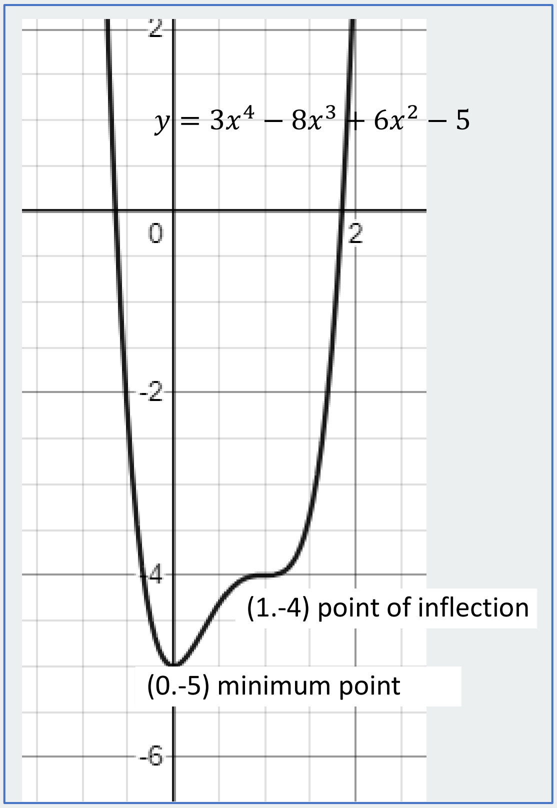
1. Locate the stationary points $(-3,54)$ and $(3,-54)$ on the curve $y = x^3 - 27x$ and determine the nature of each one. Hence sketch the graph
2. Find the stationary values of $y = 3x^4 - 8x^3 + 6x^2 - 5$. and determine the nature of the points and hence sketch the graph

ANSWER

1.



2.



4.4 SOLVE RATE OF CHANGE

The rate of change of a quantity refers to the rate of increase or decrease of the quantity with respect to time, t.

$$\frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx}$$

$\frac{dy}{dt}$ *and* $\frac{dt}{dx}$ are the rates of change of y and x

1. Identify the information given. State the symbol to represent the information.

⬇

2. Determine the suitable formula or equation that relates all the variables

⬇

3. Find the derivative of the equation

⬇

4. Solve the problem by substituting the given information

EXAMPLE

1. The surface area of a spherical ball bearing increases at a rate of $15.5cm^2s^{-1}$ when heated.

(a) Find the rate of change in the radius of the ball bearing when its radius is 3.5 cm.

$\frac{dA}{dt} = 15.5cm^2s^{-1}$ $\quad r = 3.5cm$ $\quad \frac{dr}{dt} =?$

$A = 4\pi r^2$ $\quad \frac{dA}{dr} = 8\pi r$

$$\frac{dA}{dt} = \frac{dA}{dr} \times \frac{dr}{dt}$$

$$15.5 = 8\pi r \times \frac{dr}{dt}$$

$$\frac{dr}{dt} = \frac{15.5}{8\pi r}$$

$$\frac{dr}{dt} = \frac{15.5}{8\pi(3.5)}$$

$$\frac{dr}{dt} = 0.176cms^{-1}$$

(b) Find the corresponding rate of change in the volume of the ball bearing.

$$\frac{dV}{dt} =?$$

$V = \frac{4}{3}\pi r^3$ $\quad \frac{dV}{dr} = 4\pi r^2$

$$\frac{dV}{dt} = \frac{dV}{dr} \times \frac{dr}{dt}$$

$$\frac{dV}{dt} = 4\pi(3.5)^2 \times 0.176$$

$$\frac{dV}{dt} = 8.624\pi cm^3s^{-1}$$

2. The area of a circle is increasing at a uniform rate of $10cm^2s^{-1}$. Calculate the rate of increase of the radius when the circumference of the circle is $110cm$.

$\frac{dA}{dt} = 10cm^2s^{-1}$ $\quad c = 2\pi r = 110$

$$\frac{dr}{dt} =?$$

$A = \pi r^2$

$\frac{dA}{dr} = 2\pi r$

$2\pi r = 110$

$r = \frac{110}{2\pi} = \frac{55}{\pi}$

$\frac{dA}{dt} = \frac{dA}{dr} \times \frac{dr}{dt}$

$10 = \ 2\pi r \times \frac{dr}{dt}$

$\frac{dr}{dt} \ = \ \frac{10}{2\pi r}$

$\frac{dr}{dt} \ = \ \frac{10}{2\pi(\frac{55}{\pi})} = \frac{1}{11} cms^{-1}$

3. A spherical ball is being inflated at a rate of $20cm^3s^{-1}$. Find the rate of increase of its radius when its surface area is $100\pi cm^2$.

$\frac{dV}{dt} = 20cm^3s^{-1}$ $\quad \frac{dr}{dt} = ?$

$A = 100\pi cm^2$

$V = \frac{4}{3}\pi r^3$

$\frac{dV}{dr} = 4\pi r^2$

$4\pi r^2 = 100\pi$

$r^2 = \frac{100\pi}{4\pi}$

$r = 5$

FORMATIVE PRACTICE 13

1. A spherical balloon is inflated at a rate of $3cm^3/s$. Find the increment rate of the radius when the radius is 2cm and 4cm.

Answer: $\frac{dr}{dt} = 0.0597 \frac{cm}{s}$

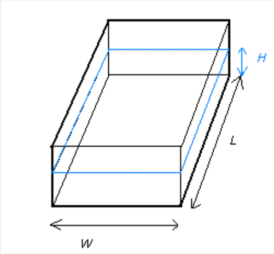
$\frac{dr}{dt} = 0.0149 \ cm/s$

2. The length of sides of a cube increases at the rate of $2 \ cms^{-1}$. When the side of the cube is $5 \ cm$, find the rate of change of its :

   a) Volume

   b) Surface area

3. A rectangular water tank (see figure below) is being filled at the constant rate of 20 liters / second. The base of the tank has dimensions w = 1 meter and L = 2 meters. What is the rate of change of the height of water in the tank?(express the answer in cm / sec).



5.0 SOLVE PARAMETRIC EQUATION

5.1 USE THE CHAIN RULE TO FIND DERIVATIVE OF PARAMETRIC EQUATION FOR ALGEBRAIC PROBLEM

- Parametric equation is an equation where the variables (usually x and y) are expressed in terms of a third parameter, usually expressed as t.

  For example:

  $x = 2t + 5 \quad y = 3t^2$
- t is known as the parameter $x(t)$ and $y(t)$ are known as parametric equations
- $\therefore \frac{dy}{dx}$ can be found using the chain rule:

$$\boldsymbol{\frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx}}$$

EXAMPLE

Find $\frac{dy}{dx}$ in terms of the parameter for each of the following:

1. $x = t^4 \quad , \quad y = t^3 - 2$

$\frac{dx}{dt} = 4t^3$ and $\frac{dy}{dt} = 3t^2$

$$\therefore \frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx}$$

$$\frac{dy}{dx} = 3t^2 \times \frac{1}{4t^3}$$

$$\frac{dy}{dx} = \frac{3}{4t}$$

2. $x = \frac{4}{t} \quad , \quad y = 6t^2$

$\frac{dx}{dt} = -\frac{4}{t^2}$ and $\frac{dy}{dt} = 12t$

$$\therefore \frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx}$$

$$\frac{dy}{dx} = 12t \times -\frac{t^2}{4}$$

$$\frac{dy}{dx} = -3t^3$$

3. $x = 4t^2 - t \quad , \quad y = 6t^2 + t + 3$

$\frac{dx}{dt} = 8t - 1$ and $\frac{dy}{dt} = 12t + 1$

$$\therefore \frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx}$$

$$\frac{dy}{dx} = (12t + 1) \times \frac{1}{(8t - 1)}$$

$$\frac{dy}{dx} = \frac{12t + 1}{8t - 1}$$

4. $x = 2\sqrt{t} \quad , \quad y = 5\sqrt{t} - 4t$

$\frac{dx}{dt} = \frac{1}{t^{\frac{1}{2}}}$ and $\frac{dy}{dt} = \frac{5-8t^{\frac{1}{2}}}{2t^{\frac{1}{2}}}$

$$\therefore \frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx}$$

$$\frac{dy}{dx} = \left(\frac{5 - 8t^{\frac{1}{2}}}{2t^{\frac{1}{2}}}\right) \times (t^{\frac{1}{2}})$$

$$\frac{dy}{dx} = \frac{5 - 8t^{\frac{1}{2}}}{t^{\frac{1}{2}}}$$

5. $x = 6t + 5 \quad , \quad y = (2t^2 + 3)^3$

$\frac{dx}{dt} = 6$ and $\frac{dy}{dt} = 3(2t^2 + 3)^2(4t)$

$\frac{dy}{dt} = 12t(2t^2 + 3)^2$

Extended Power Rule

$$\therefore \frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx}$$

$$\frac{dy}{dx} = 12t(2t^2 + 3)^2 \times \frac{1}{6}$$

$$\frac{dy}{dx} = 2t(2t^2 + 3)^2$$

6. $x = \frac{t}{1-2t}$, $y = \frac{t}{1+3t}$

Applying the Quotient Rule for both x and y

$$\frac{dx}{dt} = \frac{(1-2t)(1) - t(-2)}{(1-2t)^2}$$

$$\frac{dx}{dt} = \frac{1 - 2t + 2t}{(1-2t)^2}$$

$$\frac{dx}{dt} = \frac{1}{(1-2t)^2}$$

And

$$\frac{dy}{dt} = \frac{(1+3t)(1) - t(3)}{(1+3t)^2}$$

$$\frac{dy}{dt} = \frac{1 + 3t - 3t}{(1+3t)^2}$$

$$\frac{dy}{dt} = \frac{1}{(1+3t)^2}$$

$$\therefore \frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx}$$

$$\frac{dy}{dx} = \frac{1}{(1+3t)^2} \times \frac{(1-2t)^2}{1}$$

$$\frac{dy}{dx} = \frac{(1-2t)^2}{(1+3t)^2}$$

$$\frac{dy}{dx} = \left(\frac{1-2t}{1+3t}\right)^2$$

FORMATIVE PRACTICE 14

Find $\frac{dy}{dx}$ in terms of the parameter for each of the following:

1. $x = t^5$, $y = 3t - 1$
2. $x = \frac{2}{t}$, $y = t^2 + 5t - 3$
3. $x = 7t^2 - 1$, $y = t^3 - 2t$
4. $x = 6\sqrt{t} - t$, $y = 5t^2 - 4\sqrt{t}$
5. $x = 2t(t-4)$, $y = (t-3)^4$
6. $x = \frac{5+t}{1-2t}$, $y = \frac{1+2t}{1-5t}$

ANSWER

1. $\frac{dy}{dx} = \frac{3}{5t^2}$
2. $\frac{dy}{dx} = -\frac{1}{2}t^2(2t+5)$
3. $\frac{dy}{dx} = \frac{3t^2-2}{14t}$
4. $\frac{dy}{dx} = \frac{10\sqrt{t^3}-2}{3-\sqrt{t}}$
5. $\frac{dy}{dx} = \frac{(t-3)^3}{(t-2)}$
6. $\frac{dy}{dx} = \frac{7}{11}\left(\frac{1-2t}{1-5t}\right)^2$

6.0 APPLY IMPLICIT DIFFERENTIATION

6.1 SOLVE FIRST ORDER IMPLICIT DIFFERENTIATION FOR TWO VARIABLES FUNCTION

Implicit differentiation method is used when the relationship between x and y are impossible to express explicitly.

Differentiate both sides of the equation with respect to x. Derivative of any term involving y must include the $\frac{dy}{dx}$ factor.

EXAMPLE

Find $\frac{dy}{dx}$ for each of the following functions.

Differentiate every term with respect to x

1. $3x^2 + y^2 = 5$

   $6x + 2y(\frac{dy}{dx}) = 0$

   $2y\left(\frac{dy}{dx}\right) = -6x$

   $\left(\frac{dy}{dx}\right) = -\frac{6x}{2y} = -\frac{3x}{y}$

2. $x^2 - y^2 = 3x$

   $2x - 2y(\frac{dy}{dx}) = 3$

   $-2y\left(\frac{dy}{dx}\right) = 3 - 2x$

   $\left(\frac{dy}{dx}\right) = \frac{3-2x}{-2y}$

3. $xy^2 = 4$

   xy^2 is a product of two variables. Apply Product Rules.

   $u = x$ $\qquad v = y^2$

   $\frac{du}{dx} = 1$ $\qquad \frac{dv}{dx} = 2y\left(\frac{dy}{dx}\right)$

   $x2y\left(\frac{dy}{dx}\right) + y^2 = 0$

   $x2y\left(\frac{dy}{dx}\right) = -y^2$

   $\left(\frac{dy}{dx}\right) = \frac{-y^2}{x2y} = \frac{-y}{2x}$

4. $2x^2y^3 = -x + 3$

   $u = 2x^2$ $\qquad v = y^3$

   $\frac{du}{dx} = 4x$ $\qquad \frac{dv}{dx} = 3y^2\left(\frac{dy}{dx}\right)$

   $2x^2y^3 = -x + 3$

   $2x^2 3y^2\left(\frac{dy}{dx}\right) + 4xy^3 = -1$

   $2x^2 3y^2\left(\frac{dy}{dx}\right) = -1 - 4xy^3$

   $\left(\frac{dy}{dx}\right) = \frac{-1-4xy^3}{2x^2 3y^2}$

   $\left(\frac{dy}{dx}\right) = \frac{-1-4xy^3}{6x^2y^2}$

5. $2x^2 + 4xy - 2y^2 = 9$

   $u = 4x$ $\qquad v = y$

   $\frac{du}{dx} = 4$ $\qquad \frac{dv}{dx} = 1\left(\frac{dy}{dx}\right)$

   $2x^2 + 4xy - 2y^2 = 9$

   $4x + \left[4x\left(\frac{dy}{dx}\right) + 4y\right] - 4y\left(\frac{dy}{dx}\right) = 0$

   $4x + 4x\left(\frac{dy}{dx}\right) + 4y - 4y\left(\frac{dy}{dx}\right) = 0$

   $4x\left(\frac{dy}{dx}\right) - 4y\left(\frac{dy}{dx}\right) + 4x + 4y = 0$

   $\left(\frac{dy}{dx}\right)(4x - 4y) = -4x - 4y$

   $\left(\frac{dy}{dx}\right) = \frac{-4x-4y}{4x-4y}$

   $\left(\frac{dy}{dx}\right) = -\frac{(x+y)}{(x-y)}$

6. $y^2 + \sin y = x^2$

   $2y\left(\frac{dy}{dx}\right) + \cos y\left(\frac{dy}{dx}\right) = 2x$

   $\left(\frac{dy}{dx}\right)(2y + \cos y) = 2x$

   $\left(\frac{dy}{dx}\right) = \frac{2x}{2y+\cos y}$

FORMATIVE PRACTICE 15

Differentiate the following equations.

1. $x^2 - 3xy + 2y^2 = 7$
2. $x^2 + y^2 = 9x$
3. $7x^2 - \cos y = x^3$
4. $x^2 + 2y = e^3 + \ln y$
5. $\cos x \sin y = 10$
6. $6x^2 + y^4 = 2$

ANSWER

1. $\frac{dy}{dx} = \frac{3y-2x}{-3x+4y}$
2. $\frac{dy}{dx} = -\frac{2x-9}{2y}$
3. $\frac{dy}{dx} = \frac{3x^2-14x}{\sin y}$
4. $\frac{dy}{dx} = -\frac{2xy}{2y-1}$
5. $\frac{dy}{dx} = \frac{\sin y \sin x}{\cos x \cos y}$
6. $\frac{dy}{dx} = -\frac{3x}{y^3}$

7.0 CONSTRUCT PARTIAL DIFFERENTIATION

7.1 DEFINE PARTIAL DIFFERENTIATION

Partial differentiation is used when differentiating a function of more than one variable.

7.2 EXECUTE PARTIAL DIFFERENTIATION TO FIND:

a) First order partial differentiation

$\frac{\partial z}{\partial x}$ assume y as a constant

$\frac{\partial z}{\partial y}$ assume x as a constant

EXAMPLE

Determine $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ of the following functions.

1. $z = 2x^2 + x^2y - y^3$

$\frac{\partial z}{\partial x}$ assume y as a constant

$$\frac{\partial z}{\partial x} = 4x + 2xy$$

$\frac{\partial z}{\partial y}$ assume x as a constant

$$\frac{\partial z}{\partial y} = x^2 - 3y^2$$

2. $z = (x^3 - y)^3$

$\frac{\partial z}{\partial x}$ assume y as a constant

$$\frac{\partial z}{\partial x} = 3(x^3 - y)^2 \frac{\partial}{\partial x}(x^3 - y)$$

$$\frac{\partial z}{\partial x} = 3(x^3 - y)^2 3x^2$$

$$\frac{\partial z}{\partial x} = 9x^2(x^3 - y)^2$$

$\frac{\partial z}{\partial y}$ assume x as a constant

$$\frac{\partial z}{\partial y} = 3(x^3 - y)^2 \frac{\partial}{\partial y}(x^3 - y)$$

$$\frac{\partial z}{\partial y} = 3(x^3 - y)^2(-1)$$

$$\frac{\partial z}{\partial y} = -3(x^3 - y)^2$$

3. $z = \sin(2x^2 + y)$

$$\frac{\partial z}{\partial x} = \cos(2x^2 + y) \frac{\partial}{\partial x}(2x^2 + y)$$

$$\frac{\partial z}{\partial x} = \cos(2x^2 + y)4x$$

$$\frac{\partial z}{\partial x} = 4x\cos(2x^2 + y)$$

$$\frac{\partial z}{\partial y} = \cos(2x^2 + y) \frac{\partial}{\partial y}(2x^2 + y)$$

$$\frac{\partial z}{\partial y} = \cos(2x^2 + y)(1)$$

$$\frac{\partial z}{\partial y} = \cos(2x^2 + y)$$

4. $z = 6x^3 - e^{-xy^2}$

$$\frac{\partial z}{\partial x} = 18x^2 - e^{-xy^2}\frac{\partial}{\partial x}(-xy^2)$$

$$\frac{\partial z}{\partial x} = 18x^2 - e^{-xy^2}(-y^2)$$

$$\frac{\partial z}{\partial x} = 18x^2 + y^2e^{-xy^2}$$

$$\frac{\partial z}{\partial y} = 18x^2 - e^{-xy^2}\frac{\partial}{\partial y}(-xy^2)$$

$$\frac{\partial z}{\partial y} = 18x^2 - e^{-xy^2}[-x(2y)]$$

$$\frac{\partial z}{\partial y} = 18x^2 + 2xye^{-xy^2}$$

5. $z = \cos 2y + \sin 3x$

$$\frac{\partial z}{\partial x} = 3\cos 3x$$

$$\frac{\partial z}{\partial y} = -2\sin 2y$$

6. $z = x^2e^{2y} + ye^{2x}$

$$\frac{\partial z}{\partial x} = 2xe^{2y} + ye^{2x}\frac{\partial}{\partial x}(2x)$$

$$\frac{\partial z}{\partial x} = 2xe^{2y} + 2ye^{2x}$$

$$\frac{\partial z}{\partial y} = x^2e^{2y}\frac{\partial}{\partial y}(2y) + e^{2x}$$

$$\frac{\partial z}{\partial y} = 2x^2e^{2y} + e^{2x}$$

7. $z = \frac{x-y}{x+2y}$

Apply Quotient Rules to obtain the derivative.

$u = x - y \qquad v = x + 2y$

$\frac{\partial u}{\partial x} = 1 \qquad \frac{\partial v}{\partial x} = 1$

$$\frac{\partial z}{\partial x} = \frac{(x+2y)-(x-y)}{(x+2y)^2}$$

$$\frac{\partial z}{\partial x} = \frac{x+2y-x+y}{(x+2y)^2}$$

$$\frac{\partial z}{\partial x} = \frac{3y}{(x+2y)^2}$$

$u = x - y \qquad v = x + 2y$

$\frac{\partial u}{\partial y} = -1 \qquad \frac{\partial v}{\partial y} = 2$

$$\frac{\partial z}{\partial y} = \frac{(-1)(x+2y)-2(x-y)}{(x+2y)^2}$$

$$\frac{\partial z}{\partial y} = \frac{-x-2y-2x+2y}{(x+2y)^2}$$

$$\frac{\partial z}{\partial y} = \frac{-3x}{(x+2y)^2}$$

FORMATIVE PRACTICE 16

Compute $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$

1. $z = 3x^3y + x\sin 2y$
2. $z = (x+y)(x-y)$
3. $z = (8x+y)(2x-5y)$
4. $z = \sin 3x + y^2\sin x$

ANSWER

1. $\frac{\partial z}{\partial x} = 9x^2y + \sin 2y \qquad \frac{\partial z}{\partial y} = 3x^3 + 2x\cos 2y$
2. $\frac{\partial z}{\partial x} = 2x \qquad \frac{\partial z}{\partial y} = -2y$
3. $\frac{\partial z}{\partial x} = 32x - 38y \qquad \frac{\partial z}{\partial y} = -38x - 10y$
4. $\frac{\partial z}{\partial x} = 3\cos 3x + y^2\cos x \qquad \frac{\partial z}{\partial y} = 2y\sin x$

b) Second order partial differentiation

$$\frac{\partial^2 z}{\partial x^2} = \frac{\partial}{\partial x}(\frac{\partial z}{\partial x})$$

$$\frac{\partial^2 z}{\partial y^2} = \frac{\partial}{\partial y}(\frac{\partial z}{\partial y})$$

$$\frac{\partial^2 z}{\partial x \partial y} = \frac{\partial}{\partial x}(\frac{\partial z}{\partial y})$$

$$\frac{\partial^2 z}{\partial y \partial x} = \frac{\partial}{\partial y}(\frac{\partial z}{\partial x})$$

EXAMPLE

Determine $\frac{\partial^2 z}{\partial x^2}, \frac{\partial^2 z}{\partial y^2}, \frac{\partial^2 z}{\partial x \partial y}, and \ \frac{\partial^2 z}{\partial y \partial x}$ of the following functions.

1. $z = x^3 - 4xy$

$$\frac{\partial z}{\partial x} = 3x^2 - 4y$$

$$\frac{\partial z}{\partial y} = -4x$$

$$\frac{\partial^2 z}{\partial x^2} = \frac{\partial}{\partial x}(\frac{\partial z}{\partial x})$$

$$\frac{\partial^2 z}{\partial x^2} = \frac{\partial}{\partial x}(3x^2 - 4y)$$

$$\frac{\partial^2 z}{\partial x^2} = 6x$$

$$\frac{\partial^2 z}{\partial y^2} = \frac{\partial}{\partial y}(\frac{\partial z}{\partial y})$$

$$\frac{\partial^2 z}{\partial y^2} = \frac{\partial}{\partial y}(-4x)$$

$$\frac{\partial^2 z}{\partial y^2} = 0$$

$$\frac{\partial^2 z}{\partial x \partial y} = \frac{\partial}{\partial x}(\frac{\partial z}{\partial y})$$

$$\frac{\partial^2 z}{\partial x \partial y} = \frac{\partial}{\partial x}(-4x)$$

$$\frac{\partial^2 z}{\partial x \partial y} = -4$$

$$\frac{\partial^2 z}{\partial y \partial x} = \frac{\partial}{\partial y}(\frac{\partial z}{\partial x})$$

$$\frac{\partial^2 z}{\partial y \partial x} = \frac{\partial}{\partial y}(3x^2 - 4y)$$

$$\frac{\partial^2 z}{\partial y \partial x} = -4$$

2. $z = 2x^2 + 3xy - 2y^2$

$$\frac{\partial z}{\partial x} = 4x + 3y$$

$$\frac{\partial z}{\partial y} = 3x - 4y$$

$$\frac{\partial^2 z}{\partial x^2} = \frac{\partial}{\partial x}(\frac{\partial z}{\partial x})$$

$$\frac{\partial^2 z}{\partial x^2} = \frac{\partial}{\partial x}(4x + 3y)$$

$$\frac{\partial^2 z}{\partial x^2} = 4$$

$$\frac{\partial^2 z}{\partial y^2} = \frac{\partial}{\partial y}(\frac{\partial z}{\partial y})$$

$$\frac{\partial^2 z}{\partial y^2} = \frac{\partial}{\partial y}(3x - 4y)$$

$$\frac{\partial^2 z}{\partial y^2} = -4$$

$$\frac{\partial^2 z}{\partial x \partial y} = \frac{\partial}{\partial x}(\frac{\partial z}{\partial y})$$

$$\frac{\partial^2 z}{\partial x \partial y} = \frac{\partial}{\partial x}(3x - 4y)$$

$$\frac{\partial^2 z}{\partial x \partial y} = 3$$

$$\frac{\partial^2 z}{\partial y \partial x} = \frac{\partial}{\partial y}(\frac{\partial z}{\partial x})$$

$$\frac{\partial^2 z}{\partial y \partial x} = \frac{\partial}{\partial y}(4x + 3y)$$

$$\frac{\partial^2 z}{\partial y \partial x} = 3$$

3. $z = x^3y^2 + xy^3$

$$\frac{\partial z}{\partial x} = 3x^2y^2 + y^3$$

$$\frac{\partial z}{\partial y} = 2x^3y + 3xy^2$$

$$\frac{\partial^2 z}{\partial x^2} = \frac{\partial}{\partial x}(\frac{\partial z}{\partial x})$$

$$\frac{\partial^2 z}{\partial x^2} = \frac{\partial}{\partial x}(3x^2y^2 + y^3\ )$$

$$\frac{\partial^2 z}{\partial x^2} = 6xy^2$$

$$\frac{\partial^2 z}{\partial y^2} = \frac{\partial}{\partial y}(\frac{\partial z}{\partial y})$$

$$\frac{\partial^2 z}{\partial y^2} = \frac{\partial}{\partial y}(2x^3y + 3xy^2)$$

$$\frac{\partial^2 z}{\partial y^2} = 2x^3 + 6xy$$

$$\frac{\partial^2 z}{\partial x\partial y} = \frac{\partial}{\partial x}(\frac{\partial z}{\partial y})$$

$$\frac{\partial^2 z}{\partial x\partial y} = \frac{\partial}{\partial x}(2x^3y + 3xy^2\ )$$

$$\frac{\partial^2 z}{\partial x\partial y} = 6x^2y + 3y^2$$

$$\frac{\partial^2 z}{\partial y\partial x} = \frac{\partial}{\partial y}(\frac{\partial z}{\partial x})$$

$$\frac{\partial^2 z}{\partial y\partial x} = \frac{\partial}{\partial y}(3x^2y^2 + y^3)$$

$$\frac{\partial^2 z}{\partial y\partial x} = 6x^2y + 3y^2$$

4. $z = 2\sin x + x\cos y$

$$\frac{\partial z}{\partial x} = 2\cos x + \cos y$$

$$\frac{\partial z}{\partial y} = -x\sin y$$

$$\frac{\partial^2 z}{\partial x^2} = \frac{\partial}{\partial x}(\frac{\partial z}{\partial x})$$

$$\frac{\partial^2 z}{\partial x^2} = \frac{\partial}{\partial x}(2\cos x + \cos y)$$

$$\frac{\partial^2 z}{\partial x^2} = -2\sin x$$

$$\frac{\partial^2 z}{\partial y^2} = \frac{\partial}{\partial y}(\frac{\partial z}{\partial y})$$

$$\frac{\partial^2 z}{\partial y^2} = \frac{\partial}{\partial y}(-x\sin y)$$

$$\frac{\partial^2 z}{\partial y^2} = -x\cos y$$

$$\frac{\partial^2 z}{\partial x\partial y} = \frac{\partial}{\partial x}(\frac{\partial z}{\partial y})$$

$$\frac{\partial^2 z}{\partial x\partial y} = \frac{\partial}{\partial x}(-x\sin y)$$

$$\frac{\partial^2 z}{\partial x\partial y} = -\sin y$$

$$\frac{\partial^2 z}{\partial y\partial x} = \frac{\partial}{\partial y}(\frac{\partial z}{\partial x})$$

$$\frac{\partial^2 z}{\partial y\partial x} = \frac{\partial}{\partial y}(2\cos x + \cos y)$$

$$\frac{\partial^2 z}{\partial y\partial x} = -\sin y$$

FORMATIVE PRACTICE 17

Determine $\frac{\partial z}{\partial x}, \frac{\partial z}{\partial y}, \frac{\partial^2 z}{\partial x^2}, \frac{\partial^2 z}{\partial y^2}, \frac{\partial^2 z}{\partial x\partial y}, and\ \frac{\partial^2 z}{\partial y\partial x}$ of the following functions.

1. $z = 2x^2 + xy^2 - 3y^2$
2. $z = 5x^3 - 3x^2y^3 - 2y$
3. $z = y - \sin y + 5x^2$
4. $z = y^2\cos x + x^2\sin y$

ANSWER

1. $\frac{\partial z}{\partial x} = 4x + y^2$ $\quad \frac{\partial^2 z}{\partial x^2} = 4$ $\quad \frac{\partial^2 z}{\partial x\partial y} = 2y$

   $\frac{\partial z}{\partial y} = 2xy - 6y$ $\quad \frac{\partial^2 z}{\partial y^2} = 2x - 6$ $\quad \frac{\partial^2 z}{\partial y\partial x} = 2y$

2. $\frac{\partial z}{\partial x} = 15x^2 - 6xy^3$ $\quad \frac{\partial^2 z}{\partial x^2} = 30x - 6y^3$ $\quad \frac{\partial^2 z}{\partial x\partial y} = 18xy^2$

   $\frac{\partial z}{\partial y} = 9x^2y^2 - 2$ $\quad \frac{\partial^2 z}{\partial y^2} = 18x^2y$ $\quad \frac{\partial^2 z}{\partial y\partial x} = 18xy^2$

3. $\frac{\partial z}{\partial x} = 10x$ $\quad \frac{\partial^2 z}{\partial x^2} = 10$ $\quad \frac{\partial^2 z}{\partial x\partial y} = 0$

   $\frac{\partial z}{\partial y} = 1 - \cos y$ $\quad \frac{\partial^2 z}{\partial y^2} = \sin y$ $\quad \frac{\partial^2 z}{\partial y\partial x} = 0$

4. $\frac{\partial z}{\partial x} = -y^2\sin x + 2x\sin y$ $\quad \frac{\partial^2 z}{\partial x^2} = -y^2\cos x + 2\sin y$

   $\frac{\partial z}{\partial y} = 2y\cos x + x^2\cos y$ $\quad \frac{\partial^2 z}{\partial y^2} = 2\cos x - x^2\sin y$

   $\frac{\partial^2 z}{\partial x\partial y} = -2y\sin x + 2x\cos y$ $\quad \frac{\partial^2 z}{\partial y\partial x} = -2y\sin x + 2x\cos y$

8.0 APPLY THE TECHNIQUE OF TOTAL DIFFERENTIATION

8.1 EXECUTE TOTAL DIFFERENTIATION FOR TWO VARIABLES FUNCTION

Formula for total derivative :

$$D(z) = \frac{\partial z}{\partial x}(dx) + \frac{\partial z}{\partial y}(dy)$$

EXAMPLE

1. Determine the total differentiation of the following function:

   $z = 6x^2 - 4xy + y^3$

   $\frac{\partial z}{\partial x} = 12x - 4y$

   $\frac{\partial z}{\partial y} = -4x + 3y^2$

   Substitute the values in the formula :

   $D(z) = \frac{\partial z}{\partial x}(dx) + \frac{\partial z}{\partial y}(dy)$

   $D(z) = (12x - 4y)(dx) + (-4x + 3y^2)(dy)$

2. If $z = x^3y + 6x$, find the change of z if (x, y) change from $(2,3)\ to\ (2.01, 2.98)$

   $dx = 2.01 - 2 = 0.01$

   $dy = 2.98 - 3 = -0.02$

   $\frac{\partial z}{\partial x} = 3x^2y + 6$

   $\frac{\partial z}{\partial y} = x^3$

   Substitute the values in the formula :

   $D(z) = \frac{\partial z}{\partial x}(dx) + \frac{\partial z}{\partial y}(dy)$

   $D(z) = (3x^2y + 6\ )(dx) + (x^3)(dy)$

   $D(z) = [3(2)^2(3) + 6](0.01) + (2^3)(-0.02)$

   $D(z) = [3(2)^2(3) + 6](0.01) + (2^3)(-0.02)$

   $D(z) = 0.26$ ∴ z increase by 0.26 units

3. A right circular cone radius increases at the rate of 2cm/min. Calculate how fast is the cone's volume changing when the radius is 15cm and the height is 23cm. $V_{cone} = \frac{1}{3}\pi r^2 h$

   $V_{cone} = \frac{1}{3}\pi r^2 h$

   $\frac{\partial V}{\partial r} = \frac{2}{3}\pi rh$

   $\frac{\partial V}{\partial h} = \frac{1}{3}\pi r^2$

   $\frac{dV}{dt} = \frac{dr}{dt} \times \frac{dV}{dr}$

   $\frac{dV}{dt} = 3 \times \frac{2}{3}\pi rh$

   $\frac{dV}{dt} = 3 \times \frac{2}{3}\pi(15)(23)$

   $\frac{dV}{dt} = 690\pi\ cm^3/min$

4. The height of a cylinder is 5cm and increases at the rate of 0.7 cm/s. The radius of the cylinder is 3cm and decreases at the rate of 0.5 cm/s. Calculate the rate of change for the volume of the cylinder.

   $V = \pi r^2 h$

   $\frac{\partial V}{\partial r} = 2\pi rh$

   $\frac{\partial V}{\partial h} = \pi r^2$

   $\frac{dV}{dt} = \frac{\partial V}{\partial r}\left(\frac{dr}{dt}\right) + \frac{\partial V}{\partial h}\left(\frac{dh}{dt}\right)$

   $\frac{dV}{dt} = (2\pi rh)\left(\frac{dr}{dt}\right) + (\pi r^2)\left(\frac{dh}{dt}\right)$

   $\frac{dV}{dt} = [2\pi(3)(5)(-0.5)] + [\pi(3)^2(0.7)]$

   $\frac{dV}{dt} = -15\pi + 6.3\pi$

   $\frac{dV}{dt} = -8.7\pi\ cm^3/s$

5. The radius and height of a cylinder is 2.5cm and 6cm respectively with the error of measurement is ±0.02cm. Find the maximum error of its volume.

$V = \pi r^2 h$

$\frac{\partial V}{\partial r} = 2\pi rh$

$\frac{\partial V}{\partial h} = \pi r^2$

$\frac{dV}{dt} = \frac{\partial V}{\partial r}\left(\frac{dr}{dt}\right) + \frac{\partial V}{\partial h}\left(\frac{dh}{dt}\right)$

$\frac{dV}{dt} = (2\pi rh)\left(\frac{dr}{dt}\right) + (\pi r^2)\left(\frac{dh}{dt}\right)$

$\frac{dV}{dt} = [2\pi(2.5)(6)(0.05)] + [\pi(2.5)^2(0.05)]$

$\frac{dV}{dt} = 1.5\pi + 0.3125\pi$

$\frac{dV}{dt} = 1.8125\pi\ cm^3/s$

FORMATIVE PRACTICE 18

1. Given $z = 3x^2y + e^{2x}$. Determine the total differential of z.

   Answer

   $D(z) = (6xy + 2e^{2x})(dx) + 3x^2(dy)$

2. The height of a cone is 9mm and increases at the rate of 0.4mm/s. The radius of the base is 8.5mm and decreases at the rate of 0.3mm/s. Calculate the rate of change for the volume of the cone where $V_{cone} = \frac{1}{3}\pi r^2 h$.

   Answer : -5.667mm^3/s

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