



For Polytechnics

ENGINEERING MATHEMATICS 2

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TA330 .E54 2019



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ENGINEERING MATHEMATICS 2: INTEGRATION

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NO. PENGHUN	TA320 - ESU 2019

BUKU INI DISUMBANGKAN OLEH
PERPUSTAKAAN PORT DICKSON



PUBLISHED BY:

MATHEMATICS, SCIENCE & COMPUTER DEPARTMENT
POLITEKNIK PORT DICKSON

KM14, JALAN PANTAI, SI RUSA, 71050 PORT DICKSON, NEGERI SEMBILAN

www.polipd.edu.my

PREFACE

Assalamualaikum & Peace be Upon You

Thank to Almighty God who has given His bless to the writers for finishing this mathematic module entitled "Mathematics Engineering 2: Integration".

The writers also wishes to express their deep and sincere gratitude for those who have guided in completing this module. This module contains parts of the Engineering Mathematics 2 course syllabus of polytechnic which covers the topic of Integration. There are some useful learning material and examples that can help the students to learn Integration.

Those learning material can also be used as the guidance for those who didn't take additional mathematics in secondary school and as a reference for newly trained lecturer in this field.

Hopefully, this module can help the students and lecturer to expand their knowledge about mathematics.

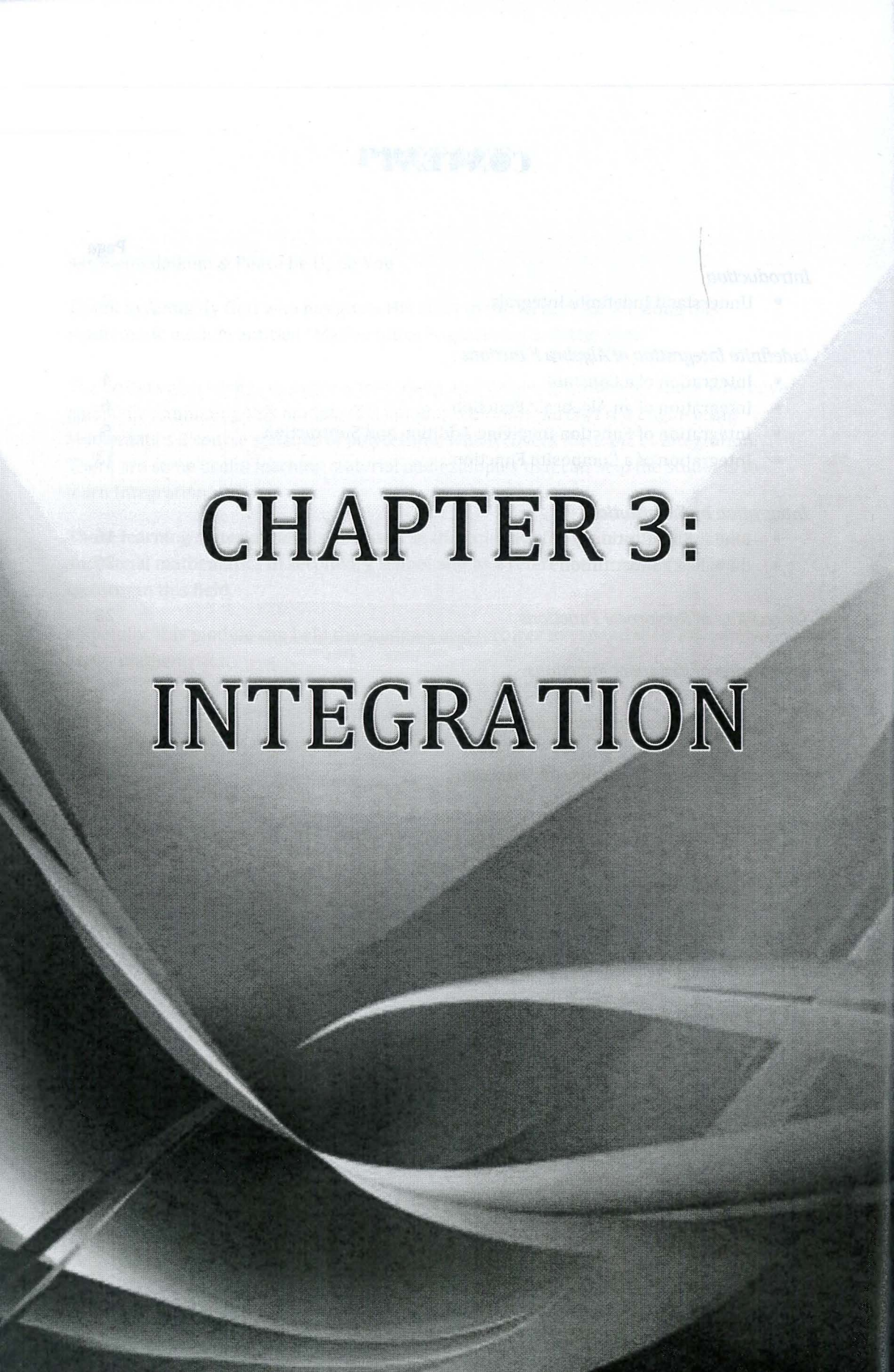
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CHAPTER 3:

INTEGRATION



INTRODUCTION

Integration is the inverse of differentiation and also called as anti-derivative or anti differentiation.

Example:

$$y = 5x$$

$$\therefore \frac{dy}{dx} = 5$$

$$\text{if } \frac{dy}{dx} = 5$$

$$y = ?$$

$$\therefore y = \int 5 \, dx$$

$$y = 5x + c$$

C is called the constant of integration.

Understand Indefinite Integrals

$$\int f(x) \, dx$$

The definition of the symbol:

- i) $\int f(x) \, dx$ is a collection of functions
- ii) $\int$ is an integral sign
- iii) f is called integrand
- iv) x variable of integration
- v) dx with respect to x

Indefinite Integrals Formulas

No	Derivative Formula	Indefinite Integrals Formula
1	$\frac{d}{dx}(c) = 0$	$\int 0 \, dx = c$
2	$\frac{d}{dx}(x^n) = nx^{n-1}$	$\int x^n \, dx = \frac{x^{n+1}}{n+1} + c, \quad n \neq -1$
3	$\frac{d}{dx}(ax^n) = anx^{n-1}$	$\int ax^n \, dx = \frac{ax^{n+1}}{n+1} + c, \quad n \neq -1$
4	$\frac{d}{dx}(ax+b)^n = n(ax+b)^{n-1} \cdot a$ $= an(ax+b)^{n-1}$	$\int (ax+b)^n \, dx = \frac{(ax+b)^{n+1}}{(a)(n+1)} + c, \quad n \neq -1$
5	$\frac{d}{dx}(kf(x)) = k \frac{d}{dx}(f(x))$	$\int kf(x) \, dx = k \int f(x) \, dx$
6	$\frac{d}{dx}(f(x) \pm g(x)) = \frac{d}{dx}(f(x)) \pm \frac{d}{dx}(g(x))$	$\int [f(x) \pm g(x)] \, dx = \int f(x) \, dx \pm \int g(x) \, dx$
7	$\frac{d}{dx}(\ln x) = \frac{1}{x}$	$\int \frac{1}{x} \, dx = \ln x + c$
8	$\frac{d}{dx}(e^{ax}) = ae^{ax}$	$\int ae^{ax} \, dx = \frac{ae^{ax}}{a} + c = e^{ax} + c$
9	$\frac{d}{dx}(\sin x) = \cos x$	$\int \cos x \, dx = \sin x + c$
10	$\frac{d}{dx}(\cos x) = -\sin x$	$\int \sin x \, dx = -\cos x + c$
11	$\frac{d}{dx}(\tan x) = \sec^2 x$	$\int \sec^2 x \, dx = \tan x + c$
12	$\frac{d}{dx}(e^x) = e^x$	$\int e^x \, dx = e^x + c$



INDEFINITE INTEGRATION OF ALGEBRAIC FUNCTIONS

Integration of a constant

$$\int a \, dx = ax + c$$



EXAMPLE 1

Find the integrals:

a) $\int 8 \, dx$

b) $\int -5 \, dz$

c) $\int \sqrt{3} \, dy$

d) $\int \frac{2}{7} \, dx$

e) $\int \sqrt{\frac{3}{4}} \, dx$

f) $\int -\frac{6}{9} \, dm$

g) $\int 0 \, dk$

h) $\int db$

SOLUTION:

$$\begin{aligned} \text{a) } \int 8 \, dx \\ = 8x + c \end{aligned}$$

$$\begin{aligned} \text{b) } \int -5 \, dz \\ = -5z + c \end{aligned}$$

$$\begin{aligned} \text{c) } \int \sqrt{3} \, dy \\ = \sqrt{3}y + c \end{aligned}$$

$$\begin{aligned} \text{d) } \int \frac{2}{7} \, dx \\ = \frac{2}{7}x + c \end{aligned}$$

$$\begin{aligned} \text{e) } \int \sqrt{\frac{3}{4}} \, dx \\ = \sqrt{\frac{3}{4}}x + c \end{aligned}$$

$$\begin{aligned} \text{f) } \int -\frac{6}{9} \, dm \\ = -\frac{6}{9}m + c \\ = -\frac{2}{3}m + c \end{aligned}$$

$$\begin{aligned} \text{g)} \quad \int 0 \, dk \\ = 0k + c \\ = c \end{aligned}$$

$$\begin{aligned} \text{h)} \quad \int db \\ = \int 1 \, db \\ = 1b + c \\ = b + c \end{aligned}$$

$$\int \frac{2n^1}{2} dn$$



EXERCISE 1

Find the integrals:

$$n^2 + c^2$$

$\begin{aligned} \text{a)} \quad \int 2 \, dx \\ = 2x + c \end{aligned}$	$\begin{aligned} \text{b)} \quad \int -9 \, dy \\ = -9y + c \end{aligned}$
$\begin{aligned} \text{c)} \quad \int \frac{7}{4} \, dm \\ = \frac{7}{4} m + c \end{aligned}$	$\begin{aligned} \text{d)} \quad \int -\frac{1}{3} \, dx \\ = -\frac{1}{3} x + c \end{aligned}$
$\begin{aligned} \text{e)} \quad \int \pi \, dz \\ = \pi z + c \end{aligned}$	$\begin{aligned} \text{f)} \quad \int 0.56 \, dn \\ = 0.56n + c \end{aligned}$
$\begin{aligned} \text{g)} \quad \int 0 \, dn \\ = 0n + c \\ = c \end{aligned}$	$\begin{aligned} \text{h)} \quad \int \sqrt{\frac{125}{5}} \, dv \\ = \sqrt{\frac{125}{5}} + c \\ = 5 + c \end{aligned}$
$\begin{aligned} \text{i)} \quad \int -\sqrt{9} \, dx \\ = -\sqrt{9} x + c \\ = -3x + c \end{aligned}$	$\begin{aligned} \text{j)} \quad \int df \\ = f + c \end{aligned}$

Integration of an algebraic function

The integrals of $\int ax \, dx$, $\int x^n \, dx$ and $\int ax^n \, dx$ where a and n is a constant real number.

i. $\int ax \, dx = \frac{ax^{n+1}}{n+1} + c, \quad n \neq -1$

ii. $\int x^n \, dx = \frac{x^{n+1}}{n+1} + c, \quad n \neq -1$

iii. $\int ax^n \, dx = \frac{ax^{n+1}}{n+1} + c, \quad n \neq -1$

**EXAMPLE 2**

Find the integrals:

a) $\int x^6 \, dx$

b) $\int 9x^3 \, dx$

c) $\int \frac{dx}{2x^2}$

d) $\int \frac{-5}{2} x^{-6} \, dx$

e) $\int 4\sqrt{x} \, dx$

f) $\int \frac{dx}{\sqrt{x}}$

SOLUTION:

a) $\int x^6 dx$
 $= \frac{x^{6+1}}{6+1} + c$
 $= \frac{x^7}{7} + c$

b) $\int 9x^3 dx$
 $= \frac{9x^4}{4} + c$

c) $\int \frac{dx}{2x^3}$
 $= \int \frac{x^{-3}}{2} dx$
 $= \frac{x^{-3+1}}{2(-1)} + c$
 $= \frac{x^{-2}}{-2} + c$
 $= -\frac{1}{2x^2} + c$

d) $\int \frac{-5}{2} x^{-6} dx$
 $= -\frac{5x^{-5}}{2(-5)} + c$
 $= \frac{x^{-5}}{2} + c$
 $= \frac{1}{2x^5} + c$

e) $\int 4\sqrt{x} dx$
 $= \int 4x^{\frac{1}{2}} dx$
 $= \frac{4x^{\frac{1}{2}+1}}{\frac{1}{2}+1} + c$
 $= \frac{4x^{\frac{3}{2}}}{\frac{3}{2}} + c$
 $= \frac{8}{3} \sqrt{x^3} + c$

f) $\int \frac{dx}{\sqrt{x}}$
 $= \int x^{-\frac{1}{2}} dx$
 $= \frac{x^{\frac{1}{2}}}{\frac{1}{2}} + c$
 $= 2\sqrt{x} + c$



EXERCISE 2

Find the integrals:

a) $\int x^3 dx$ [Ans: $\frac{x^4}{4} + c$] $= \frac{x^{3+1}}{3+1}$ $= \frac{x^4}{4} + c$	b) $\int 6x^3 dx$ [Ans: $\frac{3x^4}{2} + c$]
c) $\int x^{-2} dx$ [Ans: $-\frac{1}{x} + c$] $= \frac{x^{-2+1}}{-2+1}$ $= \frac{x^{-1}}{-1} = -\frac{x^{-1}}{1} + c$	d) $\int -\frac{2}{\sqrt{x}} dx$ [Ans: $-4\sqrt{x} + c$] $= -\frac{2}{x^{\frac{1}{2}}}$ $= -2x^{-\frac{1}{2}+1} = -2x^{\frac{1}{2}}$ $= -\frac{2}{-\frac{1}{2}+1} = -\frac{2}{\frac{1}{2}} = -4\sqrt{x} + c$
e) $\int \frac{5m^4}{7} dm$ [Ans: $\frac{m^5}{7} + c$] $= \frac{5m^{4+1}}{7(4+1)}$ $= \frac{5m^5}{35} + c$	f) $\int 6y^{-7} dy$ [Ans: $-\frac{1}{y^6} + c$] $= \frac{6y^{-7+1}}{-7+1}$ $= -\frac{6y^{-6}}{6} = -\frac{1}{y^6} + c$
g) $\int \frac{1}{k^8} dk$ [Ans: $-\frac{1}{7k^7} + c$] $= \frac{1}{k^8}$ $= \frac{k^{-8+1}}{-8+1}$ $= -\frac{1}{7k^7} + c$	h) $\int -3y dy$ [Ans: $-\frac{3y^2}{2} + c$] $= \frac{-3y^{1+1}}{1+1}$ $= -\frac{3y^2}{2} + c$
i) $\int \sqrt{n} dn$ [Ans: $\frac{2}{3}\sqrt{n^3} + c$] $= \frac{n^{\frac{1}{2}+1}}{\frac{1}{2}+1}$ $= \frac{2\sqrt{n}}{3} + c$	j) $\int \frac{3}{4}\pi x dx$ [Ans: $\frac{3}{8}\pi x^2 + c$] $= \frac{3}{4}\pi x^{1+1}$ $= \frac{3}{4(2)}\pi x^2$ $= \frac{3}{8}\pi x^2 + c$

Integration of a function involving addition and subtraction

i. Addition of integration

$$\int [f(x) + g(x)] dx = \int f(x) dx + \int g(x) dx$$

ii. Subtraction of integration

$$\int [f(x) - g(x)] dx = \int f(x) dx - \int g(x) dx$$

**EXAMPLE 3**

Find the integrals:

a) $\int (3x^2 + 4) dx$

b) $\int \left(7a^2 - \frac{4a}{5} \right) da$

c) $\int (2m - 8)(9m + 5) dm$

d) $\int \left(\frac{6x^{-2} - 3x^4}{x} \right) dx$

e) $\int \left(-9 - \frac{1}{x^2} \right) dx$

f) $\int \left(3\sqrt{x} + \frac{2}{\sqrt{x}} \right) dx$

SOLUTION :

$$\begin{aligned} \text{a) } \int (3x^2 + 4) dx &= \int 3x^2 dx + \int 4 dx \\ &= \frac{3x^3}{3} + 4x + c \\ &= x^3 + 4x + c \end{aligned}$$

$$\begin{aligned} \text{b) } \int \left(7a^2 - \frac{4a}{5} \right) da &= \int 7a^2 da - \int \frac{4}{5} a da \\ &= \int 7a^2 da - \int \frac{4}{5} a da \\ &= \frac{7a^3}{3} + \frac{4a^2}{5 \cdot 2} + c \\ &= \frac{7a^3}{3} + \frac{4a^2}{10} + c \end{aligned}$$

$$\begin{aligned}
 \text{c) } & \int (2m-8)(9m+5) \, dm \\
 & \int (18m^2 + 10m - 72m - 40) \, dm \\
 & \int (18m^2 - 62m - 40) \, dm \\
 & \int 18m^2 \, dm - \int 62m \, dm - \int 40 \, dm \\
 & = \frac{18m^3}{3} - \frac{62m^2}{2} - 40m + c \\
 & = 6m^3 - 31m^2 - 40m + c
 \end{aligned}$$

$$\begin{aligned}
 \text{d) } & \int \left(\frac{6x^{-2} - 3x^4}{x} \right) dx \\
 & = \int \frac{6x^{-2}}{x} dx - \int \frac{3x^4}{x} dx \\
 & = \int 6x^{-3} dx - \int 3x^3 dx \\
 & = \frac{6x^{-2}}{-2} - \frac{3x^4}{4} + c \\
 & = -\frac{3}{x^2} - \frac{3x^4}{4} + c
 \end{aligned}$$

$$\begin{aligned}
 \text{e) } & \int \left(-9 - \frac{1}{x^2} \right) dx \\
 & = \int -9 \, dx - \int x^{-2} \, dx \\
 & = -9x - \frac{x^{-1}}{(-1)} + c \\
 & = -9x + \frac{1}{x} + c
 \end{aligned}$$

$$\begin{aligned}
 \text{f) } & \int 3\sqrt{x} + \frac{2}{\sqrt{x}} \, dx \\
 & = \int 3x^{\frac{1}{2}} \, dx + \int 2x^{-\frac{1}{2}} \, dx \\
 & = \frac{3x^{\frac{3}{2}}}{\frac{3}{2}} + \frac{2x^{\frac{1}{2}}}{\frac{1}{2}} + c \\
 & = \frac{2}{3} \cdot 3x^{\frac{3}{2}} + 2 \cdot 2x^{\frac{1}{2}} + c \\
 & = 2\sqrt{x^3} + 4\sqrt{x} + c
 \end{aligned}$$



EXERCISE 3

Find the integrals:

$$\begin{aligned}
 \text{a) } & \int (8x^2 + 2x + 6) \, dx \quad [\text{Ans: } \frac{8}{3}x^3 + x^2 + 6x + c] \\
 & = \int 8x^2 \, dx + \int 2x \, dx + \int 6 \, dx \\
 & = \frac{8x^3}{3} + \frac{2x^2}{2} + 6x \\
 & = \frac{8}{3}x^3 + x^2 + 6x + c
 \end{aligned}$$

$$\begin{aligned}
 \text{b) } & \int (3x^4 + 2x^2 - 7) \, dx \quad [\text{Ans: } \frac{3}{5}x^5 + \frac{2x^3}{3} - 7x + c] \\
 & = \int 3x^4 \, dx + \int 2x^2 \, dx - \int 7 \, dx \\
 & = \frac{3x^{4+1}}{5} + \frac{2x^3}{3} - 7x + c \\
 & = \frac{3}{5}x^5 + \frac{2}{3}x^3 - 7x + c
 \end{aligned}$$

c) $\int (\frac{3}{2}x^3 - x^5 + 4x) dx$ [Ans: $\frac{3}{8}x^4 - \frac{x^6}{6} + 2x^2 + c$] $= \int \frac{3}{2}x^3 dx + \int x^5 dx + \int 4x dx$ $= \frac{3}{2} \frac{x^4}{4} - \frac{x^6}{6} + \frac{4x^2}{2}$ $= \frac{3}{8}x^4 - \frac{x^6}{6} + 2x^2 + c$	d) $\int (5x^3 - 2\sqrt{x} - \frac{3}{x^2}) dx$ [Ans: $\frac{5x^4}{4} - \frac{8}{5}\sqrt{x^5} + \frac{3}{x} + c$] $= \int 5x^3 dx - \int 2\sqrt{x} dx - \int \frac{3}{x^2} dx$ $= \frac{5x^4}{4} - \frac{2x^{\frac{5}{2}}}{\frac{5}{2}} - \frac{3x^{-1}}{-1}$ $= \frac{5x^4}{4} - \frac{8}{5}\sqrt{x^5} + \frac{3}{x} + c$
e) $\int (\frac{x^2-2}{x^2}) dx$ [Ans: $x + \frac{2}{x} + c$] $= \int \frac{x^2}{x^2} dx - \int \frac{2}{x^2} dx$ $= \int 1 dx - \int \frac{2}{x^2} dx$ $= x - 2x^{-1}$ $= x - \frac{2}{x} + c$	f) $\int x(6-x^5) dx$ [Ans: $3x^2 - \frac{x^6}{6} + c$] $= \int (6x - x^6) dx$ $= \int 6x dx - \int x^6 dx$ $= \frac{6x^2}{2} - \frac{x^6}{6} + c$ $= 3x^2 - \frac{x^6}{6} + c$
g) $\int (x+1)(3x-4) dx$ [Ans: $x^3 - \frac{x^2}{2} - 4x + c$] $= \int (3x^2 - 4x + 3x - 4) dx$ $= \int (3x^2 - x - 4) dx$ $= \int 3x^2 dx - \int x dx - \int 4 dx$ $= \frac{3x^3}{3} - \frac{x^2}{2} - 4x$ $= x^3 - \frac{x^2}{2} - 4x + c$	h) $\int (\frac{x^2-1}{\sqrt{x}}) dx$ [Ans: $\frac{2}{5}\sqrt{x^5} - 2\sqrt{x} + c$] $= \int \frac{x^2}{\sqrt{x}} dx - \int \frac{1}{\sqrt{x}} dx$ $= \int x^{\frac{3}{2}} dx - \int x^{-\frac{1}{2}} dx$ $= \frac{2x^{\frac{5}{2}}}{\frac{5}{2}} - 2x^{\frac{1}{2}} + c$ $= \frac{2}{5}\sqrt{x^5} - 2\sqrt{x} + c$

$$(x^2 - 2x)(x^2 - 2x)$$

$$x^2 - 2x^3 - 2x^3 + 4x^2$$

$$x^2 - 2x^3 + 4x^2 - 2x^3$$

$$x^2 - 4x^3 + 4x^2$$

i) $\int (8x-1)^2 dx$ [Ans: $\frac{64x^3}{3} - 8x^2 + x + c$] $= \int (8x-1)^2 dx$ $= \int (64x^2 - 8x + 1) dx$ $= \int 64x^2 dx - \int 8x dx + \int 1 dx$ $= \frac{64x^3}{3} - \frac{8x^2}{2} + x + c$ $= \frac{64}{3}x^3 - 4x^2 + x + c$	j) $\int (x-2x^2)^2 dx$ [Ans: $\frac{x^3}{3} - x^4 + \frac{4x^5}{5} + c$] $= \int (x^2 - 2x^3 + 4x^4) dx$ $= \int x^2 dx - \int 2x^3 dx + \int 4x^4 dx$ $= \frac{x^3}{3} - \frac{2x^4}{4} + \frac{4x^5}{5} + c$ $= \frac{x^3}{3} - \frac{x^4}{2} + \frac{4}{5}x^5 + c$
k) $\int 6\sqrt{x} + \frac{1}{\sqrt{3x}} dx$ [Ans: $4\sqrt{x^3} + \frac{2\sqrt{x}}{\sqrt{3}} + c$] $\int 6x^{\frac{1}{2}} + \frac{1}{\sqrt{3}x^{\frac{1}{2}}} dx$	l) $\int \frac{(1-x)(1+x)}{x^2} dx$ [Ans: $-\frac{1}{x} - x + c$]
m) $\int \frac{(x+2)(x-4x^2)}{x} dx$ $[Ans: -\frac{7x^2}{2} - \frac{4x^3}{3} + 2x + c]$	n) $\int \frac{2(x^2+3)(x-2x^3)}{x^3} dx$ $[Ans: -10x - \frac{4x^3}{3} + \frac{6}{x} + c]$

Integration of a composite function

Integration of a composite function is also known as integrals by using **Formula Method**.

Refer to the following formula, the integrals of $\int (ax + b)^n dx$, a and n is a constant real number.

$$\int (ax + b)^n dx = \frac{(ax + b)^{n+1}}{a(n+1)} + c, \quad n \neq -1$$

**EXAMPLE 4**

Find the integrals:

a) $\int (3x + 1) dx$

c) $\int (5 - 8k)^{-3} dk$

e) $\int \frac{1}{(6m - 4)^2} dm$

b) $\int (2x - 4)^2 dx$

d) $\int 9(2t + 6)^{\frac{3}{4}} dt$

f) $\int \frac{8}{3\sqrt{2-7x}} dx$

SOLUTION :

$$\begin{aligned} \text{a) } \int (3x + 1) dx &= \frac{(3x + 1)^{1+1}}{3 \cdot 2} + c \\ &= \frac{(3x + 1)^2}{6} + c \end{aligned}$$

$$\begin{aligned} \text{c) } \int (5 - 8k)^{-3} dk &= \frac{(5 - 8k)^{-3+1}}{-8(-2)} + c \\ &= \frac{(5 - 8k)^{-2}}{16} + c \\ &= \frac{1}{16(5 - 8k)^2} + c \end{aligned}$$

$$\begin{aligned} \text{b) } \int (2x - 4)^2 dx &= \frac{(2x - 4)^{2+1}}{2 \cdot 3} + c \\ &= \frac{(2x - 4)^3}{6} + c \end{aligned}$$

$$\begin{aligned} \text{d) } \int 9(2t + 6)^{\frac{3}{4}} dt &= \frac{9(2t + 6)^{\frac{3}{4}+1}}{2 \cdot \frac{7}{4}} + c \\ &= \frac{9(2t + 6)^{\frac{7}{4}}}{\frac{7}{2}} + c \\ &= \frac{18^{\frac{7}{4}} \sqrt[4]{(2t + 6)^7}}{7} + c \end{aligned}$$

$$e) \int \frac{1}{(6m-4)^2} dm$$

$$= \int (6m-4)^{-2} dm$$

$$= \frac{(6m-4)^{-1}}{6(-1)} + c$$

$$= \frac{1}{-6(6m-4)} + c$$

$$f) \int \frac{8}{3\sqrt{2-7x}} dx$$

$$= \frac{8}{3} \int (2-7x)^{-\frac{1}{2}} dx$$

$$= \frac{8}{3} \frac{(2-7x)^{\frac{1}{2}}}{(-7)(\frac{1}{2})} + c$$

$$= \frac{8}{3} \frac{\sqrt{2-7x}}{(-\frac{7}{2})} + c$$

$$= \frac{8}{3} \cdot \left(-\frac{2}{7}\right) \sqrt{2-7x} + c$$

$$= \frac{-16\sqrt{2-7x}}{21} + c$$



EXERCISE 4

Find the integrals:

$$a) \int (x+3)^4 dx$$

$$[Ans: \frac{(x+3)^5}{5} + c]$$

$$b) \int (7-5x)^2 dx$$

$$[Ans: \frac{(7-5x)^3}{-15} + c]$$

$$c) \int 6(4z-1)^{-3} dz$$

$$[Ans: -\frac{3}{4(4z-1)^2} + c]$$

$$d) \int \sqrt{2y+9} dy$$

$$[Ans: \frac{\sqrt{(2y+9)^3}}{3} + c]$$

e) $\int \frac{8}{6(t-1)^2} dt$ [Ans : $-\frac{4}{3(t-1)} + c$]	f) $\int \frac{dz}{(2-7z)^{\frac{2}{3}}}$ [Ans : $-\frac{\sqrt[3]{2-7z}}{7} + c$]
g) $\int \frac{\pi}{\sqrt[3]{(5x-6)^4}} dx$ [Ans : $-\frac{3\pi}{5\sqrt[3]{(5x-6)}} + c$]	h) $\int \frac{3dx}{\sqrt{1-9x}}$ [Ans : $-\frac{2\sqrt{(1-9x)}}{3} + c$]



INTEGRATION BY SUBSTITUTION

Linear Substitution

Integration by substitution is the introduction of a new variable to convert the integrand to a standard form which can deal with.

$$\int f[g(x)] \times g'(x) \, dx$$

STEP 1 : Substitute $u = g(x)$

STEP 2 : Differentiate $u = \frac{du}{dx} = g'(x) \, dx$ and $dx = \frac{du}{g'(x)}$

STEP 3 : Substitute $g(x) = u$ and $dx = \frac{du}{g'(x)}$ into the original function

STEP 4 : Solve the integral

STEP 5 : Substitute $u = g(x)$ for the final answer

Student should be able to visualize the integrand as a product of a power of a differentiation function $g(x)$ and its derivative $g'(x)$

**EXAMPLE 5**

Find the following integrals by using substitution method:

a) $\int (4x-3)^3 dx$

b) $\int (5x+4)^8 dx$

c) $\int \left(\frac{1}{(1-2k)^6} \right) dk$

d) $\int \theta^2 (\theta^3 - 2)^4 d\theta$

e) $\int -\frac{2}{3(8-x)^4} dx$

f) $\int x \sqrt{4+x^2} dx$

SOLUTION:

a) $\int (4x-3)^3 dx$

$$u = 4x - 3$$

$$\frac{du}{dx} = 4$$

$$dx = \frac{du}{4}$$

$$= \int u^3 \frac{du}{4}$$

$$= \frac{1}{4} \int u^3 du$$

$$= \frac{1}{4} \left[\frac{u^4}{4} \right] + c$$

$$= \frac{(4x-3)^4}{16} + c$$

b) $\int (5x+4)^8 dx$

$$u = 5x + 4$$

$$\frac{du}{dx} = 5$$

$$dx = \frac{du}{5}$$

$$= \int u^8 \frac{du}{5}$$

$$= \frac{1}{5} \int u^8 du$$

$$= \frac{1}{5} \left[\frac{u^9}{9} \right] + c$$

$$= \frac{(5x+4)^9}{45} + c$$

$$\text{c) } \int \left(\frac{1}{(1-2k)^6} \right) dk$$

$$u = 1 - 2k$$

$$\frac{du}{dk} = -2$$

$$dk = -\frac{du}{2}$$

$$= \int \frac{1}{(u)^6} \left(-\frac{du}{2} \right)$$

$$= -\frac{1}{2} \int (u)^{-6} du$$

$$= -\frac{1}{2} \left[\frac{u^{-5}}{-5} \right] + c$$

$$= \frac{1}{10(1-2k)^5} + c$$

$$\text{d) } \int \theta^2 (\theta^3 - 2)^4 d\theta$$

$$u = \theta^3 - 2$$

$$\frac{du}{d\theta} = 3\theta^2$$

$$d\theta = \frac{du}{3\theta^2}$$

$$= \int \theta^2 (u)^4 \left(\frac{du}{3\theta^2} \right)$$

$$= \frac{1}{3} \int (u)^4 du$$

$$= \frac{1}{3} \left[\frac{u^5}{5} \right] + c$$

$$= \frac{(\theta^3 - 2)^5}{15} + c$$

$$\text{g) } \int -\frac{2}{3(8-x)^4} dx$$

$$u = 8 - x$$

$$\frac{du}{dx} = -1$$

$$dx = -du$$

$$= \int -\frac{2}{3(u)^4} (-du)$$

$$= \frac{2}{3} \int (u)^{-4} du$$

$$= \frac{2}{3} \left[\frac{u^{-3}}{-3} \right] + c$$

$$= -\frac{2}{9(8-x)^3} + c$$

$$\text{h) } \int x \sqrt{4+x^2} dx = \int x(4+x^2)^{\frac{1}{2}} dx$$

$$u = 4 + x^2$$

$$\frac{du}{dx} = 2x$$

$$dx = \frac{du}{2x}$$

$$= \int x(u)^{\frac{1}{2}} \frac{du}{2x}$$

$$= \frac{1}{2} \int u^{\frac{1}{2}} du$$

$$= \frac{1}{2} \left[\frac{u^{\frac{3}{2}}}{\frac{3}{2}} \right] + c$$

$$= \frac{(4+x^2)^{\frac{3}{2}}}{3} + c$$



EXERCISE 5

Find each of the following integrals by using a suitable substitution:

a) $\int 6p (3p^2 - 2)^8 dp$ [Ans: $\frac{(3p^2 - 2)^9}{9} + c$]	b) $\int \frac{4 + 2x}{x^2 + 4x} dx$ [Ans: $\ln(x^2 + 4x) + c$]
c) $\int (2x - 4)^{\frac{1}{4}} dx$ [Ans: $\frac{2(2x - 4)^{\frac{5}{4}}}{5} + c$]	d) $\int -\frac{6}{(4x + 3)^3} dx$ [Ans: $\frac{3}{4(4x + 3)^2} + c$]
e) $\int \frac{6m^2 - 4}{2m^3 - 4m - 5} dm$ [Ans: $\ln(2m^3 - 4m - 5) + c$]	f) $\int 3x (1 - 2x^2) dx$ [Ans: $-\frac{3(1 - 2x^2)^2}{8} + c$]

$du = f'(x)$ Substitution

$$du = f'(x)dx$$

Let $u = f(x)$ for some function f . The differential of u , denoted du .

Characteristic of u

- the term under a root or raised to a power
- the exponent of an exponential function
- the logarithm
- the term in the denominator
- the term inside a trigonometry function

**EXAMPLE 6**

Find each of the following indefinite integrals by using substitution method:

a) $\int x e^{x^2} dx$

b) $\int 2x(x^2 - 1)^4 dx$

c) $\int \frac{4dt}{\sqrt{4t+3}}$

d) $\int x\sqrt{1-x^2} dx$

e) $\int x\sqrt{x-3} dx$

f) $\int \frac{dx}{(4x-3)^2}$

SOLUTION :

$$\text{a) } \int x e^{x^2} dx$$

$$u = x^2$$

$$du = 2x dx$$

$$\int x e^{x^2} dx = \int \left(\frac{1}{2}\right) e^{x^2} (2x dx)$$

$$= \frac{1}{2} \int e^u du$$

$$= \frac{1}{2} [e^u] + c$$

$$= \frac{1}{2} [e^{x^2}] + c$$

$$\text{b) } \int 2x(x^2 - 1)^4 dx$$

$$u = x^2 - 1$$

$$du = 2x dx$$

$$\int 2x(x^2 - 1)^4 dx = \int (x^2 - 1)^4 (2x dx)$$

$$= \int u^4 du$$

$$= \left[\frac{u^5}{5} \right] + c$$

$$= \left[\frac{(x^2 - 1)^5}{5} \right] + c$$

$$\text{c) } \int \frac{4 dt}{\sqrt{4t+3}}$$

$$u = 4t + 3$$

$$du = 4 dt$$

$$\int \frac{4 dt}{\sqrt{4t+3}} = \int \frac{(4 dt)}{(4t+3)^{\frac{1}{2}}}$$

$$= \int \frac{(du)}{u^{\frac{1}{2}}} = \int \frac{1}{u^{\frac{1}{2}}} du = \int u^{-\frac{1}{2}} du$$

$$= \left[\frac{u^{\frac{1}{2}}}{\frac{1}{2}} \right] + c$$

$$= 2 \left[(4t+3)^{\frac{1}{2}} \right] + c$$

$$\text{d) } \int x \sqrt{1-x^2} dx$$

$$u = 1 - x^2$$

$$du = -2x dx$$

$$\int x \sqrt{1-x^2} dx = -\frac{1}{2} \int x (u)^{\frac{1}{2}} (-2x dx)$$

$$= -\frac{1}{2} \int u^{\frac{1}{2}} du$$

$$= -\frac{1}{2} \left[\frac{u^{\frac{3}{2}}}{\frac{3}{2}} \right] + c$$

$$= -\frac{1}{3} \left[(1-x^2)^{\frac{3}{2}} \right] + c$$

e) $\int x\sqrt{x-3} \, dx$

$$u = x - 3$$

$$u + 3 = x$$

$$du = 1 \, dx$$

$$\begin{aligned} \int x\sqrt{x-3} \, dx &= \int (u+3)(u)^{\frac{1}{2}} \, du \\ &= \int (u)^{\frac{3}{2}} + 3(u)^{\frac{1}{2}} \, du \\ &= \left[\frac{2u^{\frac{5}{2}}}{5} + 2u^{\frac{3}{2}} \right] + c \\ &= \frac{2(x-3)^{\frac{5}{2}}}{5} + 2(x-3)^{\frac{3}{2}} + c \end{aligned}$$

f) $\int \frac{dx}{(4x-3)^2}$

$$u = 4x - 3$$

$$du = 4 \, dx$$

$$\begin{aligned} \int \frac{dx}{(4x-3)^2} &= \int \frac{1}{4} \left(\frac{1}{(4x-3)^2} \right) (4 \, dx) \\ &= \frac{1}{4} \int \frac{1}{u^2} \, du = \frac{1}{4} \int u^{-2} \, du \\ &= \frac{1}{4} \left[\frac{u^{-1}}{-1} \right] + c \\ &= -\frac{1}{4} \left[\frac{1}{(4x-3)} \right] + c \end{aligned}$$



EXERCISE 6

Find each of the following integrals by using a suitable substitution:

a) $\int 3(8y-1)e^{4y^2-y} \, dy$

[Ans: $3e^{4y^2-y} + c$]

b) $\int \frac{3y}{5y^2+4} \, dy$

[Ans: $\frac{1}{10} \ln(5y^2+4) + c$]

c) $\int x^2(3-10x^3)^4 dx$ [Ans: $-\frac{(3-10x^3)^5}{150} + c$]	d) $\int \frac{x}{\sqrt{1-4x^2}} dx$ [Ans: $-\frac{(1-4x^2)^{\frac{1}{2}}}{4} + c$]
e) $\int \frac{2a^3+1}{(a^4+2a)^3} da$ [Ans: $-\frac{1}{4(a^4+2a)^2} + c$]	f) $\int z(4z^2-5)^4 dz$ [Ans: $\frac{1}{40}(4z^2-5)^5 + c$]
g) $\int \frac{2z}{z^2+5} dz$ [Ans: $\ln(z^2+5) + c$]	h) $\int \frac{2x}{\sqrt{x-1}} dx$ [Ans: $\frac{4(x+1)^{\frac{3}{2}}}{3} - 4(x+1)^{\frac{1}{2}} + c$]



EXERCISE 7

Find each of the following integrals by using Formula Method and Substitution Method:

a) $\int (6x + 4)^4 \, dx$ [Ans: $\frac{(6x + 4)^5}{30} + c$]	
FORMULA METHOD	SUBSTITUTION METHOD
b) $\int (7x - 3)^2 \, dx$ [Ans: $\frac{(7x - 3)^3}{21} + c$]	
FORMULA METHOD	SUBSTITUTION METHOD

c) $\int \frac{7}{2x-5} dx$

[Ans: $\frac{7}{2} \ln(2x-5) + c$]

FORMULA METHOD

SUBSTITUTION METHOD

d) $\int \sqrt[5]{8x-2} dx$

[Ans: $\frac{5(8x-2)^{\frac{6}{5}}}{48} + c$]

FORMULA METHOD

SUBSTITUTION METHOD

e) $\int \frac{\pi}{(1+3t)^2} dt$ [Ans: $-\frac{\pi}{3(1+3t)} + c$]	
<i>FORMULA METHOD</i>	<i>SUBSTITUTION METHOD</i>
f) $\int (3x^2 - 2x + 1)^3 (6x - 2) dx$ [Ans: $\frac{(3x^2 - 2x + 1)^4}{4} + c$]	
<i>SUBSTITUTION METHOD</i>	$du = f'(x)dx$

g) $\int \frac{-9}{2(9c+5)^4} dc$	$[Ans: -\frac{1}{(9c+5)^3} + c]$
<i>SUBSTITUTION METHOD</i>	$du = f'(x)dx$
h) $\int 3x(1-2x^2) dx$	$[Ans: -\frac{3}{8}(-2x^2)^2 + c]$
<i>SUBSTITUTION METHOD</i>	$du = f'(x)dx$



INTEGRATION OF RECIPROCAL FUNCTIONS

$$\text{i. } \int \frac{1}{x} dx = \ln|x| + c,$$

$|x|$ must be positive

$$\text{ii. } \int \frac{1}{(ax+b)} dx = \frac{1}{a} \ln|ax+b| + c, \quad a \text{ and } c \text{ are constant}$$

$$\text{iii. } \int \frac{f'(x)}{f(x)} dx = \ln|f(x)| + c \quad f'(x) \text{ is a derivative of } f(x)$$



EXAMPLE 8

Find the integrals:

$$\text{a) } \int \frac{1}{2x} dx$$

$$\text{b) } \int \frac{1}{9z+3} dz$$

$$\text{c) } \int \frac{6y}{3y^2+6} dy$$

$$\text{d) } \int \frac{x}{x^2+4} dx$$

SOLUTION:

$$\text{a) } \int \frac{1}{2x} dx$$

$$\begin{aligned} &= \frac{1}{2} \int \frac{1}{x} dx \\ &= \frac{1}{2} \ln|x| + c \end{aligned}$$

$$\text{b) } \int \frac{1}{9z+3} dz$$

$$= \frac{1}{9} \ln|9z+3| + c$$

$$\text{c) } \int \frac{6y}{3y^2+6} dy$$

$$f(y) = 3y^2 + 6$$

$$f'(y) = 6y \quad (\text{proved})$$

$$\therefore \int \frac{f'(y)}{f(y)} dy$$

$$= \ln|f(y)| + c$$

$$= \ln|3y^2 + 6| + c$$

$$\text{d) } \int \frac{x}{x^2+4} dx$$

$$f(x) = x^2 + 4$$

$$f'(x) = 2x$$

$$\therefore \int \frac{f'(x)}{f(x)} dx$$

$$= \frac{1}{2} \ln|f(x)| + c$$

$$= \frac{1}{2} \ln|x^2 + 4| + c$$



EXERCISE 8

Find the integrals:

a) $\int \frac{6}{x} dx$ [Ans : $6 \ln x + c$]	b) $\int \frac{1}{4m} dm$ [Ans : $\frac{1}{4} \ln m + c$]
c) $\int \frac{4y}{8y^2} dy$ [Ans : $\frac{1}{2} \ln y + c$]	d) $\int \frac{5}{2x} dx$ [Ans : $\frac{5}{2} \ln x + c$]
e) $\int \frac{1}{7-x} dx$ [Ans : $-\ln 7-x + c$]	f) $\int -\frac{2}{4x-2} dx$ [Ans : $-\frac{1}{2} \ln 4x-2 + c$]

g) $\int -\frac{1}{9-x} dx$ [Ans : $\ln 9-x + c$]	h) $\int \frac{1}{8k-1} dk$ [Ans : $\frac{1}{8} \ln 8k-1 + c$]
i) $\int \frac{10x}{5x^2-1} dx$ [Ans : $\ln 5x^2-1 + c$]	j) $\int \frac{4s}{8-2s^2} ds$ [Ans : $-\ln 8-2s^2 + c$]
k) $\int \frac{3b}{3b^2+3} db$ [Ans : $\frac{1}{2} \ln 3b^2+3 + c$]	l) $\int -\frac{8m^2}{2-4m^3} dm$ [Ans : $\frac{2}{3} \ln 2-4m^3 + c$]
m) $\int \frac{4x}{x^2+2x} dx$ [Ans : $4 \ln x+2 + c$]	n) $\int \frac{2p^2}{5p^3-7p^2} dp$ [Ans : $\frac{2}{5} \ln 5p-7 + c$]



3.5 INTEGRATION OF EXPONENT FUNCTIONS

Integration of exponent functions can be simplified as below:

1. $\int e^x dx = e^x + c$
2. $\int ae^x dx = ae^x + c$
3. $\int e^{ax} dx = \frac{1}{a}e^{ax} + c$
4. $\int e^{ax+b} dx = \frac{1}{a}e^{ax+b} + c$
5. $\int ae^{bx+d} dx = \frac{1}{b}ae^{bx+d} + c$
6. $\int f'(x) e^{f(x)} dx = e^{f(x)} + c$



EXAMPLE 9

Integrate the following with respect to x function:

a) $\int e^{-3x} dx$

b) $\int 2e^{2x} dx$

c) $\int e^{2x-1} dx$

d) $\int e^{\frac{-x}{2}} dx$

e) $\int 4e^{\frac{x}{2}} dx$

f) $\int 5e^{2x} + \frac{1}{e^{3x}} dx$

g) $\int \frac{1}{\sqrt{x}} e^{\sqrt{x}} dx$

h) $\int 3xe^{x^2} dx$

SOLUTION:

$$\begin{aligned}\text{a) } \int e^{-3x} dx \\ = -\frac{1}{3}e^{-3x} + c\end{aligned}$$

$$\begin{aligned}\text{b) } \int 2e^{2x} dx \\ = \frac{2}{2}e^{2x} + c \\ = e^{2x} + c\end{aligned}$$

$$\begin{aligned}\text{c) } \int e^{2x-1} dx \\ = \frac{e^{2x-1}}{2} + c\end{aligned}$$

$$\begin{aligned}\text{d) } \int e^{\frac{-x}{2}} dx \\ = \frac{e^{\frac{-x}{2}}}{-\frac{1}{2}} + c \\ = -2e^{\frac{-x}{2}} + c\end{aligned}$$

$$\begin{aligned}\text{e) } \int 4e^{\frac{x}{2}} dx \\ = \frac{4e^{\frac{x}{2}}}{\frac{1}{2}} + c \\ = 8e^{\frac{x}{2}} + c\end{aligned}$$

$$\begin{aligned}\text{f) } \int 5e^{2x} + \frac{1}{e^{3x}} dx \\ = \int 5e^{2x} + e^{-3x} dx \\ = \frac{5e^{2x}}{2} + \frac{e^{-3x}}{-3} + c \\ = \frac{5e^{2x}}{2} - \frac{1}{3e^{3x}} + c\end{aligned}$$

$$\begin{aligned}\text{g) } \int \frac{1}{\sqrt{x}} e^{\sqrt{x}} dx \\ = 2 \int \frac{1}{2\sqrt{x}} e^{\sqrt{x}} dx \\ = 2e^{\sqrt{x}} + c\end{aligned}$$

$$\begin{aligned}\text{h) } \int 3xe^{x^2} dx \\ = \frac{3}{2} \int 2xe^{x^2} dx \\ = \frac{3}{2}e^{x^2} + c\end{aligned}$$



EXERCISE 9

Find each of the following integrals:

a) $\int e^{3x+1} dx$ [Ans: $\frac{1}{3}e^{3x+1} + c$]	b) $\int 6e^{3-2x} dx$ [Ans: $-3e^{3-2x} + c$]
c) $\int \left(e^x + \frac{1}{e^x} \right)^2 dx$ [Ans: $\frac{1}{2}e^{2x} + 2x - \frac{1}{2}e^{-2x} + c$]	d) $\int x^2 e^{x^3} dx$ [Ans: $\frac{1}{3}e^{x^3} + c$]
e) $\int \frac{e^x}{x^2} dx$ [Ans: $-\frac{1}{x}e^x + c$]	f) $\int \frac{e^{\sqrt{x}}}{\sqrt{x}} dx$ [Ans: $2e^{\sqrt{x}} + c$]
g) $\int e^{\frac{1}{2}x+1} dx$ [Ans: $2e^{\frac{1}{2}x+1} + c$]	h) $\int \cos 2x e^{\sin 2x} dx$ [Ans: $\frac{1}{2}e^{\sin 2x} + c$]

i) $\int (2 - e^{2x})^2 dx$ [Ans: $4x - 2e^{2x} + \frac{1}{4}e^{4x} + c$]	j) $\int e^x (2 - e^x)^3 dx$ [Ans: $-\frac{(2 - e^x)^4}{4} + c$]
k) $\int \frac{2e^{2x} - e^x}{e^{-3x}} dx$ [Ans: $\frac{2}{5}e^{5x} - \frac{1}{4}e^{4x} + c$]	l) $\int \frac{3e^{-2x}}{(1 + e^{-2x})^3} dx$ [Ans: $\frac{3}{4(1 + e^{-2x})^2} + c$]
m) $\int 4x^3 e^{x^4} dx$ [Ans: $e^{x^4} + c$]	n) $\int \frac{-7e^{1-5y}}{21} dy$ [Ans: $\frac{1}{15}e^{1-5y} + c$]
o) $\int \frac{e^{-\frac{3}{2}p}}{3} dp$ [Ans: $-\frac{2}{3}e^{-\frac{3}{2}p} + c$]	p) $\int \frac{e^x}{\sqrt{1 + e^x}} dx$ [Ans: $2\sqrt{1 + e^x} + c$]



INTEGRATION OF TRIGONOMETRIC FUNCTIONS

Integration of trigonometric functions can be simplified as below:

$$1. \int \sin x \, dx = -\cos x + c$$

$$2. \int \cos x \, dx = \sin x + c$$

$$3. \int \sec^2 x \, dx = \tan x + c$$

$$4. \int \sec x \tan x \, dx = \sec x + c$$

$$5. \int \sin ax \, dx = -\frac{1}{a} \cos ax + c$$

$$6. \int \cos ax \, dx = \frac{1}{a} \sin ax + c$$

$$7. \int \sec^2 ax \, dx = \frac{1}{a} \tan ax + c$$

$$8. \int \sin (ax + b) \, dx = -\frac{1}{a} \cos (ax + b) + c$$

$$9. \int \cos (ax + b) \, dx = \frac{1}{a} \sin (ax + b) + c$$

$$10. \int \sec^2 (ax + b) \, dx = \frac{1}{a} \tan (ax + b) + c$$

$$11. \int a \sin (bx + d) \, dx = \frac{-a \cos (bx + d)}{b} + c$$

$$12. \int a \cos (bx + d) \, dx = \frac{a \sin (bx + d)}{b} + c$$

$$13. \int a \sec^2 (bx + d) \, dx = \frac{a \tan (bx + d)}{b} + c$$

$$14. \int \cos^2 x \, dx = \int \frac{1}{2} (1 + \cos 2x) \, dx = \frac{1}{2} (x + \frac{1}{2} \sin 2x) + c$$

$$15. \int \sin^2 x \, dx = \int \frac{1}{2} (1 - \cos 2x) \, dx = \frac{1}{2} (x - \frac{1}{2} \sin 2x) + c$$

$$16. \int \tan^2 x \, dx = \int (\sec^2 x - 1) \, dx = \tan x - x + c$$

**EXAMPLE 10**

Find each of the following indefinite trigonometric function:

a) $\int \sin 7x \, dx$

b) $\int \cos(-3x) \, dx$

c) $\int \sec^2 \frac{x}{2} \, dx$

d) $\int \sin(4x-3) \, dx$

e) $\int \cos(8-x) \, dx$

f) $\int \sec^2 \left(x + \frac{\pi}{2} \right) \, dx$

g) $\int 2 \sin 3x \, dx$

h) $\int 2 \sec^2(3x+9) \, dx$

i) $\int (1 + \cos 3x) \, dx$

j) $\int \sin x - \cos x \, dx$

k) $\int (5 \sec^2 x - 2 \cos x) \, dx$

l) $\int (3 \cos x - 5x^3) \, dx$

SOLUTION :

a) $\int \sin 7x \, dx$

$$= -\frac{1}{7} \cos 7x + c$$

b) $\int \cos(-3x) \, dx$

$$= -\frac{1}{3} \sin(-3x) + c$$

c) $\int \sec^2 \frac{x}{2} \, dx$

$$= 2 \tan \frac{x}{2} + c$$

d) $\int \sin(4x-3) \, dx$

$$= \frac{1}{4} \cos(4x-3) + c$$

e) $\int \cos(8-x) \, dx$

$$= -\sin(8-x) + c$$

f) $\int \sec^2 \left(x + \frac{\pi}{2} \right) \, dx$

$$= \tan \left(x + \frac{\pi}{2} \right) + c$$

$$\begin{aligned}
 \text{g) } \int 2 \sin 3x \, dx &= 2 \int \sin 3x \, dx \\
 &= 2 \left(\frac{1}{3} \right) (-\cos 3x) \, dx \\
 &= -\frac{2}{3} \cos 3x + c
 \end{aligned}$$

$$\begin{aligned}
 \text{h) } \int 2 \sec^2(3x+9) \, dx &= 2 \int \sec^2(3x+9) \, dx \\
 &= 2 \left(\frac{1}{3} \right) \tan(3x+9) + c \\
 &= \frac{2}{3} \tan(3x+9) + c
 \end{aligned}$$

$$\begin{aligned}
 \text{i) } \int (1 + \cos 3x) \, dx &= \int 1 \, dx + \int \cos 3x \, dx \\
 &= x + \frac{1}{3} \sin 3x + c
 \end{aligned}$$

$$\begin{aligned}
 \text{j) } \int (\sin x - \cos x) \, dx &= \int \sin x \, dx - \int \cos x \, dx \\
 &= -\cos x - \sin x + c
 \end{aligned}$$

$$\begin{aligned}
 \text{k) } \int (5 \sec^2 x - 2 \cos x) \, dx &= \int 5 \sec^2 x \, dx - \int 2 \cos x \, dx \\
 &= 5 \int \sec^2 x \, dx - 2 \int \cos x \, dx \\
 &= 5 \tan x - 2 \sin x + c
 \end{aligned}$$

$$\begin{aligned}
 \text{l) } \int (3 \cos x - 5x^3) \, dx &= \int 3 \cos x \, dx - \int 5x^3 \, dx \\
 &= 3 \int \cos x \, dx - \int 5x^3 \, dx \\
 &= 3 \sin x - \frac{5x^4}{4} + c
 \end{aligned}$$



EXERCISE 10

Integrate the following with respect to x :

$$\text{a) } \int \sin 4x \, dx \quad \left[\text{Ans: } -\frac{1}{4} \cos 4x + c \right]$$

$$\text{b) } \int 2 \cos 11x \, dx \quad \left[\text{Ans: } \frac{2}{11} \sin 11x + c \right]$$

c) $\int 6\sec^2(4-2x) dx$ [Ans: $-3\tan(4-2x)+c$]	d) $\int \sin\left(\frac{\pi}{4}-2x\right) dx$ [Ans: $\frac{1}{2}\cos\left(\frac{\pi}{4}-2x\right)+c$]
e) $\int 4\cos(5x+1) dx$ [Ans: $\frac{4}{5}\sin(5x+1)+c$]	f) $\int \sin\frac{13x}{3} dx$ [Ans: $-\frac{3}{13}\cos\frac{13x}{3}+c$]
g) $\int (5-4\sin(\pi-3x)) dx$ [Ans: $5x-\frac{4}{3}\cos(\pi-3x)+c$]	h) $\int (1+2\cos x) dx$ [Ans: $x-2\sin x+c$]

i) $\int (7 \cos 5x + 4 \sin 3x) dx$ [Ans: $\frac{7}{5} \sin 5x - \frac{4}{3} \cos(-3x) + c$]	j) $\int \sec x (\tan x + \sec x) dx$ [Ans: $\sec x + \tan x + c$]
k) $\int \left(\cos \frac{3x}{2} + \sin \frac{x}{2} \right) dx$ [Ans: $\frac{2}{3} \sin \frac{3x}{2} + 2 \cos \frac{x}{2} + c$]	l) $\int (\cos(x + \alpha) - \sin(7x - \alpha)) dx$ [Ans: $\sin(x + \alpha) + \frac{1}{7} \cos(7x - \alpha) + c$]

Trigonometry Identity:

To find the integrals of $\sin^2 x$ and $\cos^2 x$, we use the following trigonometric identities

i. Double Angle Formulae:

$$\begin{aligned}\sin 2A &= 2 \sin A \cos A \\ \cos 2A &= \cos^2 A - \sin^2 A \\ &= 2\cos^2 A - 1 \\ &= 1 - 2\sin^2 A \\ \tan 2A &= \frac{2 \tan A}{1 - \tan^2 A}\end{aligned}$$

ii. Basic Identities:

$$\begin{aligned}\sin^2 A + \cos^2 A &= 1 \\ 1 + \tan^2 A &= \sec^2 A \\ 1 + \cot^2 A &= \operatorname{cosec}^2 A\end{aligned}$$

iii. Addition Formulae:

$$\begin{aligned}\sin(A \pm B) &= \sin A \cos B \pm \cos A \sin B \\ \cos(A \pm B) &= \cos A \cos B \mp \sin A \sin B \\ \tan(A \pm B) &= \frac{\tan A \pm \tan B}{1 \mp \tan A \tan B}\end{aligned}$$

Integration by using Substitution involved Trigonometric Identity:

- For the trigonometric question that have index (power), use the trigonometric identity.
- Step to solve:
 - a. Change the functions using suitable trigonometric identity.
 - b. Integrate the new function.

**EXAMPLE 11**

Find each of the following indefinite trigonometric function:

a) $\int \sin^2 3x \, dx$

b) $\int \cos^2 5x \, dx$

c) $\int \tan^2 x \, dx$

d) $\int \sin \frac{1}{2}\theta + \cos \frac{1}{2}\theta \, d\theta$

SOLUTION :

a) $\int \sin^2 3x \, dx$

$$\cos 2A = 1 - 2\sin^2 A$$

$$2\sin^2(3x) = 1 - \cos 2(3x)$$

$$\sin^2(3x) = \frac{1 - \cos(6x)}{2}$$

$$\sin^2(3x) = \frac{1}{2}(1 - \cos 6x)$$

$$\int \sin^2 x \, dx = \int \frac{1}{2}(1 - \cos 2x) \, dx = \frac{1}{2}\left(x - \frac{1}{2}\sin 2x\right) + c$$

$$\int \sin^2 3x \, dx = \int \frac{1}{2}(1 - \cos 2(3x)) \, dx = \int \frac{1}{2}(1 - \cos 6x) \, dx$$

$$\int \frac{1}{2}(1 + \cos 6x) \, dx = \frac{1}{2} \int 1 \, dx - \int \cos 6x \, dx$$

$$\int \frac{1}{2}(1 + \cos 6x) \, dx = \frac{1}{2} \int 1 \, dx - \int \cos 6x \, dx$$

$$= \frac{1}{2} \int 1 \, dx - \int \cos 6x \, dx$$

$$= \frac{1}{2} \left(x - \frac{1}{6} \sin 6x \right) + c$$

$$= \frac{1}{2}x - \frac{1}{12} \sin 6x + c$$

b) $\int \cos^2 5x \, dx$

$$\cos 2A = 2\cos^2 A - 1$$

$$2\cos^2(5x) = \cos 2(5x) + 1$$

$$\cos^2(5x) = \frac{\cos(10x) + 1}{2}$$

$$\cos^2(5x) = \frac{1}{2}(\cos 10x + 1)$$

$$\int \cos^2 x \, dx = \int \frac{1}{2}(1 + \cos 2x) \, dx = \frac{1}{2}\left(x + \frac{1}{2}\sin 2x\right) + c$$

$$\int \cos^2 5x \, dx = \int \frac{1}{2}(1 + \cos 2(5x)) \, dx = \int \frac{1}{2}(1 + \cos 10x) \, dx$$

$$\int \frac{1}{2}(\cos 10x + 1) \, dx = \frac{1}{2} \int \cos 10x \, dx + \int \frac{1}{2} \, dx \quad \int \frac{1}{2}(1 + \cos 10x) \, dx = \frac{1}{2} \int 1 \, dx + \int \cos 10x \, dx$$

$$= \frac{1}{2} \int \cos 10x \, dx + \int \frac{1}{2} \, dx$$

$$= \frac{1}{2} \left(\frac{1}{10} \sin 10x + x \right) + c$$

$$= \frac{1}{20} \sin 10x + \frac{1}{2} x + c$$

c) $\int \tan^2 x \, dx$

$$1 + \tan^2 A = \sec^2 A$$

$$\tan^2 x = \sec^2 x - 1$$

$$= \int \sec^2 x \, dx - \int 1 \, dx$$

$$= \tan x - x + c$$

d) $\int \sin \frac{1}{2} \theta + \cos \frac{1}{2} \theta \, d\theta$

$$= \int \sin \frac{1}{2} \theta \, d\theta + \int \cos \frac{1}{2} \theta \, d\theta$$

$$= \frac{-\cos \frac{1}{2} \theta}{\frac{1}{2}} + \frac{\sin \frac{1}{2} \theta}{\frac{1}{2}} + c$$

$$= -2 \cos \frac{1}{2} \theta + 2 \sin \frac{1}{2} \theta + c$$



EXERCISE 11

Find each of the following indefinite integrals:

a) $\int 3 \sin^2 2x \, dx$

[Ans: $\frac{1}{2} \left(3x - \frac{3}{4} \sin 4x \right) + c$]

b) $\int \frac{1}{2} \cos^2 \frac{1}{2} x \, dx$

[Ans: $\frac{1}{4} (\sin x + x) + c$]

c) $\int \cos^3 x \sin^2 x \, dx$

[Ans: $\frac{1}{3} \sin^3 x - \frac{1}{5} \sin^5 x + c$]

d) $\int \cos^2 x \sin^3 x \, dx$

[Ans: $\frac{1}{5} \cos^5 x - \frac{1}{3} \cos^3 x + c$]



INTEGRAL OF INVERSE TRIGONOMETRIC FUNCTION

The following is a list of integration formula for inverse trigonometric function.

$$\text{i. } \int \frac{1}{\sqrt{a^2 - u^2}} du = \sin^{-1} \frac{u}{a} + c$$

$$\text{ii. } \int \frac{-1}{\sqrt{a^2 - u^2}} du = \cos^{-1} \frac{u}{a} + c$$

$$\text{iii. } \int \frac{1}{a^2 + u^2} du = \frac{1}{a} \tan^{-1} \frac{u}{a} + c$$

$$\text{iv. } \int \frac{-1}{a^2 + u^2} du = \frac{1}{a} \cot^{-1} \frac{u}{a} + c$$

$$\text{v. } \int \frac{1}{u\sqrt{u^2 - a^2}} du = \frac{1}{a} \sec^{-1} \left| \frac{u}{a} \right| + c$$

$$\text{vi. } \int \frac{-1}{u\sqrt{u^2 - a^2}} du = \frac{1}{a} \operatorname{cosec}^{-1} \left| \frac{u}{a} \right| + c$$

**EXAMPLE 12**

Find the integrals:

a) $\int \frac{-dx}{\sqrt{16-x^2}}$

b) $\int \frac{dx}{x\sqrt{4x^4-49}}$

SOLUTION :

a) $\int \frac{-dx}{\sqrt{16-x^2}}$

$a^2 = 16$

$a = 4$

$u^2 = x^2$

$u = x$

$\frac{du}{dx} = 1$

$du = dx$

$\therefore \int \frac{-dx}{\sqrt{4^2-x^2}}$

$a = 4, u = x \text{ and } du = dx \rightarrow \int \frac{-1}{\sqrt{a^2-u^2}} du$

$= \cos^{-1} \frac{u}{a} + c$

$= \cos^{-1} \frac{x}{4} + c$

$$\text{b) } \int \frac{dx}{x\sqrt{4x^4 - 49}}$$

$$a^2 = 49$$

$$u^2 = 4x^4$$

$$a = 7$$

$$u = 2x^2$$

$$\frac{du}{dx} = 4x$$

$$\frac{du}{4x} = dx$$

$$\therefore \int \frac{dx}{x\sqrt{(2x^2)^2 - 7^2}}$$

$$a = 7, u = 2x^2 \text{ and } dx = \frac{du}{4x} \rightarrow \int \frac{1}{x\sqrt{u^2 - a^2}} \frac{du}{4x}$$

$$= \frac{1}{4} \int \frac{1}{x^2 \sqrt{u^2 - a^2}} du$$

$$= \frac{1}{4} \int \frac{1}{\frac{u}{2} \sqrt{u^2 - a^2}} du$$

$$= \frac{1}{4} \int \frac{2}{u \sqrt{u^2 - a^2}} du$$

$$= \frac{1}{2} \int \frac{1}{u \sqrt{u^2 - a^2}} du$$

$$= \frac{1}{2} \left(\frac{1}{a} \sec^{-1} \left| \frac{u}{a} \right| + c \right)$$

$$= \frac{1}{2} \bullet \frac{1}{7} \sec^{-1} \left| \frac{2x^2}{7} \right| + c$$

$$= \frac{1}{14} \sec^{-1} \left| \frac{2x^2}{7} \right| + c$$

**EXERCISE 12**

Find the integrals:

a) $\int \frac{2}{\sqrt{4-x^2}} dx$

[Ans: $2 \sin^{-1} \frac{x}{2} + c$]

b) $\int \frac{-z \, dz}{\sqrt{49 - 3z^4}}$

[Ans: $\frac{1}{2\sqrt{3}} \cos^{-1} \frac{\sqrt{3}z^2}{7} + c$]

c) $\int \frac{e^{3x}}{\sqrt{7-e^{6x}}} dx$

[Ans: $\frac{1}{3} \sin^{-1} \frac{e^{3x}}{\sqrt{7}} + c$]

d) $\int \frac{4x^2}{x^6 + 6} dx$

[Ans: $\frac{4}{3\sqrt{6}} \tan^{-1} \frac{x^3}{\sqrt{6}} + c$]



INTEGRATION BY PART

Integration by parts is a rule that transforms the integral of products of functions into other (ideally simpler) integrals. The rule can be derived in one line by simply integrating the product rule of differentiation.

If $u = f(x)$, $v = g(x)$ and the differentials $du = f'(x) dx$ and $dv = g'(x) dx$, then

integration by parts states that $\int f(x)g'(x) dx = f(x)g(x) - \int f'(x)g(x) dx$

$$\int u dv = uv - \int v du$$

When we using the integration by parts, we use a rule proposed by Herbert Kasube of Bradley University advises that whichever function comes first in the following list should be **u**.

- Logarithmic functions: $\ln x$, $\log_b x$, etc
- Inverse trigonometric functions: $\arctan x$, $\operatorname{arcsec} x$, etc
- Algebraic functions: x^2 , $5x^{17}$, etc
- Trigonometric functions: $\sin x$, $\tan x$, etc
- Exponential functions: e^x , etc

And the function which is to be dv is whichever comes last in the list.

EXAMPLE

FUNCTIONS	FROM THE RULE
$\int x \cos x dx$	$u = x$ $dv = \cos x dx$
$\int x^2 e^{2x} dx$	$u = x^{2x}$ $dv = e^{2x} dx$
$\int x^2 \ln 2x dx$	$u = \ln 2x$ $dv = x^2 dx$
$\int x \tan^{-1} x dx$	$u = \tan^{-1} x$ $dv = x dx$

**EXAMPLE 13**

Find the following indefinite integrals by using integration by parts:

a) $\int x e^x dx$

b) $\int x \cos x dx$

c) $\int x \ln x dx$

d) $\int x^2 \sin 2x dx$

SOLUTION :

a) $\int x e^x dx$

$u = x$

$dv = e^x dx$

$\frac{du}{dx} = 1$

$v = \int e^x dx$

$du = dx$

$= e^x$

$$\int u dv = u v - \int v du$$

$$= x(e^x) - \int e^x dx$$

$$= x(e^x) - e^x + c$$

b) $\int x \cos x dx$

$u = x$

$dv = \cos x dx$

$\frac{du}{dx} = 1$

$v = \int \cos x dx$

$du = dx$

$= \sin x$

$$\int u dv = u v - \int v du$$

$$= x(\sin x) - \int (\sin x) dx$$

$$= x(\sin x) - (-\cos x) + c$$

$$= x(\sin x) + \cos x + c$$

c) $\int x \ln x \, dx$

$$u = \ln x$$

$$\frac{du}{dx} = \frac{1}{x}$$

$$du = \frac{dx}{x}$$

$$dv = x \, dx$$

$$v = \int x \, dx$$

$$= \frac{x^2}{2}$$

$$\int u \, dv = u \, v - \int v \, du$$

$$= (\ln x) \left(\frac{x^2}{2} \right) - \int \left(\frac{x^2}{2} \right) \frac{dx}{x}$$

$$= (\ln x) \left(\frac{x^2}{2} \right) - \frac{1}{2} \int (x) \, dx$$

$$= \frac{x^2}{2} \ln x - \frac{x^2}{4} + c$$

d) $\int x^2 \sin 2x \, dx$

$$u = x^2$$

$$dv = \sin 2x \, dx$$

$$\frac{du}{dx} = 2x$$

$$v = \int \sin 2x \, dx$$

$$du = 2x \, dx$$

$$= -\frac{1}{2} \cos 2x$$

$$\int u \, dv = u \, v - \int v \, du$$

$$= x^2 \left(-\frac{1}{2} \cos 2x \right) - \int \left(-\frac{1}{2} \cos 2x \right) 2x \, dx$$

$$= x^2 \left(-\frac{1}{2} \cos 2x \right) + \int x \cos 2x \, dx$$

$$u = x$$

$$dv = \cos 2x$$

$$\frac{du}{dx} = 1$$

$$v = \int \cos 2x \, dx$$

$$du = dx$$

$$= \frac{1}{2} \sin 2x$$

$$= \left(-\frac{x^2}{2} \cos 2x \right) + \left[x \left(\frac{1}{2} \sin 2x \right) - \int \left(\frac{1}{2} \sin 2x \right) dx \right]$$

$$= -\frac{x^2}{2} \cos 2x + \frac{x}{2} \sin 2x + \frac{\cos 2x}{4} + c$$

**EXERCISE 13**

Solve the integrals by using integration by parts:

a) $\int x \sin 2x \, dx$

[Ans: $-\frac{x \cos x}{2} + \frac{\sin 2x}{4} + c$]

b) $\int \ln(2+3x) \, dx$

[Ans: $x \ln(2+3x) - x + \frac{2}{3} \ln(2+3x) + c$]

c) $\int 2xe^x dx$

[Ans: $2e^x[x-1] + c$]

d) $\int 3x \sin x dx$

[Ans: $-3x \cos x + 3 \sin x + c$]

e) $\int x^2 e^{2x} dx$

[Ans: $\frac{1}{2}e^{2x}\left[x^2 - x + \frac{1}{2}\right] + c$]

f) $\int x^2 e^{-x} dx$

[Ans: $e^{-x}(x^2 + 2x + 2) + c$]

g) $\int x^3 e^{x^2} dx$

$$[\text{Ans: } \frac{x^2 e^{x^2}}{2} - \frac{e^{x^2}}{2} + c]$$

h) $\int e^x \cos x dx$

$$[\text{Ans: } \frac{1}{2} e^x (\cos x + \sin x) + c]$$



INTEGRATION BY PARTIAL FRACTION

In algebra, the partial fraction decomposition or partial fraction expansion is a procedure used to reduce the degree of either the numerator or the denominator of a rational function (also known as a rational algebraic fraction).

In symbols, one can use partial fraction expansion to change a rational function in the form

$$\frac{f(x)}{g(x)} = \frac{n}{d} = \frac{\text{numerator}}{\text{denominator}}$$

The numerator and denominator are called the terms of the algebraic fraction

- If the degree of numerator is less than denominator in a rational algebraic function, $f(x) < g(x)$, It is called **PROPER FRACTION**
- If the degree of numerator is greater than denominator in a rational algebraic function, $f(x) > g(x)$, It is called **IMPROPER FRACTION**

Technique of partial fractions $\frac{f(x)}{g(x)}$, which can be written as the sum of partial fractions as follows

$$\frac{f(x)}{g(x)} = F_1 + F_2 + F_3 + \dots, \text{ where}$$

- Each F has the form $\frac{A_i}{(ax+b)^n}$ or $\frac{B_1x+C_1}{(ax^2+bx+c)^m}$, $m, n \geq 1$.
- $g(x)$ is usually expressed as the product of linear factors $(ax+b)$ or quadratic factors (ax^2+bx+c)
- If the degree of $f(x) >$ degree of $g(x)$, use long division to get the proper form of
$$\frac{f(x)}{g(x)} = h(x) + \frac{z(x)}{g(x)}$$

Table of Partial Fraction Decomposition

Algebraic fraction can be expressed as partial fraction as following:

$\frac{f(x)}{g(x)}$	Term in Partial Fraction Decomposition	Partial Fraction Decomposition
$\frac{2x}{(x+1)(x-3)}$	$\frac{A}{(x+1)} + \frac{B}{(x-3)}$	Proper fraction with Linear Factor
$\frac{3-x}{(x+1)^2}$	$\frac{A}{(x+1)} + \frac{B}{(x+1)^2}$	Proper fraction with Repeated Linear Factor
$\frac{x^2+1}{(1-x)(x^2+3)}$	$\frac{A}{(1-x)} + \frac{Bx+C}{(x^2+3)}$ $\uparrow$ quadratic	Proper fraction with Quadratic Factor

Integration by partial fractions has two stages:

- i) Get the form of partial fractions to be integrated
- ii) Each partial fraction was integrated separately

**EXAMPLE 14**

Find each of the following indefinite integrals by using integration by partial fraction:

a) $\int \frac{2x}{(x+1)(x-3)} dx$

b) $\int \frac{3x+2}{(4x-3)^2} dx$

c) $\int \frac{x^2+1}{(x^2+3)(1-x)} dx$

d) $\int \frac{x^3-2}{(x+1)(x-2)} dx$

SOLUTION :

a) $\int \frac{2x}{(x+1)(x-3)} dx$

$$\frac{2x}{(x+1)(x-3)} = \frac{A}{(x+1)} + \frac{B}{(x-3)}$$

$$2x = A(x-3) + B(x+1)$$

when $x = -1$

$$2(-1) = A(-1-3)$$

$$-2 = -4A$$

$$A = \frac{1}{2}$$

when $x = 3$

$$2(3) = B(3+1)$$

$$6 = 4B$$

$$B = \frac{3}{2}$$

$$\int \frac{2x}{(x+1)(x-3)} dx = \int \frac{A}{(x+1)} dx + \int \frac{B}{(x-3)} dx$$

$$= \int \frac{\frac{1}{2}}{(x+1)} dx + \int \frac{\frac{3}{2}}{(x-3)} dx$$

$$= \frac{1}{2} \ln(x+1) + \frac{3}{2} \ln(x-3) + c$$

$$\text{b) } \int \frac{3x+2}{(4x-3)^2} dx$$

$$\frac{3x+2}{(4x-3)^2} = \frac{A}{(4x-3)} + \frac{B}{(4x-3)^2}$$

$$3x+2 = A(4x-3) + B$$

exp and

$$3x+2 = 4Ax - 3A + B$$

$$\text{when } x = \frac{3}{4}$$

coefficients x^0

$$2 = -3A + B$$

$$3\left(\frac{3}{4}\right) + 2 = B$$

$$2 = -3A + \left(\frac{17}{4}\right)$$

$$B = \frac{17}{4}$$

$$A = \frac{3}{4}$$

$$\int \frac{3x+2}{(4x-3)^2} dx = \int \frac{A}{(4x-3)} dx + \int \frac{B}{(4x-3)^2} dx$$

$$= \int \frac{\frac{3}{4}}{(4x-3)} dx + \int \frac{\frac{17}{4}}{(4x-3)^2} dx$$

$$= -\frac{3}{16} \ln(4x-3) - \frac{17}{16(4x-3)} + c$$

c) $\int \frac{x^2+1}{(x^2+3)(1-x)} dx$

$$\frac{x^2+1}{(1-x)(x^2+3)} = \frac{A}{(1-x)} + \frac{Bx+C}{(x^2+3)}$$

$$x^2+1 = A(x^2+3) + (Bx+C)(1-x)$$

exp and

$$x^2+1 = Ax^2 + 3A + Bx - Bx^2 + C - Cx$$

when $x=1$

$$(1)^2+1 = A((1)^2+3)$$

$$A = \frac{1}{2}$$

coefficients x^2

$$1 = A - B$$

$$1 = \left(\frac{1}{2}\right) - B$$

$$B = -\frac{1}{2}$$

coefficients x

$$0 = B - C$$

$$0 = \left(-\frac{1}{2}\right) - C$$

$$C = -\frac{1}{2}$$

$$\int \frac{x^2+1}{(1-x)(x^2+3)} dx = \int \frac{A}{(1-x)} dx + \int \frac{Bx+C}{(x^2+3)} dx$$

$$= \int \frac{\frac{1}{2}}{(1-x)} dx - \int \frac{\frac{1}{2}x + \frac{1}{2}}{(x^2+3)} dx$$

$$= -\frac{1}{2} \ln(1-x) - \frac{1}{2} \int \frac{x+1}{x^2+3} dx$$

$$= -\frac{1}{2} \ln(1-x) - \frac{1}{2} \left[\int \frac{x}{x^2+3} dx + \int \frac{1}{x^2+3} dx \right]$$

$$= -\frac{1}{2} \ln(1-x) - \frac{1}{4} \ln(x^2+3) - \frac{1}{2\sqrt{3}} \tan^{-1}\left(\frac{x}{\sqrt{3}}\right) + c$$

$$d) \int \frac{x^3 - 2}{(x+1)(x-2)} dx = \int \frac{x^3 - 2}{x^2 - x - 2} dx$$

$$\begin{aligned} x^2 - x - 2 \overline{) \frac{x+1}{x^3 - 2}} \\ \underline{(-)x^3 - x^2 - 2x} \\ (-)x^2 - x - 2 \\ \underline{3x} \end{aligned} = (x+1) + \frac{3x}{x^2 - x - 2}$$

$$\int \frac{3x}{(x+1)(x-2)} dx = \frac{A}{(x+1)} + \frac{B}{(x-2)}$$

$$3x = A(x-2) + B(x+1)$$

$$\text{when } x = 2$$

$$3(2) = B(2+1)$$

$$6 = 3B$$

$$B = 2$$

$$\text{when } x = -1$$

$$3(-1) = A(-1-2)$$

$$-3 = -3A$$

$$A = 1$$

$$\int \frac{3-x}{(x+1)^2} dx = \int \frac{A}{(x+1)} dx + \int \frac{B}{(x+1)^2} dx$$

$$\int \frac{3x}{(x+1)(x-2)} dx = \int (x+1) dx + \int \frac{A}{(x+1)} dx + \int \frac{B}{(x-2)} dx$$

$$= \left(\frac{x^2}{2} + x \right) + \ln(x+1) + 2\ln(x-2)$$



EXERCISE 14

Find each of the following integrals by using integration by partial fraction:

a) $\int \frac{x-1}{(x-2)(3x-2)} dx$ [Ans: $\frac{1}{4} \ln(x-2)(3x-2) + c$]

b) $\int \frac{6}{x^2 + 2x - 8} dx$ [Ans: $\ln \left(\frac{x-2}{x+4} \right) + c$]

c) $\int \frac{6x-1}{x^2-4x^3} dx$

[Ans: $2 \ln x + \frac{1}{x} - 2 \ln(1-4x) + c$]

d) $\int \frac{dx}{x^2-9}$

[Ans: $\frac{1}{6} \ln \left(\frac{x-3}{x+3} \right) + c$]

e) $\int \frac{x^2 + 3x - 10}{x^2 - 2x - 3} dx$

[Ans : $x + 3\ln(x+1) + 2\ln(x-3) + c$]

Find each of the following integrals by using integration by partial fractions

f) $\int \frac{3x^3 - x^2 - 13x - 13}{x^2 - x - 6} dx$

[Ans : $\frac{3}{2}x^2 + 2x + 3\ln(x+2) + 4\ln(x-3) + c$]

g) $\int \frac{2x-3}{x(x^2+1)} dx$

[Ans: $-3 \ln x + \frac{3}{2} \ln(x^2+1) + 2 \tan^{-1} x + c$]

h) $\int \frac{1}{x(1+x^2)} dx$

[Ans: $\ln x - \frac{1}{2} \ln(1+x^2) + c$]

e) $\int \frac{x^2 + 3x - 10}{x^2 - 2x - 3} dx$

[Ans: $x + 3\ln(x+1) + 2\ln(x-3) + c$]

For each of the following integrals by using integration by partial fractions:

f) $\int \frac{3x^3 - x^2 - 13x - 13}{x^2 - x - 6} dx$

[Ans: $\frac{3}{2}x^2 + 2x + 3\ln(x+2) + 4\ln(x-3) + c$]

g) $\int \frac{2x-3}{x(x^2+1)} dx$

[Ans: $-3\ln x + \frac{3}{2}\ln(x^2+1) + 2\tan^{-1}x + c$]

h) $\int \frac{1}{x(1+x^2)} dx$

[Ans: $\ln x - \frac{1}{2}\ln(1+x^2) + c$]



DEFINITE INTEGRALS

Definite integral is integrals with the limits of integration. Refer to the following equation, **a** is a lower limit of integration and **b** is an upper limit of integration. $\int_a^b f(x) dx$ is read as 'the integral from a to b of $f(x)$ with respect to x '.

$$\int_a^b f(x) dx$$

State the Fundamental Theorem of Calculus

If a function $f(x)$ is continuous on the interval $[a, b]$, then

$$\int_a^b f(x) dx = F(b) - F(a)$$

State the Properties of Definite Integrals

No	Properties of Definite Integrals
1	$\int_a^a f(x) dx = 0$
2	$\int_b^a f(x) dx = -\int_a^b f(x) dx$
3	$\int_a^b kf(x) dx = k \int_a^b f(x) dx$, where k is a constant
4	$\int_a^b [f(x) + g(x)] dx = \int_a^b f(x) dx + \int_a^b g(x) dx$
5	$\int_a^c f(x) dx = \int_a^b f(x) dx + \int_b^c f(x) dx$, where $a < b < c$

**EXAMPLE 15**

Evaluate definite integral by using properties of definite integral:

a) $\int_{-4}^{-2} \left(x^2 + \frac{1}{x^3} \right) dx$

b) $\int_0^5 \frac{3x^2}{\sqrt{x^3 + 2}} dx$

c) $\int_0^{2\pi} \sin x \cos x dx$

d) $\int_{\frac{\pi}{2}}^{2\pi} \sin 2\theta dx$

e) $\int_1^8 y \left(\sqrt[3]{5y^2} \right) dy$

f) $\int_{\pi}^{\frac{3\pi}{2}} \sqrt{\sin x + 1} \cos x dx$

g) $\int_0^4 e^{\sin x} \cos x dx$

h) $\int_{\frac{1}{2}}^2 \frac{e^x dx}{\sqrt{9 - e^{2x}}}$

SOLUTION:

$$\text{a) } \int_{-4}^{-2} \left(x^2 + \frac{1}{x^3} \right) dx$$

$$= \int_{-4}^{-2} (x^2 + x^{-3}) dx$$

$$= \left[\frac{x^3}{3} + \frac{x^{-2}}{(-2)} \right]_{-4}^{-2}$$

$$= \left[\frac{(-2)^3}{3} + \frac{(-2)^{-2}}{(-2)} \right] - \left[\frac{(-4)^3}{3} + \frac{(-4)^{-2}}{(-2)} \right]$$

$$= \left[-\frac{8}{3} + \left(-\frac{1}{8} \right) \right] - \left[-\frac{64}{3} + \left(-\frac{1}{32} \right) \right]$$

$$= \frac{1783}{96}$$

$$\text{b) } \int_0^5 \frac{3x^2}{\sqrt{x^3+2}} dx = \int_0^5 3x^2 (x^3+2)^{-\frac{1}{2}} dx$$

$$u = x^3 + 2$$

$$\frac{du}{dx} = 3x^2 \quad dx = \frac{du}{3x^2}$$

$$= \int_0^5 3x^2 (u)^{-\frac{1}{2}} \frac{du}{3x^2}$$

$$= \int_0^5 u^{-\frac{1}{2}} du$$

$$= \left[\frac{u^{\frac{1}{2}}}{\frac{1}{2}} \right]_0^5$$

$$= \left[2\sqrt{x^3+2} \right]_0^5$$

$$= \left[2\sqrt{5^3+2} - 2\sqrt{0^3+2} \right]$$

$$= 19.7105$$

$$\text{c) } \int_0^{2\pi} \sin x \cos x dx$$

$$u = \sin x$$

$$\frac{du}{dx} = \cos x$$

$$dx = \frac{du}{\cos x}$$

$$= \int_0^{2\pi} u (\cos x) \left(\frac{du}{\cos x} \right)$$

$$= \int_0^{2\pi} u du$$

$$= \left[\frac{u^2}{2} \right]_0^{2\pi} = \left[\frac{(\sin x)^2}{2} \right]_0^{2\pi}$$

$$= \left[\frac{(\sin(2\pi))^2}{2} - \frac{(\sin(0))^2}{2} \right]$$

$$= 0$$

$$\text{d) } \int_{\frac{\pi}{2}}^{2\pi} \sin 2\theta d\theta$$

$$= \left[-\frac{1}{2} \cos 2\theta \right]_{\frac{\pi}{2}}^{2\pi}$$

$$= \left[-\frac{1}{2} \cos 2(2\pi) - \left(-\frac{1}{2} \cos 2\left(\frac{\pi}{2}\right) \right) \right]$$

$$= \left[-\frac{1}{2} - \left(\frac{1}{2} \right) \right]$$

$$= -1$$

$$e) \int_1^8 y \sqrt[3]{5y^2} \, dy$$

$$\int_1^8 y(5y^2)^{\frac{1}{3}} \, dy$$

$$u = 5y^2$$

$$\frac{du}{dy} = 10y$$

$$dy = \frac{du}{10y}$$

$$= \int_1^8 y(u)^{\frac{1}{3}} \left(\frac{du}{10y} \right)$$

$$= \left(\frac{1}{10} \right) \int_1^8 (u)^{\frac{1}{3}} \, du$$

$$= \frac{1}{10} \left[\frac{u^{\frac{4}{3}}}{\frac{4}{3}} \right]_1^8$$

$$= \frac{1}{10} \left[\frac{3^{\frac{4}{3}} \sqrt[3]{(5y^2)^4}}{4} \right]_1^8$$

$$= \frac{1}{10} \left[\frac{3^{\frac{4}{3}} \sqrt[3]{(5(8)^2)^4}}{4} - \frac{3^{\frac{4}{3}} \sqrt[3]{(5(1)^2)^4}}{4} \right]$$

$$= \frac{1}{10} [1641.58 - 6.41]$$

$$= 163.517$$

$$f) \int_{\pi}^{\frac{3\pi}{2}} \sqrt{\sin x + 1} \cos x \, dx$$

$$\int_{\pi}^{\frac{3\pi}{2}} (\sin x + 1)^{\frac{1}{2}} \cos x \, dx$$

$$u = \sin x + 1$$

$$\frac{du}{dx} = \cos x$$

$$dx = \frac{du}{\cos x}$$

$$= \int_{\pi}^{\frac{3\pi}{2}} (u)^{\frac{1}{2}} \cos x \frac{du}{\cos x}$$

$$= \int_{\pi}^{\frac{3\pi}{2}} (u)^{\frac{1}{2}} \, du$$

$$= \left[\frac{u^{\frac{3}{2}}}{\frac{3}{2}} \right]_{\pi}^{\frac{3\pi}{2}}$$

$$= \frac{2}{3} \left[(\sin x + 1)^{\frac{3}{2}} \right]_{\pi}^{\frac{3\pi}{2}}$$

$$= \frac{2}{3} \left[\left(\sin \left(\frac{3\pi}{2} \right) + 1 \right)^{\frac{3}{2}} - (\sin(\pi) + 1)^{\frac{3}{2}} \right]$$

$$= \frac{2}{3} [0 - 1]$$

$$= -\frac{2}{3}$$

$$g) \int_0^4 e^{\sin x} \cos x \, dx$$

$$u = \sin x$$

$$\frac{du}{dx} = \cos x$$

$$dx = \frac{du}{\cos x}$$

$$= \int_0^4 e^u \cos x \left(\frac{du}{\cos x} \right)$$

$$= \int_0^4 e^u \, du$$

$$= [e^u]_0^4$$

$$= [e^{\sin x}]_0^4$$

$$= [e^{\sin(4)} - e^{\sin(0)}]$$

$$= [1.0722 - 1]$$

$$= 0.0722$$

$$h) \int_{1/2}^1 \frac{e^x \, dx}{\sqrt{9 - e^{2x}}}$$

$$a^2 = 9$$

$$u^2 = e^{2x}$$

$$a = 3$$

$$u = e^x$$

$$\frac{du}{dx} = e^x$$

$$dx = \frac{du}{e^x}$$

$$\int_{1/2}^1 \frac{e^x}{\sqrt{3^2 - (e^x)^2}} \, dx$$

$$a = 3, u = e^x \text{ and } dx = \frac{du}{e^x} \rightarrow \int_{1/2}^1 \frac{e^x}{\sqrt{3^2 - (e^x)^2}} \, dx$$

$$\int_{1/2}^1 \frac{e^x}{\sqrt{a^2 - u^2}} \frac{du}{e^x}$$

$$= \int_{1/2}^1 \frac{1}{\sqrt{a^2 - u^2}} \, du$$

$$= \left[\sin^{-1} \left(\frac{u}{a} \right) \right]_{1/2}^1$$

$$= \left[\sin^{-1} \left(\frac{e^x}{3} \right) \right]_{1/2}^1$$

$$= \sin^{-1} \left(\frac{e^2}{3} \right) - \sin^{-1} \left(\frac{e^{1/2}}{3} \right)$$

$$= 64.971^\circ - 33.338^\circ$$

$$= 31.63^\circ @ 0.552 \text{ rad}$$

**EXERCISE 15**

Find the integrals:

a) $\int_0^1 (x^2 - 2x) dx$

[Ans: $-\frac{2}{3}$]

b) $\int_{-1}^1 6\left(\frac{y}{2} + y^2\right) dy$

[Ans: 4]

c) $\int_{-2}^2 \frac{2(m^2 + 2)(m^2 - 1)}{2m^2} dm$

[Ans: $\frac{34}{3}$]

d) $\int_1^3 \frac{1}{8 - 2x} dx$

[Ans: 0.549]

e) $\int_{\frac{1}{2}}^1 \frac{-dx}{\sqrt{5-4x^2}}$

[Ans: -18.435°]

f) $\int_1^2 \frac{3t}{t^2+4} dt$ (Substitution) [Ans: 0.705]

g) $\int_0^1 \frac{2y}{\sqrt{4-y^2}} dy$ (Substitution) [Ans: 0.536]

h) $\int_0^{\frac{\pi}{6}} \sin^2 \theta \cos^3 \theta \, d\theta$ (Substitution)

[Ans: 0.0354]

i) $\int_0^{\frac{\pi}{3}} \frac{3 \sin x}{(4 + 2 \cos x)^2} \, dx$ (Substitution)

[Ans: 0.05]

j) $\int_{-2}^1 (x+2)e^x \, dx$

[Ans: 5.572]

k) $\int_0^2 e^{2x-3} \, dx$

[Ans: 1.334]

Solve the integrals by using integration by parts:

a) $\int_1^2 \ln 2x \, dx$

[Ans : 1.0794]

b) $\int_0^1 x e^{5x} \, dx$

[Ans : 23.786]

c) $\int_{-2}^1 (x+2) e^x \, dx$

[Ans : 5.572]

Find each of the following integrals by using integration by partial fraction:

a) $\int_1^2 \frac{2x}{(x+1)(x+2)} dx$

[Ans : 0.34]

b) $\int_0^1 \frac{12}{(3-2x)(3+2x)} dx$

[Ans : 1.6094]

**EXAMPLE 16**

Given that $\int_2^3 f(x) dx = 3$, $\int_3^4 f(x) dx = \frac{5}{2}$. Find the following integrals.

a) $\int_2^3 20f(x) dx$

b) $\int_3^2 (-3f(x) - 2) dx$

c) $\int_2^4 f(x) dx$

d) $\int_3^4 (3x - 6f(x)) dx$

SOLUTION:

$$\begin{aligned} \text{a) } \int_2^3 20f(x) dx &= 20 \int_2^3 f(x) dx \\ &= 20 \cdot 3 \\ &= 60 \end{aligned}$$

$$\begin{aligned} \text{b) } \int_3^2 (-3f(x) - 2) dx &= - \int_2^3 (-3f(x) - 2) dx \\ &= 3 \int_2^3 f(x) dx + \int_2^3 2 dx \\ &= \left[(3)(3) + [2x]_2^3 \right] \\ &= [9 + [2(3) - 2(2)]] \\ &= 11 \end{aligned}$$

$$\begin{aligned} \text{c) } \int_2^4 f(x) dx &= \int_2^3 f(x) dx + \int_3^4 f(x) dx \\ &= 3 + \frac{5}{2} \\ &= \frac{11}{2} \end{aligned}$$

$$\begin{aligned} \text{d) } \int_3^4 (3x - 6f(x)) dx &= \int_3^4 3x dx - 6 \int_3^4 f(x) dx \\ &= \left[\frac{3x^2}{2} \right]_3^4 - 6 \left(\frac{5}{2} \right) \\ &= \left[\frac{3(4)^2}{2} - \frac{3(3)^2}{2} \right] - 15 \\ &= \frac{21}{2} - 15 \\ &= -\frac{9}{2} \end{aligned}$$



EXERCISE 16

1. Given that $\int_1^5 g(x) dx = -8$, find:

a) $\int_1^5 -2g(x) dx$ [Ans: 16]	b) $\int_1^5 (5g(x) - 6) dx$ [Ans: -64]
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2. If $\int_{-2}^2 f(x) dx = \frac{8}{3}$ and $\int_2^4 f(x) dx = \frac{10}{7}$, find the following:

a) $\int_{-2}^2 9f(x) dx + \int_2^4 f(x) dx$ [Ans: $\frac{178}{7}$]	b) $-\int_{-2}^2 f(x) dx$ [Ans: $-\frac{8}{3}$]
c) $\int_2^{-2} (f(x) + 3) dx - 2 \int_4^2 f(x) dx$ [Ans: $-\frac{248}{21}$]	d) $-5 \int_{-2}^2 f(x) dx + \int_4^2 f(x) dx$ [Ans: $-\frac{310}{21}$]

3. Find the value of following if $\int_3^6 f(x) dx = 9$, $\int_0^2 g(x) dx = 3$, and $\int_{-1}^1 h(x) dx = 7$

a) $\int_3^6 4f(x) dx + \int_{-1}^1 (-2h(x) + 5) dx$ [Ans : 32]	b) $6 \left(\int_0^2 g(x) dx - \int_{-1}^1 h(x) dx \right)$ [Ans : -24]
c) $\int_6^3 f(x) dx + 8 \int_2^0 g(x) dx - \int_{-1}^1 h(x) dx$ [Ans : -40]	d) $\int_0^2 \left(g(x) + \frac{3}{2} \right) dx + \int_6^3 (-2f(x)) dx$ [Ans : 24]



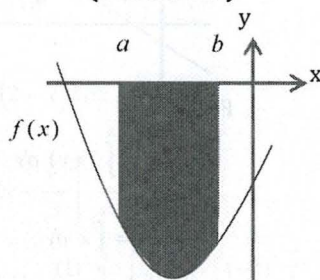
APPLY THE TECHNIQUES OF INTEGRATION

Calculate area of the graph by using integration along x-axis and y axis

Area of the Graph (x-axis)

Area of Shaded Region

(Below x-axis)



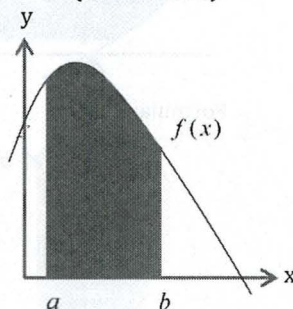
Formula:

$$A = -\int_a^b f(x) \, dx$$

$$A = \left| \int_a^b y \, dx \right|$$

Area of Shaded Region

(Above x-axis)



Formula:

$$A = \int_a^b f(x) \, dx$$

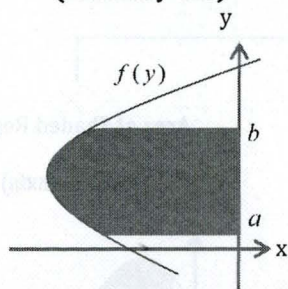
$$A = \int_a^b y \, dx$$

If the area of shaded region is below x-axis, then the value is negative. Therefore the formula of area is written as $A = \left| \int_a^b f(x) \, dx \right|$ since the area is always positive. While the value is positive if the area of shaded region is above x-axis.

Area of the Graph (y-axis)

Area of Shaded Region

(Left side y-axis)



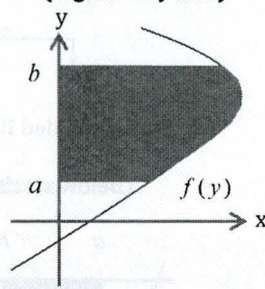
Formula:

$$A = - \int_a^b f(y) dy$$

$$A = \left| \int_a^b x dy \right|$$

Area of Shaded Region

(Right side y-axis)



Formula:

$$A = \int_a^b f(y) dy$$

$$A = \int_a^b x dy$$

If the area of shaded region is left side y-axis, then the value is negative. Therefore the formula of area is written as $A = \left| \int_a^b x dy \right|$. While the value is positive if the area of shaded region is right side y-axis.



EXAMPLE 17

Find the area of the region bounded by the curve $y = 2 - x^2$, x -axis with the limit given $x = 1$ and $x = -1$

SOLUTION:

$$A = \int_a^b y \, dx$$

$$A = \int_{-1}^1 (2 - x^2) \, dx$$

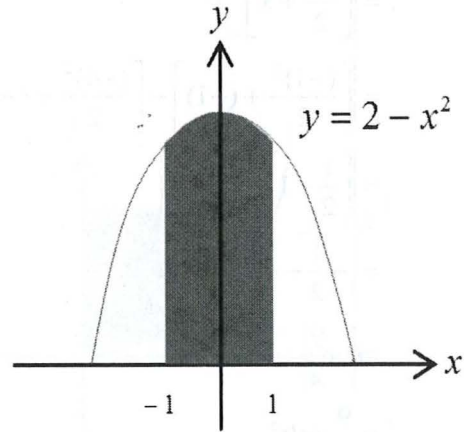
$$= \left[2x - \frac{x^3}{3} \right]_{-1}^1$$

$$= \left[2(1) - \frac{(1)^3}{3} \right] - \left[2(-1) - \frac{(-1)^3}{3} \right]$$

$$= \left[2 - \frac{1}{3} \right] - \left[-2 + \frac{1}{3} \right]$$

$$= \frac{5}{3} - \left(-\frac{5}{3} \right)$$

$$= \frac{10}{3} \text{ unit}^2$$



Find the area of the region bounded by the curve $y = x + 1$, x -axis with the limit given $x = -4$ and $x = -1$.

SOLUTION:

$$A = \left| \int_a^b y \, dx \right|$$

$$A = \left| \int_{-4}^{-1} (x + 1) \, dx \right|$$

$$= \left| \left[\frac{x^2}{2} + x \right]_{-4}^{-1} \right|$$

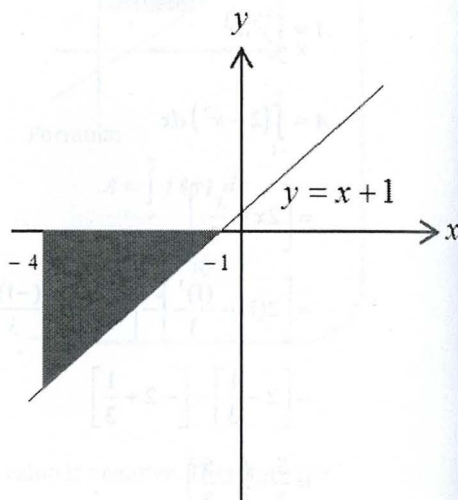
$$= \left| \left[\frac{(-1)^2}{2} + (-1) \right] - \left[\frac{(-4)^2}{2} + (-4) \right] \right|$$

$$= \left| \left[\frac{1}{2} - 1 \right] - [8 - 4] \right|$$

$$= \left| -\frac{1}{2} - 4 \right|$$

$$= \left| -\frac{9}{2} \right|$$

$$= \frac{9}{2} \text{ unit}^2$$



**EXAMPLE 18**

Find the area of the region bounded by the curve $y = x^2$, y -axis with the limit given $y = 1$ and $y = 0$.

SOLUTION:

$$y = x^2$$

$$\therefore x = y^{1/2}$$

$$A = \int_a^b x \, dy$$

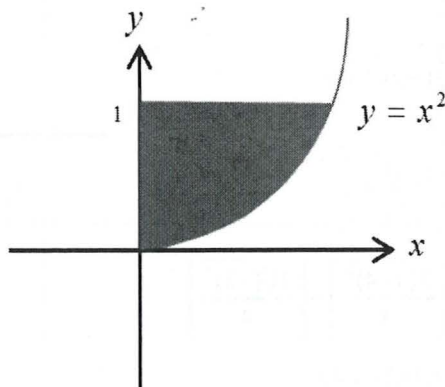
$$A = \int_0^1 y^{1/2} \, dy$$

$$= \left[\frac{y^{3/2}}{3/2} \right]_0^1$$

$$= \left[\frac{2\sqrt{1^3}}{3} \right] - \left[\frac{2\sqrt{0^3}}{3} \right]$$

$$= \frac{2}{3} - 0$$

$$= \frac{2}{3} \text{ unit}^2$$



Find the area of the region bounded by the curve $y = -x^3$, y-axis with the limit given $y = 2$ a $y = 4$.

SOLUTION:

$$y = -x^3$$

$$\therefore x = (-y)^{1/3}$$

$$A = \int_a^b x \, dy$$

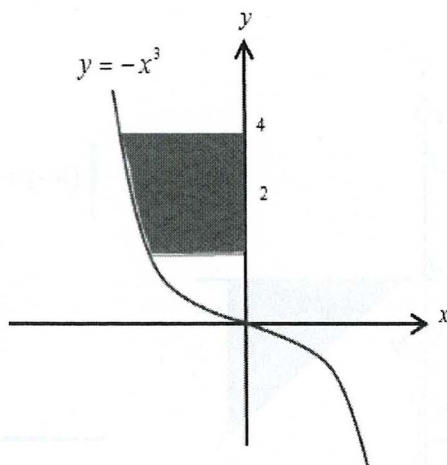
$$A = \int_2^4 (-y)^{1/3} \, dy$$

$$= \left[\frac{(-y)^{4/3}}{4/3} \right]_2^4$$

$$= \left[\frac{3\sqrt[3]{(-4)^4}}{4} \right] - \left[\frac{3\sqrt[3]{(-2)^4}}{4} \right]$$

$$= 4.762 - 1.89$$

$$= 2.8721 \text{ unit}^2$$

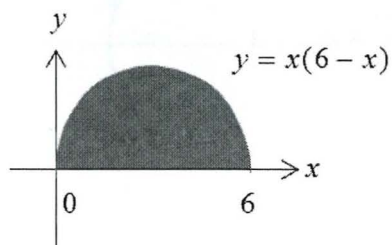




EXERCISE 17

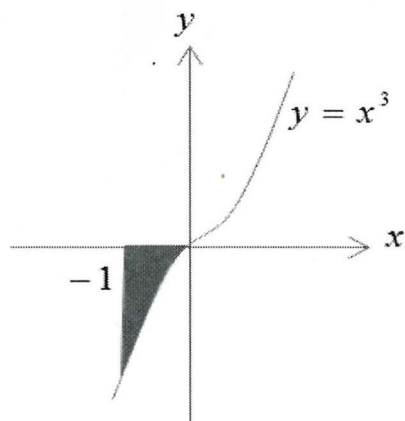
Find the area of the region bounded by the curve of the following figures.

a)



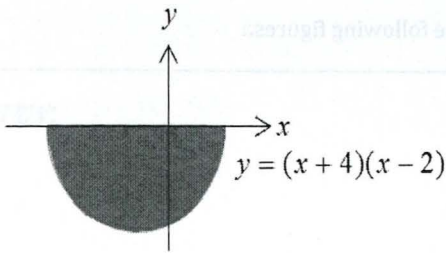
[Ans : 36 unit^2]

b)



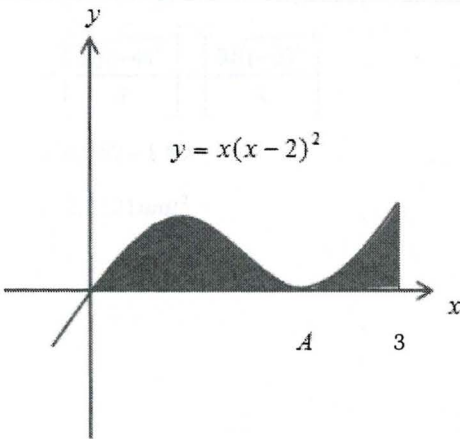
[Ans : $\frac{1}{4} \text{ unit}^2$]

c)



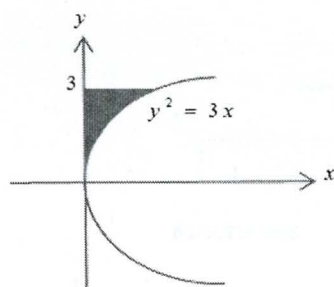
[Ans : 36 unit^2]

d)

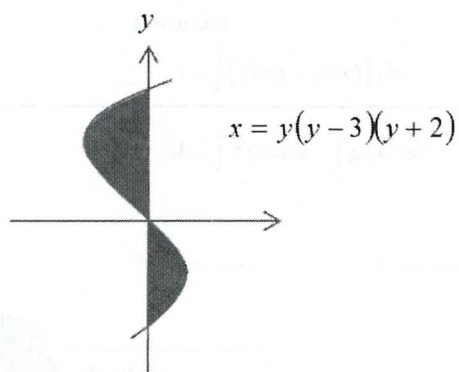


[Ans : $\frac{9}{4} \text{ unit}^2$]

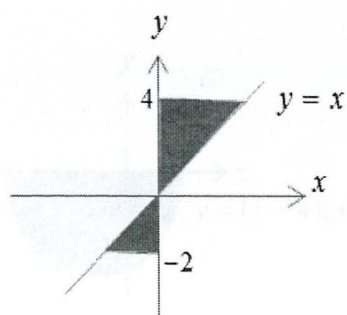
e)

[Ans: 3 unit^2]

f)

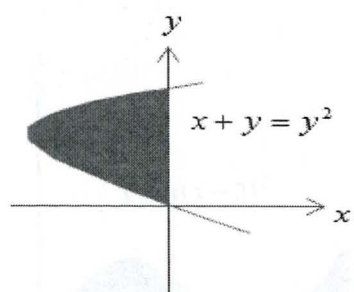
[Ans: $\frac{253}{12} \text{ unit}^2$]

g)



[Ans: 10 unit^2]

h)

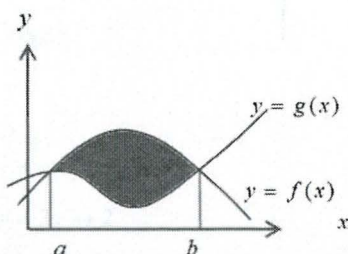


[Ans: $\frac{1}{6} \text{ unit}^2$]

Area bounded by a two function

Area Between Curves

About x-axis



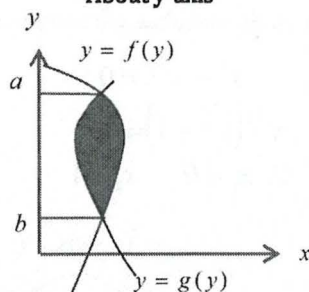
Formula:

$$A = \int_a^b [f(x) - g(x)] dx$$

OR

$$A = \int_a^b f(x) dx - \int_a^b g(x) dx$$

About y-axis



Formula:

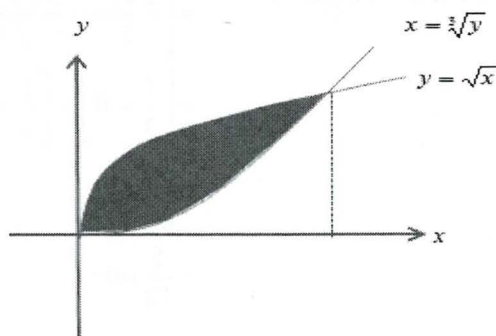
$$A = \int_a^b [f(y) - g(y)] dy$$

OR

$$A = \int_a^b f(y) dy - \int_a^b g(y) dy$$

EXAMPLE 19

- a) Find the area between the curve $x = \sqrt[3]{y}$ and $y = \sqrt{x}$.



SOLUTION:

$$x = \sqrt[3]{y}$$

$$y = x^3$$

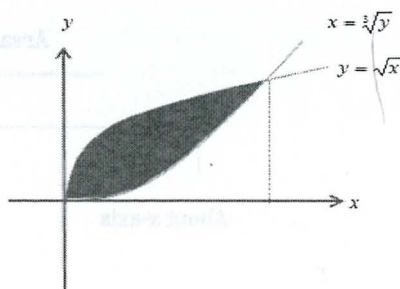
$$y = x^3 \rightarrow y = x^{1/2}$$

$$x^3 = x^{1/2}$$

$$x^3 - x^{1/2} = 0$$

$$x^{1/2}(x^{5/2} - 1) = 0$$

$$\therefore x_1 = 0 \quad x_2 = 1$$



$$A = \int_0^1 x^{1/2} - x^3 dx$$

OR

$$= \left[\frac{x^{3/2}}{3/2} - \frac{x^4}{4} \right]_0^1$$

$$= \left[\frac{2\sqrt{x^3}}{3} - \frac{x^4}{4} \right]_0^1$$

$$= \left[\frac{2\sqrt{1^3}}{3} - \frac{1^4}{4} \right] - \left[\frac{2\sqrt{0^3}}{3} - \frac{0^4}{4} \right]$$

$$= \left[\frac{2}{3} - \frac{1}{4} \right] - 0$$

$$= \left[\frac{2}{3} - \frac{1}{4} \right]$$

$$= \frac{5}{12} \text{ unit}^2$$

$$A = \int_0^1 x^{1/2} dx - \int_0^1 x^3 dx$$

$$= \left[\frac{x^{3/2}}{3/2} \right]_0^1 - \left[\frac{x^4}{4} \right]_0^1$$

$$= \left[\frac{2\sqrt{1^3}}{3} - \frac{2\sqrt{0^3}}{3} \right] - \left[\frac{1^4}{4} - \frac{0^4}{4} \right]$$

$$= \left[\frac{2}{3} - 0 \right] - \left[\frac{1}{4} - 0 \right]$$

$$= \left[\frac{2}{3} - \frac{1}{4} \right]$$

$$= \frac{5}{12} \text{ unit}^2$$

The diagram shows the area bounded by the curve $y = x(6 - x)$ and the line $2x - y = 0$ about the x-axis.

Find:

- the coordinates of points A and B
- the area of the shaded region
- the volume of revolution when the shaded region is rotated through 360° about the x-axis.

$$y = x(6 - x) \rightarrow 1$$

$$y = 2x \rightarrow 2$$

$$1 \rightarrow 2$$

$$x(6 - x) = 2x$$

$$6x - x^2 = 2x$$

$$x^2 - 6x + 2x = 0$$

$$x^2 - 4x = 0$$

$$x(x - 4) = 0$$

$$x_1 = 0 \quad x_2 = 4$$

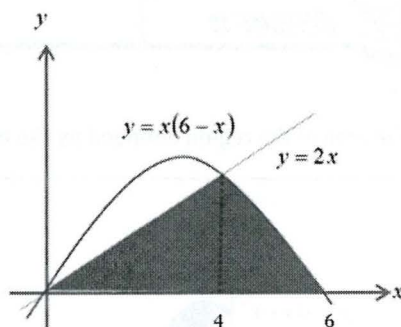
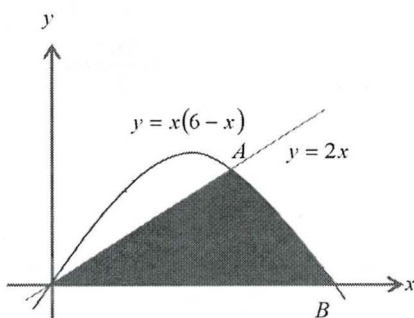
$$\therefore A = (4, 0)$$

$$y = 0 \rightarrow y = x(6 - x)$$

$$0 = x(6 - x)$$

$$x_1 = 0 \quad x_2 = 6$$

$$\therefore B = (6, 0)$$



$$\begin{aligned}
 A_1 &= \int_0^4 y \, dx \\
 &= \int_0^4 2x \, dx \\
 &= \left[\frac{2x^2}{2} \right]_0^4 \\
 &= [(4)^2 - (0)^2] \\
 &= 16 \text{ unit}^2
 \end{aligned}$$

$\therefore$ Area of shaded region $= A_1 + A_2$

$$= 16 + \frac{28}{3}$$

$$= \frac{76}{3} \text{ unit}^2$$

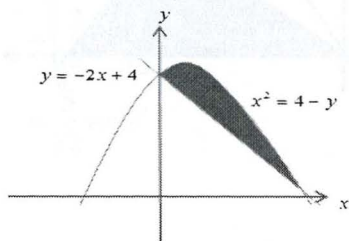
$$\begin{aligned}
 A_2 &= \int_4^6 y \, dx \\
 &= \int_4^6 (6x - x^2) \, dx \\
 &= \left[\frac{6x^2}{2} - \frac{x^3}{3} \right]_4^6 \\
 &= \left[\left(3(6)^2 - \frac{(6)^3}{3} \right) - \left(3(4)^2 - \frac{(4)^3}{3} \right) \right] \\
 &= 36 - \frac{80}{3} \\
 &= \frac{28}{3} \text{ unit}^2
 \end{aligned}$$



EXERCISE 18

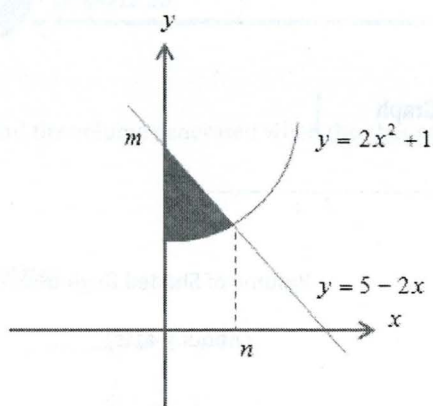
Find the area of the region bounded by the two functions of the following figures.

a)

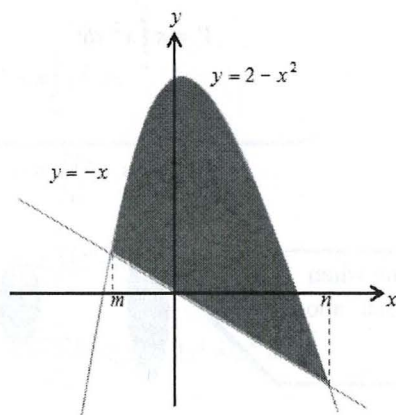


[Ans : $\frac{4}{3} \text{ unit}^2$]

b)

[Ans: $\frac{7}{3} \text{ unit}^2$]

c)

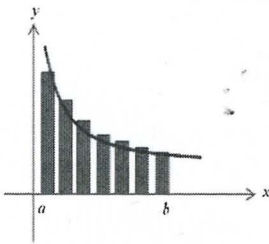
[Ans: $\frac{9}{2} \text{ unit}^2$]

Calculate volume of object generated when the bounded region is revolved along x-axis and y-axis

Volume of the Graph

Volume of Shaded Region

(About x-axis)

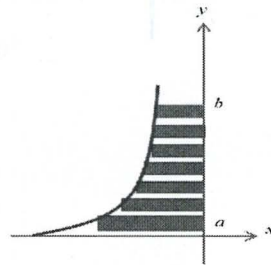


Formula:

$$V = \pi \int_a^b y^2 dx$$

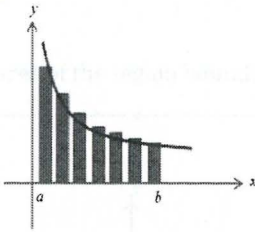
Volume of Shaded Region

(About y-axis)

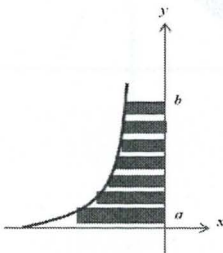
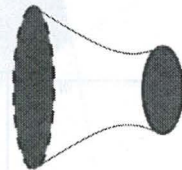


Formula:

$$V = \pi \int_a^b x^2 dy$$



The shaded region when rotated through 360° about x-axis



The shaded region when rotated through 360° about y-axis



**EXAMPLE 20**

- a) Find the volume generated when the shaded region rotated 360° about the x -axis.

SOLUTION:

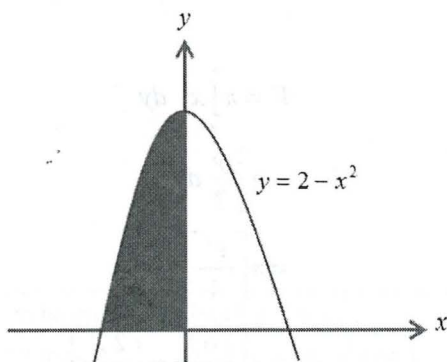
$$y = 0 \rightarrow y = 2 - x^2$$

$$0 = 2 - x^2$$

$$x^2 = 2$$

$$x = \sqrt{2}$$

$$x_1 = -1.41 \quad x_2 = 1.41$$



$$V = \pi \int_a^b y^2 dx$$

$$= \pi \int_{-1.41}^0 (2 - x^2)^2 dx$$

$$= \pi \int_{-1.41}^0 (2 - x^2)(2 - x^2) dx$$

$$= \pi \int_{-1.41}^0 4 - 2x^2 - 2x^2 + x^4 dx$$

$$= \pi \int_{-1.41}^0 4 - 4x^2 + x^4 dx$$

$$= \pi \left[4x - \frac{4x^3}{3} + \frac{x^5}{5} \right]_{-1.41}^0$$

$$= \pi \left[0 - \left(4(-1.41) - \frac{4(-1.41)^3}{3} + \frac{(-1.41)^5}{5} \right) \right]$$

$$= \pi [0 - (-5.64 + 3.738 - 1.115)]$$

$$= 3.017\pi \text{ unit}^3$$

- b) Find the volume generated when the shaded region rotated 360° about the y-axis.

SOLUTION:

$$y = 2x^2$$

$$\frac{y}{2} = x^2$$

$$V = \pi \int_a^b x^2 dy$$

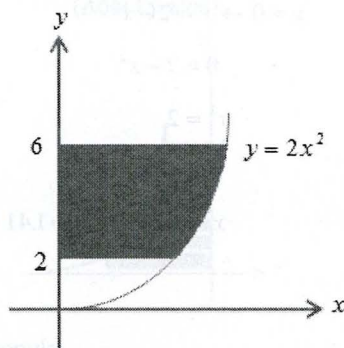
$$= \pi \int_2^6 \frac{y}{2} dy$$

$$= \pi \left[\frac{y^2}{4} \right]_2^6$$

$$= \pi \left[\frac{(6)^2}{4} - \frac{(2)^2}{4} \right]$$

$$= \pi [9 - 1]$$

$$= 8\pi \text{ unit}^3$$



Find the volume of the region bounded when rotated through 360° of the following questions.

- a) Find the volume of the solid of revolution when the region bounded by the curve $y - 1 = x^2$, and the lines $x = 3$, $x = 1$ is rotated through 360° about x-axis. [Ans : $67.7\pi \text{ unit}^3$]

- b) Find the volume of the solid of revolution when the region bounded by the curve $y = \sqrt{x}$, and the lines $x = 4$, $x = 1$ is rotated through 360° about x-axis. [Ans : $\frac{15}{2}\pi \text{ unit}^3$]

- c) Find the volume of the solid of revolution when the region bounded by the curve $y = x^{1/2}$ and the lines $y = 2$, $y = -2$ is rotated through 360° about y-axis. [Ans: $\frac{16}{3}\pi \text{ unit}^3$]

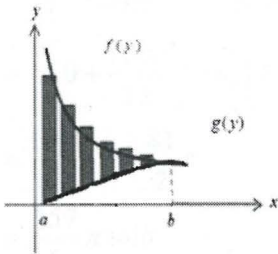
- d) Find the volume of revolution when the area bounded by the curve $(y-2)^2 = x$, the lines $y = 2$ and $y = 0$ is rotated through 360° about y-axis. [Ans: $\frac{32}{5}\pi \text{ unit}^3$]

Volume between a curve and a straight line

Volume of Revolution Between Two Curve

Volume of Shaded Region

(About x-axis)



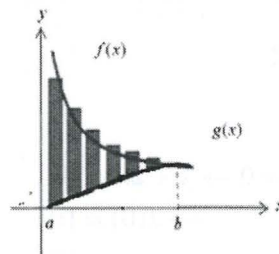
Formula:

$$V = \pi \int_a^b [f(x)^2 - g(x)^2] dx$$

$$V = \pi \left[\int_a^b f(x)^2 dx - \int_a^b g(x)^2 dx \right]$$

Volume of Shaded Region

(About y-axis)



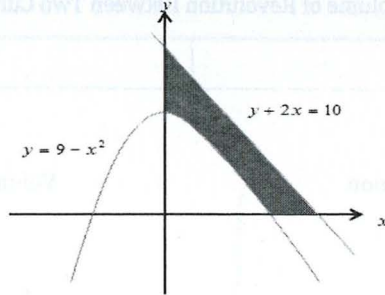
Formula:

$$V = \pi \int_a^b [f(y)^2 - g(y)^2] dy$$

$$V = \pi \left[\int_a^b f(y)^2 dy - \int_a^b g(y)^2 dy \right]$$

**EXAMPLE 21**

Find the volume generated when the shaded region rotated 360° about the y -axis.



SOLUTION:

a)

$$x = 0 \rightarrow y + 2x = 10$$

$$y + 2(0) = 10$$

$$y = 10$$

$$x = 0 \rightarrow y = 9 - x^2$$

$$y = 9 - (0)^2$$

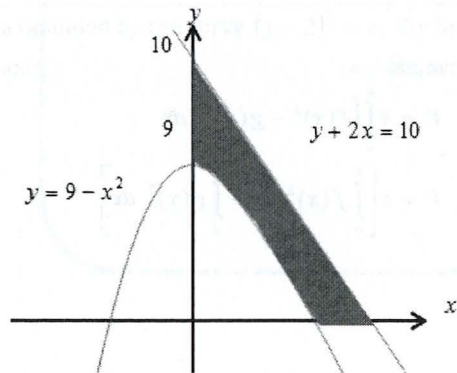
$$y = 9$$

$$y + 2x = 10$$

$$x = \frac{10 - y}{2}$$

$$y = 9 - x^2$$

$$x^2 = 9 - y$$



$$\begin{aligned}
 V &= \pi \left[\int_0^{10} x^2 dy - \int_0^9 x^2 dy \right] \\
 &= \pi \left[\int_0^{10} \left(\frac{10-y}{2} \right)^2 dy - \int_0^9 9-y dy \right] \\
 &= \pi \left[\int_0^{10} \frac{(10-y)^2}{4} dy - \int_0^9 9-y dy \right] \\
 &= \pi \left[\frac{(10-y)^3}{4(-1)(3)} \right]_0^{10} - \pi \left[9y - \frac{y^2}{2} \right]_0^9 \\
 &= \pi \left[\frac{(10-10)^3}{-12} - \frac{(10-0)^3}{(-12)} \right] - \pi \left[\left(9(9) - \frac{9^2}{2} \right) - \left(9(0) - \frac{0^2}{2} \right) \right] \\
 &= \pi \left[0 + \frac{1000}{12} \right] - \pi \left[\left(81 - \frac{81}{2} \right) - 0 \right] \\
 &= \frac{1000}{12} \pi - \frac{81}{2} \pi \\
 &= \frac{257}{6} \pi \text{ unit}^3
 \end{aligned}$$



EXERCISE 20

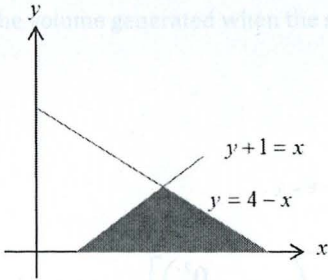
a) Find the volume of the solid of revolution when the region bounded by the curve $y = \sqrt{x}$,

$y = \frac{x}{4}$, $x = 0$ and $x = 4$ is rotated through 360° about x -axis.

[Ans: $\frac{20}{3} \pi \text{ unit}^3$]

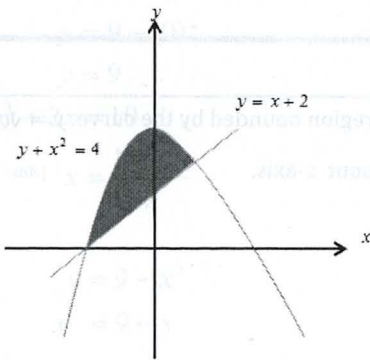
- b) Find the volume of the solid of revolution when the region bounded by the curve $y + 1 = x$, $y = 4 - x$ and the x-axis revolves 360° about x-axis.

[Ans: $\frac{9}{4} \pi \text{ unit}^3$]



- c) Find the volume of the solid of revolution when the region is rotated through 360° about x-axis.

[Ans: $\frac{42}{5} \pi \text{ unit}^3$]





TUTORIALS

1. By using a suitable method, find the following integrals:

a) $\int -\frac{\pi}{3} dz$

[Ans: $-\frac{\pi}{3}z + c$]

b) $\int dk$

[Ans: $k + c$]

c) $\int (x+3)(x-5) dx$ [Ans: $\frac{x^3}{3} - x^2 - 15x + c$]

d) $\int_5^6 6 dz$

[Ans: 6]

e) $\int_2^4 \sqrt{x^3} - \frac{1}{2x^2} dx$

[Ans: 10.412]

f) $\int_1^8 \frac{5s^2 - 2}{s^4} ds$

[Ans: 3.7096]

g) $\int \frac{3m^2}{m^3 - 7} dm$

[Ans: $\ln|m^3 - 7| + c$]

h) $\int \frac{\pi}{n-5} dn$

[Ans: $\pi \ln|n-5| + c$]

i) $\int \frac{3z}{2z^2 - 8} dz$

[Ans: $\frac{3}{4} \ln|2z^2 - 8| + c$]

j) $\int \frac{dx}{10-x}$

[Ans: $-\ln|10-x| + c$]

k) $\int \frac{x^2 dx}{\sqrt{36-4x^9}}$

[Ans: $\frac{1}{6} \sin^{-1} \frac{x^3}{3} + c$]

l) $\int \frac{dy}{3y^2 + 18y + 30}$

[Ans: $\frac{1}{3} \tan^{-1}(y+3) + c$]

2. By using a suitable method, find the following integrals:

a) $\int (1 + \cos 7x) dx$

[Ans: $x + \frac{\sin 7x}{7} + c$]

b) $\int (3e^x + 4 \sin x) dx$

[Ans: $3e^x - 4 \cos x + c$]

c) $\int e^x - \sin x dx$

[Ans: $e^x + \cos x + c$]

d) $\int \frac{e^x}{\sqrt{1+e^x}} dx$

[Ans: $2\sqrt{1+e^x} + c$]

e) $\int e^x (1 + e^x)^3 dx$

[Ans: $\frac{(1+e^x)^4}{4} + c$]

f) $\int \frac{\sin x}{1 + \cos x} dx$

[Ans: $\ln|1 - \cos x| + c$]

g) $\int_0^{\frac{\pi}{6}} x \sec^2 2x dx$

[Ans: 0.28]

h) $\int_{\frac{\pi}{6}}^{\frac{\pi}{2}} (1 - \cos x) dx$

[Ans: 0.547]

3. By using integration by composite function, find the following integrals:

$$\begin{array}{ll}
 \text{a) } \int -\frac{3}{\sqrt{6x-8}} dx & [\text{Ans: } -\sqrt{6x-8} + c] \\
 \text{b) } \int (2-t^3)^2 dt & [\text{Ans: } -\frac{(2-t^3)^3}{9t^2} + c] \\
 \text{c) } \int \frac{\pi}{(3x+6)^2} dx & [\text{Ans: } -\frac{\pi}{3(3x+6)} + c] \\
 \text{d) } \int \frac{dz}{(4-z)^3} & [\text{Ans: } \frac{1}{2(4-z)^2} + c] \\
 \text{e) } \int (1-\cos 2x)^2 dx & [\text{Ans: } \frac{(1-\cos 2x)^3}{6\sin 2x} + c] \\
 \text{f) } \int (2+e^{2x}) dx & [\text{Ans: } \frac{(2+e^{2x})^2}{4e^{2x}} + c]
 \end{array}$$

4. By using integration by substitution method, find the following integrals:

$$\begin{array}{ll}
 \text{a) } \int \sin^5 x \, dx & [\text{Ans: } \frac{2}{3} \cos^3 x - \cos x - \frac{1}{5} \cos^5 x + c] \\
 \text{b) } \int \frac{x-1}{\sqrt{x-2}} dx & [\text{Ans: } \frac{2}{3} (x-2)^{\frac{3}{2}} + 2\sqrt{x-2} + c] \\
 \text{c) } \int \frac{1}{x \ln x} dx & [\text{Ans: } \ln(\ln x) + c] \\
 \text{d) } \int 2z\sqrt{1+z^2} dz & [\text{Ans: } \frac{2}{3} (1+z^2)^{\frac{3}{2}} + c] \\
 \text{e) } \int \frac{1}{(11-3\theta)^9} d\theta & [\text{Ans: } \frac{1}{24(11-3\theta)^8} + c] \\
 \text{f) } \int \frac{\pi}{(5+3z)^2} dz & [\text{Ans: } -\frac{\pi}{3(5+3z)} + c]
 \end{array}$$

5. Find the following integrals by using integration by part:

$$\begin{array}{ll}
 \text{a) } \int x^3 e^{x^2+1} dx & [\text{Ans: } \frac{x^2 e^{x^2+1}}{2} - \frac{e^{x^2+1}}{2} + c] \\
 \text{b) } \int \cos x \ln(\sin x) dx & [\text{Ans: } \ln \sin x - \sin x + c] \\
 \text{c) } \int_{12^0}^{20^0} x^3 \cos x^2 dx & [\text{Ans: } 0.797] \\
 \text{d) } \int_0^{\ln 2} 3x e^{3x} dx & [\text{Ans: } 3.21]
 \end{array}$$

6. Integrate each of the following with respect to x:

$$\begin{array}{ll}
 \text{a) } \int \frac{3}{x^2-4} dx & [\text{Ans: } \frac{3}{4} \ln(x-2) - \frac{3}{4} \ln(x+2) + c] \\
 \text{b) } \int \frac{x^2}{16-x^2} dx & [\text{Ans: } -x - 2 \ln(4-x) + 2 \ln(4+x) + c] \\
 \text{c) } \int \frac{2-x}{(5+x)^2} dx & [\text{Ans: } -\ln(5+x) - \frac{7}{5+x} + c] \\
 \text{d) } \int \frac{x+20}{x^2-4x-21} dx & [\text{Ans: } \frac{27}{10} \ln(x-7) - \frac{17}{10} \ln(x+3) + c]
 \end{array}$$

7. Find the area of the region bounded by the curve $y = -x^2 - 2x + 8$ about the x-axis.

$$[\text{Ans: } 36 \text{ unit}^2]$$

8. Find the area of the region bounded by the curve $x = (y-1)^2$ about the y-axis and the limit given $y = 2$ and $y = 4$. [Ans: $\frac{26}{3} \text{ unit}^2$]

9. Find the area of the region bounded by the curve $x = 25 - y^2$ and $x = y^2 + 17$.

[Ans: $\frac{64}{3} \text{ unit}^2$]

10. Find the volume of the solid revolution when the region bounded by the curve $\sqrt{y} = x$ and the limit given $x = 0$ and $x = 1$ is rotated through 360° about the x-axis. [Ans: $\frac{1}{5} \pi \text{ unit}^3$]

11. Find the volume of the solid revolution when the region bounded by the curve $x = \sqrt{9 - y^2}$ and the limit given $y = 1$, $y = 2$ is rotated through 360° about the y-axis. [Ans: $\frac{20}{3} \pi \text{ unit}^3$]

12. Find the volume of the solid revolution when the region bounded by the curve $y = 3(x^2 + 1)$ and the limit given $x = 1$, $x = 3$ is rotated through 360° about the x-axis.

[Ans: $609.597 \pi \text{ unit}^3$]

00029207

$$\int \frac{dx}{2x^2}$$

$$= \frac{x^2}{2} dx$$

$$= \frac{x^{2+1}}{2} dx$$

$$= \frac{x^{-1}}{2(-1)}$$

$$= \frac{x^{-1}}{-2}$$

$$(1) \frac{x^{-1}}{-2} + C$$

$$= -\frac{1}{2x} + C$$

TARIKH DIPULANGKAN
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ISBN 978 967 2248 05 1



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